

1-1-2010

## Pseudo simplicial groups and crossed modules

İLKER AKÇA

SEDAT PAK

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)

---

### Recommended Citation

AKÇA, İLKER and PAK, SEDAT (2010) "Pseudo simplicial groups and crossed modules," *Turkish Journal of Mathematics*: Vol. 34: No. 4, Article 5. <https://doi.org/10.3906/mat-0807-20>

Available at: <https://journals.tubitak.gov.tr/math/vol34/iss4/5>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact [academic.publications@tubitak.gov.tr](mailto:academic.publications@tubitak.gov.tr).

# Pseudo simplicial groups and crossed modules

*İ. Akça and S. Pak*

## Abstract

In this paper, we define the notion of pseudo 2-crossed module and give a relation between the pseudo 2-crossed modules and pseudo simplicial groups with Moore complex of length 2.

**Key Words:** Crossed modules, Pseudo simplicial groups, Moore complex.

## 1. Introduction

Simplicial groups occupy a place somewhere between homological group theory, homotopy theory, algebraic K-theory and algebraic geometry. In each sector they have played a significant part in developments over quite a lengthy period of time and there is an extensive literature on their homotopy theory.

Crossed modules were introduced by Whitehead in [15] with a view to capturing the relationship between  $\pi_1$  and  $\pi_2$  of a space. Homotopy systems (which would now be called free crossed complexes [5] or totally free crossed chain complexes [3], [4]) were introduced, again by Whitehead, to incorporate the action of  $\pi_1$  on the higher relative homotopy groups of a *CW*-complex. They consist of a crossed module at the base and a chain complex of modules over  $\pi_1$  further up.

Conduché [6] defined the notion of 2-crossed module, as a model of connected 3-types and showed how to obtain a 2-crossed module from a simplicial group.

Inasaridze(c.f. [8],[9]) constructed homotopy groups of pseudosimplicial groups and nonabelian derived functors with values in the category of groups.

In this paper we analysis the low dimensional parts of the Moore complex of a pseudosimplicial group. We prove that the category of crossed modules is equivalent to the category of pseudosimplicial groups with Moore complex of length 1. We extend this result to 2-dimension by defining pseudo 2-crossed modules and give the relation between the category of pseudo 2-crossed modules and the category of pseudosimplicial groups with Moore complex of length 2.

The above theorems, in some sense, are well known. We give details of the proofs as analogous proofs can be found in the literature [1], [2], [6], [10] and [13].

**2. Pseudo simplicial groups**

A *pseudo simplicial group*  $\mathbf{G}$  consists of  $\{G_n\}$  together with boundary homomorphisms  $\partial_i^n : G_n \rightarrow G_{n-1}$ ,  $0 \leq i \leq n$ , ( $n \neq 0$ ) and pseudo degeneracies  $s_i^n : G_n \rightarrow G_{n+1}$ ,  $0 \leq i \leq n$ , satisfying the following pseudosimplicial identities:

$$\begin{aligned} \partial_i^{n-1} \partial_j^n &= \partial_{j-1}^{n-1} \partial_i^n && \text{for } i < j \\ \partial_i^{n+1} s_j^n &= s_{j-1}^{n-1} \partial_i^n && \text{for } i < j \\ \partial_j^{n+1} s_j^n &= 1 = \partial_{j+1}^{n+1} s_j^n \\ \partial_i^{n+1} s_j^n &= s_j^{n-1} \partial_{i-1}^n && \text{for } i > j + 1, \end{aligned}$$

The groups  $G_n$  can be nonabelian. To obtain the definition of *simplicial group*, we must add the condition that  $s_i^{n+1} s_j^n = s_{j+1}^{n+1} s_i^n$  for  $i \leq j$  (see [11]).

A topological interpretation is, for example, the  $F$ -construction of Milnor [12], which gives the simplicial group of loops of the suspension of a complex. For an arbitrary simplicial set  $\mathbf{K}$  with pole  $\psi$ , the group of  $n$ -simplicies  $FK_n$  is the free group on a family of generators  $\sigma$  in one-to-one correspondence with the  $n$ -simplicies  $\sigma \in K_n$ , with the single relation  $(s_{n-1} s_{n-2} \dots s_0(\psi)) = e_n$ , while the boundary and degeneracy homomorphisms are induced by the corresponding mappings of the set  $\mathbf{K}$ .

For any pseudosimplicial group  $\mathbf{G}$ , put  $NG_n = G_n \cap \text{Ker } \partial_0^n \cap \dots \cap \text{Ker } \partial_{n-1}^n$ ,  $n \geq 0$ , and let  $d_n$  be the restriction of  $\partial_n^n$  to  $NG_n$ ,  $n > 0$ . Then  $\text{im } d_n$  is a normal subgroup of  $G_{n-1}$ , and  $\text{im } d_{n+1} \subset \text{Ker } d_n$  for  $n > 0$ . This determines the Moore complex  $\mathbf{NG} = \{NG_n, d_n\}$ . Clearly  $\mathbf{NG}$  is independent of the pseudodegeneracies, depending only on the boundary homomorphisms.

The  $n$ -dimensional homology group of the Moore complex  $\mathbf{NG}$  is called the *n-dimensional homotopy group*  $\pi_n(\mathbf{G})$  of the pseudosimplicial group  $\mathbf{G}$ ,  $n \geq 0$ .

A mapping  $\mathbf{f} : \mathbf{G} \rightarrow \mathbf{G}'$  induces, in a natural fashion, homomorphisms  $\pi_n(\mathbf{f}) : \pi_n(\mathbf{G}) \rightarrow \pi_n(\mathbf{G}')$ ,  $n \geq 0$ .

