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## Order continuous operators on $CD_0(K, E)$ and $CD_w(K, E)$ -spaces

Faruk Polat

### Abstract

In [2], Alpay and Ercan characterized order continuous duals of spaces  $CD_0(K, E)$  and  $CD_w(K, E)$  where  $K$  is a compact Hausdorff space without isolated points and  $E$  is a Banach lattice. In this note, we generalize their results to an arbitrary Dedekind complete Banach lattice  $F$ , that is to say, we characterize order continuous operators on these spaces taking values in an arbitrary Dedekind complete Banach lattice  $F$ .

**Key Words:**  $CD_0(K)$ -spaces, order continuous operators, isometric lattice isomorphism.

### 1. Introduction

Recall that a topological space is called *basically disconnected* if the closure of any  $F_\sigma$ -open set is open. A compact Hausdorff space that is basically disconnected will be referred to as a *quasi-Stonean space*. For a quasi-Stonean space  $K$  without isolated points, the following function spaces were introduced by Abramovich and Wickstead [1]:

$$\begin{aligned}
 l_w^\infty(K) &= \{f : f \text{ is real valued, bounded and the set} \\
 &\quad \{k : f(k) \neq 0\} \text{ is countable}\}; \\
 c_0(K) &= \{f : f \text{ is real valued and the set} \\
 &\quad \{k : |f(k)| > \varepsilon\} \text{ is finite for each } \varepsilon > 0\}.
 \end{aligned}$$

These spaces were used to define  $CD_0(K) = C(K) \oplus c_0(K)$  and  $CD_w(K) = C(K) \oplus l_w^\infty(K)$ . Both spaces  $CD_0(K)$  and  $CD_w(K)$  are  $AM$ -spaces with strong order unit  $\mathbf{1}$  under the pointwise order and supremum norm. Properties such as Cantor property, Dedekind completeness, sequential order continuity of the norm in these spaces were studied in [1]. Further, Alpay and Ercan [2] relaxed the condition on the quasi-Stonean space  $K$  and took it to be a compact Hausdorff space without isolated points and they defined the following vector-valued versions of  $l_w^\infty(K)$  and  $c_0(K)$ .

**Definition 1** For a set  $K$  and a normed space  $E$ , let  $C_0(K, E)$  be the space of all  $E$ -valued functions  $f$  on  $K$  such that for each  $\varepsilon > 0$ , the set  $\{s \in K : \|f(s)\| > \varepsilon\}$  is finite. Similarly, let  $l_w^\infty(K, E)$  be the space of all bounded  $E$ -valued functions on  $K$  with countable support.

The following vector-valued versions of the spaces  $CD_0(K)$  and  $CD_w(K)$  were given in [2].

**Definition 2** Let  $K$  be a compact Hausdorff space without isolated points and  $E$  be a normed space.  $CD_0(K, E)$  denotes the set of all  $E$ -valued functions on  $K$  of the form  $f + d$  such that  $f \in C(K, E)$  and  $d \in C_0(K, E)$ . Similarly,  $CD_w(K, E)$  denotes the set of all  $E$ -valued functions on  $K$  of the form  $f + d$  such that  $f \in C(K, E)$  but  $d \in l_w^\infty(K, E)$ .

As order continuous operators as well as order continuous duals are very much in use here, it is useful to give their definitions. For more details on order continuous operators, see [3].

**Definition 3** (1) A net  $\{x_\alpha\}$  in a Riesz space is said to be decreasing to zero (in symbols  $x_\alpha \downarrow 0$ ) whenever  $\alpha \geq \beta$  implies  $x_\alpha \leq x_\beta$  and  $\inf\{x_\alpha\} = 0$  holds.

(2) A net  $\{x_\alpha\}$  in a Riesz space is said to be order convergent to  $x$ , denoted by  $x_\alpha \rightarrow^\circ x$  whenever there exists a net  $\{y_\alpha\}$  with the same indexed set satisfying  $|x_\alpha - x| \leq y_\alpha \downarrow 0$ .

(3) A linear operator  $T : E \rightarrow F$  between two Riesz spaces is said to be order continuous whenever  $x_\alpha \rightarrow^\circ 0$  in  $E$  implies  $Tx_\alpha \rightarrow^\circ 0$  in  $F$ . The collection of all order continuous operators will be denoted by  $L_n(E, F)$ . It is useful to note that a positive operator  $T : E \rightarrow F$  is order continuous if and only if  $x_\alpha \downarrow 0$  in  $E$  implies  $Tx_\alpha \downarrow 0$  in  $F$ . The vector space  $L_n(E, \mathbb{R})$  of all order continuous linear functionals is referred to as the order continuous dual of  $E$  and denoted by  $E_n^\sim$ .

Alpay and Ercan [2] proved that the spaces  $CD_0(K, E)$  and  $CD_w(K, E)$  are Banach lattices for a Banach lattice  $E$ . They investigated order properties of these spaces and characterized their order continuous duals.

The following definitions and theorems were given in [2].

**Definition 4** Let  $K$  be a compact Hausdorff space without isolated points and  $E$  be a Banach lattice. Then  $D_0(K, E_n^\sim)$  denotes the set of all mappings  $\beta = \beta(k)$  from  $K$  into  $E_n^\sim$  satisfying

$$\sup_{\|f\| \leq 1} \sum_k |\beta(k)|(|f(k)|) < \infty$$

for each  $f \in CD_0(K, E)$  and  $\sum_k |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$  whenever  $f_\alpha \downarrow 0$ .

As usual,  $\sum_k |\beta(k)|(|f(k)|)$  is the supremum of the sums

$$\sum_S |\beta(k)|(|f(k)|),$$

where  $S \subset K$  and is finite.  $D_0(K, E_n^\sim)$  is a normed Riesz space under pointwise operations and supremum norm.

**Theorem 5** Let  $K$  and  $E$  be as above. Then  $CD_0(K, E)_n^\sim$  and  $D_0(K, E_n^\sim)$  are isometrically lattice isomorphic spaces.

**Definition 6** Let  $K$  be a compact Hausdorff space without isolated points and  $E$  be a Banach lattice. Then