Let  $\mathbf{f}$  and  $\mathbf{g}$  be two mappings from  $\mathbf{G}$  to  $\mathbf{G}'$ . The following definition is due to Inassaridze [9].  $\mathbf{f}$  is *pseudohomotopic* to  $\mathbf{g}$  if there exist homomorphisms  $h_i^n : G_n \rightarrow G'_{n+1}$ ,  $0 \leq i \leq n$ , such that

$$\begin{aligned} \partial_0^{n+1} h_0^n &= f_n && \partial_{n+1}^{n+1} h_n^n = g_n, \\ \partial_i^{n+1} h_j^n &= h_{j-1}^{n-1} \partial_i^n && \text{for } i < j, \\ \partial_{j+1}^{n+1} h_{j+1}^n &= \partial_{j+1}^{n+1} h_j^n, \\ \partial_i^{n+1} h_j^n &= h_j^{n-1} \partial_{i-1}^n && \text{for } i > j + 1. \end{aligned}$$

To obtain the definition of homotopy of  $\mathbf{f}$  to  $\mathbf{g}$ , we must add the following conditions:

$$s_i^{n+1} h_j^n = h_{j+1}^{n+1} s_i^n \text{ for } i \leq j, \text{ and } s_i^{n+1} h_j^n = h_j^{n+1} s_{i-1}^n \text{ for } i > j.$$

**Theorem 2.1** [9] *The homotopy groups  $\pi_n(\mathbf{G})$  are abelian for  $n \geq 1$ . If the mapping  $\mathbf{f} : \mathbf{G} \rightarrow \mathbf{G}'$  is pseudohomotopic to a mapping, then  $\pi_n(\mathbf{f}) = \pi_n(\mathbf{g})$ ,  $n \geq 0$ .*

A mapping  $\mathbf{f} : \mathbf{G} \rightarrow \mathbf{G}'$  of pseudosimplicial groups is called *simplicial* if it satisfies the condition  $f_{n+1} s_i^n = s_i^n f_n$  for  $n \geq 0$ ,  $0 \leq i \leq n$ . A simplicial map  $\mathbf{f} : \mathbf{G} \rightarrow \mathbf{G}'$  is called a *weak equivalence* if it induces

isomorphisms  $\pi_n(\mathbf{G}) \cong \pi_n(\mathbf{G}')$  for  $n \geq 0$ . A simplicial map  $\mathbf{f} : \mathbf{G} \rightarrow \mathbf{G}'$  called a *fibration* if  $f_n : G_n \rightarrow G'_n$  is surjective for  $n \geq 0$ .

By a  $k$ -truncated pseudosimplicial group we mean a collection of groups  $\{G_0, \dots, G_k\}$  and boundary homomorphisms  $\partial_i^n : G_n \rightarrow G_{n-1}$  for  $0 \leq i \leq n, 0 \leq n \leq k$  and pseudodegeneracies  $s_i^n : G_n \rightarrow G_{n+1}$  for  $0 \leq i \leq n, 0 \leq n \leq k$  which satisfy the pseudosimplicial identities. Clearly by forgetting higher dimensions, any pseudosimplicial group  $\mathbf{G}$  yields a  $k$ -truncated pseudosimplicial group  $tr^k \mathbf{G}$ . The functor  $tr^k$  admits a right adjoint  $\cos k^k$ , called the *k-coskeleton functor*, and a left adjoint functor  $sk^k$  called the *k-skeleton functor*. We recall from [7] a brief description of these functors.

Suppose  $tr^k(\mathbf{G}) = \{G_0, \dots, G_k\}$  is a pseudosimplicial group. A family of homomorphisms

$$\begin{array}{ccc}
 & \xrightarrow{\delta_{k+1}} & \\
 (\delta_0, \dots, \delta_{k+1}) : X_{k+1} & \vdots & G_k \\
 & \xrightarrow{\delta_0} &
 \end{array}$$

is the *simplicial kernel* of the family of boundary homomorphisms  $(\partial_0, \dots, \partial_k)$  if it has the following universal property: given any family  $(\partial_0, \dots, \partial_{k+1})$  of  $k + 2$  homomorphisms  $\partial_i : Y \rightarrow G_k$  satisfying the identities  $\partial_i \partial_j = \partial_{j-1} \partial_i$  ( $0 \leq i < j \leq k + 1$ ) with the last part of the truncated pseudosimplicial group, there exists a unique homomorphism  $f : Y \rightarrow X_{k+1}$  such that  $\delta_i f = \partial_i$ . Given the simplicial kernel  $X_{k+1}$  the family of

homomorphisms  $(\alpha_{n+1j}, \dots, \alpha_{1j}, \alpha_{0j})$ , defined by

$$\alpha_{ij} = \begin{cases} s_{j-1} & i < j \\ id & i = j, i = j + 1 \\ s_j d_{i-1} & i > j + 1, \end{cases}$$

satisfies the pseudosimplicial identities with the last part of the truncated pseudosimplicial group; hence there exists a unique  $s_j : G_k \rightarrow X_{k+1}$  such that  $\delta_i s_j = \alpha_{ij}$ . We thus have a  $(k + 1)$ -truncated pseudosimplicial group  $\{G_0, \dots, G_k, X_{k+1}\}$ . By iterating this construction we get a pseudosimplicial group  $\cos k^k(tr^k(\mathbf{G})) = \{G_0, \dots, G_k, X_{k+1}\}$  called the *coskeleton* of the truncated pseudosimplicial group. If  $\mathbf{G}, \mathbf{G}'$  are any pseudosimplicial groups, than any truncated simplicial map  $\mathbf{f} : tr^k \mathbf{G} \rightarrow tr^k \mathbf{G}'$  extends uniquely to a simplicial map  $\mathbf{f} : \mathbf{G} \rightarrow \cos k^k(tr^k(\mathbf{G}'))$ .

The  $k$ -skeleton functor can be constructed by a dual process involving pseudosimplicial cokernels

$$\begin{array}{ccc}
 & \xrightarrow{s_k} & \\
 (s_0, \dots, s_k) : G_k & \vdots & X_{k+1}. \\
 & \xrightarrow{s_0} &
 \end{array}$$

(That is, universal systems of  $k + 1$  arrows which satisfy pseudosimplicial identities.)

**3. Crossed modules and pseudo 2-crossed modules**

**3.1. Crossed modules**

J.H.C. Whitehead (1949) [15] described crossed modules in various contexts, especially in his investigation into the group structure of relative homotopy groups.

**Definition 3.1** *Let  $P$  be a group. A pre-crossed module of groups is a  $P$ -group  $M$ , and a group homomorphism*

$$\partial : M \longrightarrow P$$

such that

$$CM1) \partial(p m) = p \partial(m) p^{-1}$$

for all  $m \in M, p \in P$ . This is a crossed  $P$ -module if, in addition,

$$CM2) \partial^{(m)} m' = m m' m^{-1}$$

for all  $m, m' \in M$ . The last condition is called the Peiffer identity. We denote such a crossed module by  $(M, P, \partial)$ . A map of crossed modules

$$(\partial : M \longrightarrow P) \longrightarrow (\partial' : M' \longrightarrow P')$$

is a pair of homomorphisms  $f_0 : P \longrightarrow P', f_1 : M \longrightarrow M'$  such that  $f_0 \partial = \partial' f_1$  and  $f_1(p m) = {}^{(f_0 p)} f_1(m)$  for all  $m \in M, p \in P$ .

The Moore complex

$$\dots \longrightarrow M_n \longrightarrow \dots \longrightarrow M_1 \longrightarrow M_0$$

of a pseudosimplicial group is of length  $k$ , if  $M_n = 0$  for all  $n \geq k + 1$  (so a Moore complex of length  $k$  is also of length  $r$  for  $r \geq k$ ).

The following lemma is a straightforward modification of Theorem 1.3 in [6].

**Lemma 3.2** *Let  $\mathbf{G}$  be a pseudosimplicial group. The Moore complex of its  $k$ -coskeleton  $\text{cos } k^k(\text{tr}^k \mathbf{G})$  is of length  $k + 1$ , and is identical to the Moore complex of  $\mathbf{G}$  in dimensions  $\leq k$ . Moreover, in dimensions  $k - 1$  to  $k + 2$  the Moore complex of  $\text{cos } k^k(\text{tr}^k \mathbf{G})$  is an exact sequence*

$$1 \longrightarrow N(\text{cos } k^k(\text{tr}^k \mathbf{G}))_{k+1} \xrightarrow{\partial_{k+1}} N\mathbf{G}_k \xrightarrow{\partial_k} N\mathbf{G}_{k-1}.$$

where  $N_k$  is the  $k$ th term of the Moore complex of  $G$ .

**Proof.** The  $(k + 1)$ -dimensional part of  $\text{cos } k^k(\text{tr}^k G)$  can be identified with the subgroup of the  $(k + 2)$ -fold direct sum  $G_k^{k+2}$  consisting of those elements  $(x_0, \dots, x_{k+1})$  such that  $d_j x_k = d_{k-1} x_j$  for  $j < k$ ; the face maps are given by  $d_j(x_0, \dots, x_{k+1}) = x_j$ . Thus  $N(\text{cos } k^k(\text{tr}^k \mathbf{G}))_{k+1}$  consists of elements  $(1, \dots, 1, x_{k+1})$  such that  $d_j x_{k+1} = 1$  for all  $j$ . In other words  $N(\text{cos } k^k(\text{tr}^k \mathbf{G}))_{k+1}$  is the kernel of  $\partial_k : N\mathbf{G}_k \longrightarrow N\mathbf{G}_{k-1}$ , and hence we have the exact sequence of the lemma.

The injectivity of  $\partial_{k+1}$  and the isomorphism

$$\cos k^{n-1} (tr^{n-1} (\cos k^k (tr^k G))) \simeq \cos k^k (tr^k G)$$

for  $n \geq k + 2$  shows that the Moore complex of  $\cos k^k (tr^k G)$  is of length  $k + 1$ . □

The following theorem is well known. In [6] and [10] this theorem was proved.

**Theorem 3.3** ([6], [10]) *The category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length 1.*

Now, we shall give the pseudo version of this theorem.

**Theorem 3.4** *The category of crossed modules is equivalent to the category of pseudosimplicial groups with Moore complex of length 1.*

**Proof.** Let  $\mathbf{G}$  be a pseudosimplicial group with Moore complex of length 1. Put  $P = NG_0 = G_0$ ,  $M = NG_1 = \ker(d_0 : G_1 \rightarrow G_0)$  and  $\partial = d_1$  (restricted to  $M$ ). Then  $p \in P$  acts on  $m \in M$  by  ${}^p m = s_0(p) m s_0(p)^{-1}$ , and  $\partial({}^p m) = d_1(s_0(p) m s_0(p)^{-1})$ . Since the Moore complex  $\cdots \rightarrow 1 \rightarrow M \xrightarrow{\partial} P \rightarrow 1$  is of length 1, we have  $\partial_2 NG_2 = 1$ . It then follows that for all  $m, m' \in M$  and  $p \in P$ ,

$$\begin{aligned} (i) \quad \partial_1({}^p m) &= d_1({}^p m) \\ &= d_1(s_0(p) m s_0(p)^{-1}) \\ &= d_1 s_0(p) d_1(m) d_1 s_0(p)^{-1} \\ &= p \partial_1(m) p^{-1} \\ (ii) \quad (\partial_1 m) m' &= s_0 \partial_1(m) m' s_0 \partial_1(m)^{-1} \\ &= s_0 d_1(m) m' s_0 d_1(m)^{-1} \\ &= s_0 d_1(m) m' s_0 d_1(m)^{-1} [(m(m')^{-1} m^{-1})(mm' m^{-1})] \\ &= d_2 s_0(m) d_2 s_1(m') d_2 s_0(m)^{-1} d_2 s_1(m) d_2 s_1(m')^{-1} \\ &\quad d_2 s_1(m^{-1})(mm' m^{-1}) \\ &= d_2(s_0(m) s_1(m') s_0(m)^{-1} s_1(m) s_1(m')^{-1} s_1(m)^{-1}) \\ &\quad (mm' m^{-1}) \\ &= mm' m^{-1} \end{aligned}$$

for  $m, m' \in M$ , because  $s_0(m) s_1(m') s_0(m)^{-1} s_1(m) s_1(m')^{-1} s_1(m)^{-1}$  lies in  $\partial_2 NG_2$ . Thus  $\partial : M \rightarrow P$  is a crossed module.

Conversely, let  $\partial : M \rightarrow P$  be a crossed module. By using the action of  $P$  on  $M$  we can form the semi-direct product  $M \rtimes P = \{(m, p) : m \in M, p \in P\}$ , in which multiplication

$$(m, p) \cdot (m', p') = (m {}^p m', pp')$$

for  $m, m' \in M, p, p' \in P$ . There are homomorphisms

$$\begin{aligned} d_0 : M \rtimes P &\rightarrow P, & (m, p) &\mapsto p, \\ d_1 : M \rtimes P &\rightarrow P, & (m, p) &\mapsto (\partial m) p, \\ s_0 : P &\rightarrow M \rtimes P, & p &\mapsto (1, p). \end{aligned}$$

Let  $G_0 = P$ ,  $G_1 = M \rtimes P$ . We have a 1-truncated pseudosimplicial group  $\{G_0, G_1\}$  whose 1-coskeleton we denote by  $G^1$ . The group  $M \rtimes P$  acts on  $M$  via the action of  $P$  on  $M$  and the homomorphism  $d_1$ . We can thus form the semi-direct product  $M \rtimes (M \rtimes P)$  and construct homomorphisms

$$\begin{aligned} d_0 : M \rtimes (M \rtimes P) &\longrightarrow M \rtimes P, & (m, m', p) &\longmapsto (m', p), \\ d_1 : M \rtimes (M \rtimes P) &\longrightarrow M \rtimes P, & (m, m', p) &\longmapsto (mm', p), \\ d_2 : M \rtimes (M \rtimes P) &\longrightarrow M \rtimes P, & (m, m', p) &\longmapsto (m, (\partial m')p), \\ s_0 : M \rtimes P &\longrightarrow M \rtimes (M \rtimes P), & (m, p) &\longmapsto (1, m, p), \\ s_1 : M \rtimes P &\longrightarrow M \rtimes (M \rtimes P), & (m, p) &\longmapsto (m, 1, p). \end{aligned}$$

Conditions (i) and (ii) of a crossed module ensure that these are homomorphisms (Condition (ii) is needed for  $d_2$ ). Let  $G_2 = M \rtimes (M \rtimes P)$ . We then have a 2-truncated pseudosimplicial group  $\{G_0, G_1, G_2\}$  whose 2-coskeleton we denote by  $G^2$ . There is a unique simplicial map  $G^2 \rightarrow G^1$  which in dimensions 0 and 1 is the identity. We let  $\overline{G}^2$  denote the image of  $G^2$  in  $G^1$ . It is readily checked that the Moore complex of  $G^2$  is trivial in dimension 2; it follows from Lemma 3.2 that  $\overline{G}^2$  is a pseudosimplicial group whose Moore complex is of length 1.  $\square$

### 3.2. Pseudo 2-crossed modules

Conduché [6] in 1984 described the notion of 2-crossed module as a model for (homotopy connected) 3-types.

**Definition 3.5** *A pseudo 2-crossed module of groups consists of a complex of  $P$ -groups*

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$$

and  $\partial_2, \partial_1$  morphisms of  $P$ -groups, where the group  $P$  acts on itself by conjugation, such that

$$L \xrightarrow{\partial_2} M$$

is a crossed module. Thus  $M$  acts on  $L$  and we require that for all  $l \in L$ ,  $m \in M$  and  $p \in P$  that  ${}^p m (pl) = {}^p (ml)$ . Further, there is a  $P$ -equivariant function,

$$\{, \} : M \rtimes M \longrightarrow L$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{aligned} \mathbf{P-2CM1} \quad & \partial_2\{m, m'\} = (\partial_1 m m') m m'^{-1} m^{-1} \\ \mathbf{P-2CM2} \quad & \{\partial_2 l, \partial_2 l'\} = [l', l] \\ \mathbf{P-2CM3} \quad & (i) \{mm', m''\} = \partial_1 m \{m', m''\} \{m, m' m'' m'^{-1}\} \\ & (ii) \{m, m' m''\} = \{m, m'\}^{m m' m^{-1}} \{m, m''\} \\ \mathbf{P-2CM4} \quad & (a) \{\partial_2 l, m\} = {}^m l (l)^{-1}, \\ & (b) \{m, \partial_2 l\} = (\partial_1 m l) ({}^m l)^{-1}. \\ \mathbf{P-2CM5} \quad & \{m, \partial_2 l\} \{\partial_2 l, m\} = \partial_1 m l (l)^{-1} \end{aligned}$$

for all  $l, l' \in L$ ,  $m, m', m'' \in M$  and  $p \in P$ . We denote such a pseudo 2-crossed module of groups by  $\{L, M, P, \partial_2, \partial_1\}$ . To obtain the definition of 2-crossed modules, we must add the condition that:

$$\mathbf{2CM6)} \quad {}^p \{m, m'\} = \{{}^p m, {}^p m'\}.$$

A morphism of pseudo 2-crossed modules of groups may be pictured by diagram

$$\begin{array}{ccccc} L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & P \\ f_2 \downarrow & & f_1 \downarrow & & f_0 \downarrow \\ L' & \xrightarrow{\partial_2} & M' & \xrightarrow{\partial_1} & P' \end{array}$$

of groups and homomorphisms such that  $f_0 \partial_1 = \partial_1' f_1$ ,  $f_1 \partial_2 = \partial_2' f_2$  and such that

$$f_1 ({}^p m) = ({}^{f_0(p)}) f_1 (m), \quad f_2 ({}^p l) = ({}^{f_0(p)}) f_2 (l)$$

and

$$\{, \} = f_1 \times f_1 = f_2 \{, \},$$

for all  $l \in L$ ,  $m \in M$ ,  $p \in P$ . We thus define the category of pseudo 2-crossed modules, denoting it by  $\mathbf{pX}_2\mathbf{Mod}$ . Morphisms  $f_1$  and  $f_2$  are called equivariant if  $P = P'$  with  $f_0 = \text{identity of } P$ .

The category of simplicial groups with Moore complex of length 2 is equivalent to that of 2-crossed modules. This equivalence was proved by Conduché in [6]. Now, we shall give the pseudo version of this equivalence in the following theorem.

**Theorem 3.6** *The category of pseudo 2-crossed modules is equivalent to that of category of pseudosimplicial groups with Moore complex of length 2.*

**Proof.** Let  $\mathbf{G}$  be a pseudosimplicial group with Moore complex of length 2. We construct a pseudo 2-crossed module as follows:  $P = G_0$ ,  $M = \ker(d_0 : G_1 \rightarrow G_0)$ , and  $L = \ker(d_0 : G_2 \rightarrow G_1) \cap \ker(d_1 : G_2 \rightarrow G_1)$ . Then  $p \in P$  acts on  $m \in M$  by  ${}^p m = s_0(p) m s_0(p)^{-1}$ , and on  $l \in L$  by  $\partial_1(m) l = s_0(m) l s_0(m)^{-1}$  and  $m \in M$  acts on  $l \in L$  by  ${}^m l = s_1(m) l s_1(m)^{-1}$ . For  $m, m' \in M$ , set  $\{m, m'\} = s_0(m) s_1(m') s_0(m)^{-1} s_1(m) s_1(m')^{-1} s_1(m)^{-1}$ . Let  $\partial_1 = d_1$  (restricted to  $M$ ) and  $\partial_2 = d_2$  (restricted to  $L$ ).

$$\begin{aligned} \mathbf{P - 2CM1)} \quad \partial_2 \{m, m'\} &= \partial_2 \left( s_0(m) s_1(m') s_0(m)^{-1} s_1(m) s_1(m')^{-1} s_1(m)^{-1} \right) \\ &= d_2 s_0(m) d_2 s_1(m') d_2 s_0(m)^{-1} d_2 s_1(m) d_2 s_1(m')^{-1} d_2 s_1(m)^{-1} \\ &= d_2 s_0(m) m' d_2 s_0(m)^{-1} m(m')^{-1} (m)^{-1} \\ &= s_0 d_1(m) m' s_0 d_1(m)^{-1} m(m')^{-1} (m)^{-1} \\ &= (\partial_1 m m') m m'^{-1} m^{-1}. \end{aligned}$$

$$\begin{aligned} \mathbf{P - 2CM2)} \quad \{\partial_2 l, \partial_2 l'\} &= \{d_2 l, d_2 l'\} \\ &= s_0 d_2(l) s_1 d_2(l') s_0 d_2(l)^{-1} s_1 d_2(l) s_1 d_2(l')^{-1} s_1 d_2(l)^{-1} \\ &= d_3 s_0(l) d_3 s_1(l') d_3 s_0(l)^{-1} d_3 s_1(l) d_3 s_1(l')^{-1} d_3 s_1(l)^{-1} \\ &= d_3 s_0(l) d_3 s_1(l') d_3 s_0(l)^{-1} d_3 s_1(l) d_3 s_1(l')^{-1} d_3 s_1(l)^{-1} \end{aligned}$$



$$\begin{aligned}
 &= d_3 s_0(l) d_3 s_1(l') d_3 s_0(l)^{-1} d_3 s_1(l) d_3 s_1(l')^{-1} d_3 s_1(l)^{-1} [l, l'] [l', l] \\
 &= d_3 s_0(l) d_3 s_1(l') d_3 s_0(l)^{-1} d_3 s_1(l) d_3 s_1(l')^{-1} d_3 s_1(l)^{-1} \left( (l'(l)^{-1} (l)^{-1}) [l', l] \right) \\
 &= d_3 s_0(l) d_3 s_1(l') d_3 s_0(l)^{-1} d_3 s_1(l) d_3 s_1(l')^{-1} d_3 s_1(l)^{-1} d_3 s_2(l) d_3 s_2(l')^{-1} d_3 s_2(l)^{-1} [l', l] \\
 &= d_3 (s_0(l) s_1(l') s_0(l)^{-1} s_1(l) s_1(l')^{-1} s_1(l)^{-1} s_2(l) s_2(l')^{-1} s_2(l)^{-1}) [l', l] \\
 &= [l', l],
 \end{aligned}$$

where  $s_0(l) s_1(l') s_0(l)^{-1} s_1(l) s_1(l')^{-1} s_1(l)^{-1} s_2(l) s_2(l')^{-1} s_2(l)^{-1}$  lies in  $\partial_3 NG_3$ .

$$\begin{aligned}
 \mathbf{P - 2CM3} \quad (i) \quad \{mm', m''\} &= s_0(mm') s_1(m'') s_0(mm')^{-1} s_1(mm') s_1(m'')^{-1} \\
 &= s_0(m) s_0(m') s_1(m'') s_0(m')^{-1} s_0(m)^{-1} s_1(m) \\
 &= s_0(m) s_0(m') s_1(m'') s_0(m')^{-1} s_1(m) s_1(m'')^{-1} \\
 &= s_0(m) s_0(m') s_1(m'') s_0(m')^{-1} s_1(m) s_1(m'')^{-1} \\
 &= s_0(m) s_0(m') s_1(m'') s_0(m')^{-1} s_1(m) s_1(m'')^{-1} \\
 &= s_0(m) s_0(m') s_1(m'') s_0(m')^{-1} s_1(m) s_1(m'')^{-1} \\
 &= s_0(m) s_0(m') s_1(m'') s_0(m')^{-1} s_1(m) s_1(m'')^{-1} \\
 &= s_0(m) s_0(m') s_1(m'') s_0(m')^{-1} s_1(m) s_1(m'')^{-1} \\
 &= \partial_1 m \{m', m''\} (s_0(m) s_1(m' m'' (m')^{-1}) s_0(m)^{-1} s_1(m) \\
 &= \partial_1 m \{m', m''\} (m' m'' (m')^{-1})^{-1} s_1(m)^{-1} \\
 &= \partial_1 m \{m', m''\} \{m, m' m'' (m')^{-1}\}.
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \{m, m' m''\} &= s_0(m) s_1(m' m'') s_0(m)^{-1} s_1(m) s_1(m' m'')^{-1} s_1(m)^{-1} \\
 &= s_0(m) s_1(m') s_1(m'') s_0(m)^{-1} s_1(m) s_1(m'')^{-1} s_1(m')^{-1} s_1(m)^{-1} \\
 &= s_0(m) s_1(m') \left( s_0(m)^{-1} s_1(m) s_1(m')^{-1} s_1(m)^{-1} \right) \\
 &= \left( s_1(m) s_1(m') s_1(m)^{-1} s_0(m) \right) s_1(m'') s_0(m)^{-1} s_1(m) \\
 &= \left( s_0(m) s_1(m') s_0(m)^{-1} s_1(m) s_1(m')^{-1} s_1(m)^{-1} \right) \\
 &= \left( s_1(m) s_1(m') s_1(m)^{-1} s_0(m) s_1(m'') s_0(m)^{-1} s_1(m) \right) \\
 &= \left( s_1(m'')^{-1} s_1(m')^{-1} s_1(m)^{-1} \right) \\
 &= \{m, m'\} (s_1(m) s_1(m') s_1(m)^{-1} s_0(m) s_1(m'') s_0(m)^{-1} s_1(m) \\
 &= \{m, m'\} (s_1(m)^{-1} s_1(m)) s_1(m')^{-1} s_1(m)^{-1} \\
 &= \{m, m'\} (s_1(m) s_1(m') s_1(m)^{-1} s_0(m) s_1(m'') s_0(m)^{-1} s_1(m) \\
 &= \{m, m'\} (s_1(m)^{-1} s_1(m) s_1(m')^{-1} s_1(m)^{-1} \\
 &= \{m, m'\} (s_1(m m' (m)^{-1}) s_0(m) s_1(m'') s_0(m)^{-1} s_1(m) \\
 &= s_1(m'')^{-1} s_1(m)^{-1} s_1(m m' (m)^{-1})^{-1} \\
 &= \{m, m'\}^{m m' m^{-1}} \{m, m''\}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{P} - \mathbf{2CM4} \quad (a) \quad \{\partial_2 l, m\} &= s_0(\partial_2(l)) s_1(m) s_0(\partial_2(l)^{-1}) s_1(\partial_2(l)) s_1(m)^{-1} s_1(\partial_2(l)^{-1}) \\
 &= s_0 d_2(l) s_1(m) s_0 d_2(l)^{-1} s_1 d_2(l) s_1(m)^{-1} s_1 d_2(l)^{-1} \\
 &= d_3 s_0(l) s_1(m) d_3 s_0(l)^{-1} d_3 s_1(l) s_1(m)^{-1} d_3 s_1(l)^{-1} \\
 &\quad \left( l s_0(m) l^{-1} s_0(m)^{-1} \right) \left( s_0(m) l s_0(m)^{-1} l^{-1} \right) \\
 &= (d_3 s_0(l) d_3 s_2 s_1(m) d_3 s_0(l)^{-1} d_3 s_1(l) d_3 s_2 s_1(m)^{-1} d_3 s_1(l)^{-1} \\
 &\quad d_3 s_2(l) d_3 s_2 s_0(m) d_3 s_2(l)^{-1} d_3 s_2 s_0(m)^{-1}) \left( s_0(m) l s_0(m)^{-1} l^{-1} \right) \\
 &= d_3(s_0(l) s_2 s_1(m) s_0(l)^{-1} s_1(l) s_2 s_1(m)^{-1} s_1(l)^{-1} \\
 &\quad s_2(l) s_2 s_0(m) s_2(l)^{-1} s_2 s_0(m)^{-1}) \left( s_0(m) l s_0(m)^{-1} l^{-1} \right) \\
 &= {}^m l(l)^{-1},
 \end{aligned}$$

where  $s_0(l) s_2 s_1(m) s_0(l)^{-1} s_1(l) s_2 s_1(m)^{-1} s_1(l)^{-1} s_2(l) s_2 s_0(m) s_2(l)^{-1} s_2 s_0(m)^{-1}$  lies in  $\partial_3 NG_3$ .

$$\begin{aligned}
 (b) \quad \{m, \partial_2 l\} &= s_0(m) s_1 \partial_2(l) s_0(m)^{-1} s_1(m) s_1 \partial_2(l)^{-1} s_1(m)^{-1} \\
 &= s_0(m) s_1 d_2(l) s_0(m)^{-1} s_1(m) s_1 d_2(l)^{-1} s_1(m)^{-1} \\
 &= \left( s_0(m) l s_0(m)^{-1} \right) \left( s_1(m) (l)^{-1} s_1(m)^{-1} \right) \\
 &= (\partial_1 m l)(m(l)^{-1}).
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{P} - \mathbf{2CM5} \quad \{m, \partial_2 l\} \{\partial_2 l, m\} &= \left( s_0(m) s_1 d_2(l) s_0(m)^{-1} s_1(m) s_1 d_2(l)^{-1} s_1(m)^{-1} \right) \\
 &\quad \left( s_0 d_2(l) s_1(m) s_0 d_2(l)^{-1} s_1 d_2(l) s_1(m)^{-1} s_1 d_2(l) \right) \\
 &= \left( s_0(m) d_3 s_1(l) s_0(m)^{-1} s_1(m) d_3 s_1(l)^{-1} s_1(m)^{-1} \right) \\
 &\quad \left( d_3 s_0(l) s_1(m) d_3 s_0(l)^{-1} d_3 s_1(l) s_1(m)^{-1} d_3 s_1(l) \right) \\
 &\quad \left( l s_0(m) (l)^{-1} s_0(m)^{-1} \right) \left( s_0(m) l s_0(m)^{-1} (l)^{-1} \right) \\
 &= (d_3 s_2 s_0(m) d_3 s_1(l) d_3 s_2 s_0(m)^{-1} d_3 s_2 s_1(m) \\
 &\quad d_3 s_1(l)^{-1} d_3 s_2 s_1(m)^{-1}) (d_3 s_0(l) d_3 s_2 s_1(m) d_3 s_0(l)^{-1} \\
 &\quad d_3 s_1(l) d_3 s_2 s_1(m)^{-1} d_3 s_1(l)) (d_3 s_2(l) d_3 s_2 s_0(m) \\
 &\quad d_3 s_2(l)^{-1} d_3 s_2 s_0(m)^{-1}) \left( s_0(m) l s_0(m)^{-1} (l)^{-1} \right) \\
 &= d_3(s_2 s_0(m) s_1(l) s_2 s_0(m)^{-1} s_2 s_1(m) s_1(l)^{-1} s_2 s_1(m)^{-1} \\
 &\quad s_0(l) s_2 s_1(m) s_0(l)^{-1} s_1(l) s_2 s_1(m)^{-1} s_1(l) s_2(l) \\
 &\quad s_2 s_0(m) s_2(l)^{-1} s_2 s_0(m)^{-1}) \left( s_0(m) l s_0(m)^{-1} (l)^{-1} \right) \\
 &= \partial_1 m l(l)^{-1},
 \end{aligned}$$

where

$$\begin{aligned}
 &s_2 s_0(m) s_1(l) s_2 s_0(m)^{-1} s_2 s_1(m) s_1(l)^{-1} s_2 s_1(m)^{-1} s_0(l) s_2 s_1(m) s_0(l)^{-1} \\
 &\quad s_1(l) s_2 s_1(m)^{-1} s_1(l) s_2(l) s_2 s_0(m) s_2(l)^{-1} s_2 s_0(m)^{-1}
 \end{aligned}$$

lies in  $\partial_3 NG_3$ .

Conversely we start with a pseudo 2-crossed module  $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$ . Set  $G_0 = P$ . Using the action of  $P$  on  $M$ , we can form the semi-direct product  $G_1 = M \rtimes P$ . There are homomorphisms

$$\begin{aligned}
 d_0 : M \rtimes P &\longrightarrow P, & (m, p) &\longmapsto p, \\
 d_1 : M \rtimes P &\longrightarrow P, & (m, p) &\longmapsto (\partial m) p, \\
 s_0 : P &\longrightarrow M \rtimes P, & p &\longmapsto (1, p).
 \end{aligned}$$

There is an action of  $m \in M$  on  $l \in L$  given by

$$m.l = \{m, \partial_2 l\}^{\partial_1 m} l.$$

Using this action we can form the semi-direct product  $L \rtimes M$ . There is an action of  $(m, p) \in M \rtimes P$  on  $(l, m') \in L \rtimes M$  given by

$$(m, p) \cdot (l, m') = ({}^m l \ {}^p l, {}^m m' \ {}^p m).$$

Using this action, we form the semi-direct product  $G_2 = (L \rtimes M) \rtimes (M \rtimes P)$ . There are homomorphisms

$$\begin{aligned} d_0 &: (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (M \rtimes P), & (l, m', m, p) &\longmapsto (m', p), \\ d_1 &: (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (M \rtimes P), & (l, m', m, p) &\longmapsto ({}^m m', p), \\ d_2 &: (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (M \rtimes P), & (l, m', m, p) &\longmapsto (m, \partial_1 m' \ p), \\ s_0 &: (M \rtimes P) \longrightarrow (L \rtimes M) \rtimes (M \rtimes P), & (m, p) &\longmapsto (1, 1, m', p), \\ s_1 &: (M \rtimes P) \longrightarrow (L \rtimes M) \rtimes (M \rtimes P), & (m, p) &\longmapsto (1, m', 1, p). \end{aligned}$$

There is an action of  $(l, m) \in L \rtimes M$  on  $l' \in L$  given by

$${}^{(l, m)} l' = (ll')^m l'$$

and we can construct the semi-direct product  $L \rtimes (L \rtimes M)$ . There is an action of  $(m, p) \in M \rtimes P$  on  $(l, l', m') \in L \rtimes (L \rtimes M)$  given by

$$m \cdot l = \{m, \partial l\}^m l.$$

There is also an action of  $(l', m) \in L \rtimes M$  on  $(l, l', m') \in L \rtimes (L \rtimes M)$  given by

$$(m, p) \cdot (l, m') = ({}^m l \ {}^p l, {}^m m' \ {}^p m).$$

These last two actions combine to give an action of  $(L \rtimes M) \rtimes (M \rtimes P)$  on  $L \rtimes (L \rtimes M)$ , from which we construct the semi-direct product  $G_3 = (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P)$ . There are homomorphisms

$$\begin{aligned} d_0 &: (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow ((L \rtimes M) \rtimes (M \rtimes P)) \\ & \quad (l, l', m, l'', m', m'', p) \longmapsto (l', m', m'', p), \\ d_1 &: (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow ((L \rtimes M) \rtimes (M \rtimes P)) \\ & \quad (l, l', m, l'', m', m'', p) \longmapsto (l' l'', m m', m'', p), \\ d_2 &: (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow ((L \rtimes M) \rtimes (M \rtimes P)) \\ & \quad (l, l', m, l'', m', m'', p) \longmapsto (ll', m, m' m'', p), \\ d_3 &: (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow ((L \rtimes M) \rtimes (M \rtimes P)) \\ & \quad (l, l', m, l'', m', m'', p) \longmapsto (l, m, (\partial_2 l'') m', (\partial m') p), \\ s_1 &: (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \\ & \quad (l, m, m', p) \longmapsto (1, 1, 1, 1, m', m, p), \\ s_2 &: (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \\ & \quad (l, m, m', p) \longmapsto (1, 1, m', 1, 1, m, p), \\ s_3 &: (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \\ & \quad (l, m, m', p) \longmapsto (1, 1, m', 1, m, 1, p). \end{aligned}$$

Axioms (1) – (5) ensure that these are indeed homomorphisms. Let  $G_2$  be the 2-coskeleton of the 2-truncated pseudosimplicial groups  $\{G_0, G_1, G_2\}$ ; let  $G_3$  be the 3-coskeleton of the 3-truncated pseudosimplicial groups

$\{G_0, G_1, G_2, G_3\}$ . There is a unique simplicial map  $G^3 \rightarrow G^2$  which in dimensions 0, 1 and 2 is the identity. We let  $\overline{G}^3$  denote the image of  $G^3$  in  $G^2$ . It is readily checked that the Moore complex of  $G^3$  is trivial in dimension 3; it follows from Lemma 3.2 that  $\overline{G}^3$  is a pseudosimplicial group whose Moore complex is of length 2.

The above constructions yield the required equivalence.

We now associate to the each pseudosimplicial groups  $G$  a simplicial inclusion  $U^k G \rightarrow G$  and quotient  $G \rightarrow V^k G$  such that the following Proposition 4 holds. The inclusion and quotient are described carefully in the proof of Proposition 4, but in essence can be described in terms of Moore complexes as follows. Suppose that  $(M_n, \partial_n)$  is the Moore complex of  $G$ . Then  $U^k G$  will have the Moore complex

$$\cdots \rightarrow M_{k+3} \rightarrow M_{k+2} \rightarrow \ker(\partial_{k+1}) \rightarrow 1 \rightarrow 1 \cdots ,$$

and  $V^k G$  will have the Moore complex

$$1 \rightarrow 1 \rightarrow \text{im}(\partial_{k+1}) \rightarrow M_k \rightarrow M_{k-1} \rightarrow \cdots .$$

□

**Proposition 3.7** *For any pseudosimplicial group  $G$  and integer  $k \geq 0$ , there is a functorial short exact sequence of pseudosimplicial groups*

$$1 \rightarrow U^k G \xrightarrow{I} G \xrightarrow{\phi} V^k G \rightarrow 1$$

such that:

(i) the Moore complex of  $U^k G$  is trivial in dimensions  $0, 1, \dots, k$ , and identical with the Moore complex of  $G$  in dimensions  $\geq k + 2$ ;

(ii) the map  $\iota$  induces isomorphisms on homotopy groups  $\pi_n(U^k G) \cong \pi_n(G)$  for  $n \geq k + 1$ ;

(iii) the Moore complex of  $V^k G$  is trivial in dimensions  $\geq k + 2$ , and in dimensions  $\leq k$  is identical with the Moore complex of  $G$ .

**Proof.** First construct the  $k$ -coskeleton  $\text{cos } k^k(tr^k G)$  of  $G$ . By lemma the Moore complex of  $\text{cos } k^k(tr^k G)$  is

$$1 \rightarrow 1 \rightarrow K \xrightarrow{\partial_{k+1}} M_k \xrightarrow{\partial_k} M_{k-1} \rightarrow \cdots \xrightarrow{\partial_1} M_0 .$$

Here  $\partial_{k+1}$  is an inclusion,  $K$  is the kernel of  $\partial_k$ , and in dimensions  $\leq k$  this complex coincides with the Moore complex of  $G$ . Since  $K$  is a normal subgroup of  $G_k$ , we can quotient  $tr^k G$  by  $K$  to obtain a  $k$ -truncated pseudosimplicial groups  $tr^k G / K = \{G_0, \dots, G_{k-1}, G_k/K\}$ . We now construct the  $k$ -coskeleton  $\text{cos } k^k(tr^k G / K)$ . There is a unique simplicial map  $G \rightarrow \text{cos } k^k(tr^k G / K)$  which in dimensions less than  $k$  is the identity, and in dimension  $k$  is the quotient  $G_k \rightarrow G_k / K$ ; we let  $U^k G$  denote the kernel of this map, and  $V^k G$  denote the image of  $G$  in  $\text{cos } k^k(tr^k G / K)$ . We thus have a short exact sequence of pseudosimplicial groups

$$1 \rightarrow U^k G \rightarrow G \rightarrow V^k G \rightarrow 1$$

which induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(U^k G) \longrightarrow \pi_n(G) \longrightarrow \pi_n(V^k G) \rightarrow \pi_{n-1}(U^k G) \rightarrow \cdots .$$

The assertions of the proposition are easily checked. □

**Proposition 3.8** *Let  $G$  be a pseudosimplicial groups such that  $\pi_n G = 1$ , for  $n = 0, \dots, k$ . Then there exists a weak homotopy equivalence  $F \simeq G$  with  $F$  a free pseudosimplicial groups such that  $F_n = 1$  for  $n = 0, \dots, k$ .*

**Proposition 3.9** *It follows from Proposition 2.7(i) and the simplicial identities that the pseudosimplicial groups  $U^k G$  is trivial in dimensions  $\leq k$ . From axiom (M2) of a model category (see [14]) there is a weak equivalence  $F \simeq U^k G$  with  $F$  a free pseudosimplicial groups. We can construct  $F$  so that it meets the requirements of the propositions.*

## Acknowledgements

The authors wishes to thank the referee for helpful comments.

## References

- [1] Arvasi, Z., Porter, T.: Simplicial and crossed resolutions of commutative algebras, Journal of Algebra, 181, 426-428 (1996).
- [2] Arvasi, Z., Porter, T.: Freeness conditions of 2-crossed modules of commutative algebras, Applied Categorical Structures, **6**, 455-471 (1998).
- [3] Baues, H. J.: Combinatorial homotopy and 4-dimensional complexes, Walter de Gruyter 1991.
- [4] Baues, H. J.: Homotopy types, Handbook of Algebraic Topology, Edited by I. M. James, Elsevier, 1-72 (1995).
- [5] Brown, R., Higgins, P.J.: Colimit-theorems for relative homotopy groups, Jour. Pure Appl. Algebra, 22, 1-41 (1981).
- [6] Conduche, D.: Modules croises generalises de Longueur 2, J.Pure Appl. Algebra, 34, 155-178 (1984).
- [7] Duskin, J.: Simplicials methods and the interpretation of triple cohomology, Memoirs A.M.S., Vol.3 163, (1975).
- [8] Inasaridze, H. N.: Homotopy of pseudosimplicial groups and nonabelian derived functors, Sakharth, SSR, Mecn. Akad. Moambe, 76, 533-536 (1974).
- [9] Inasaridze, H. N.: Homotopy of pseudosimplicial groups and nonabelian derived functors and algebraic K-theory, Math. Sbornik, TOM, 98, (140), No: 3, 303-323 (1975).
- [10] Loday, J. L.: Spaces having finitely many non-trivial homotopy groups, Jour. Pure Appl. Algebra, 24, 179-202 (1982).
- [11] May, J. P.: Simplicial objects in algebraic topology, Math, Studies, 11, Van Nostrand 1967.

- [12] Milnor, J.W.: The Construction FK, Mimeographed Notes, Univ. Princeton, N.J. 1956.
- [13] Mutlu, A., Porter, T.: Freeness conditions of 2-crossed modules and complexes, Theory and Applications of Categories, Vol. 4, No. 8, 174-194 (1998).
- [14] Quillen, D.G.: Homotopical algebra, Lecture Notes in Math. 43 Springer, New York 1967.
- [15] Whitehead, J.H.C.: Combinatorial homotopy I and II, Bull. Amer. Math. Soc., 55, 231-245 and 453-496 (1949).

İ. AKÇA

Eskişehir Osmangazi University, Art and Science Faculty, Mathematics Department,  
26480, Eskişehir-TURKEY e-mail: iakca@ogu.edu.tr,

Received 04.07.2008

S. PAK

Dumlupınar University, Art and Science Faculty,  
Mathematics Department, Kütahya-TURKEY  
e-mail: sedatpak@dumlupinar.edu.tr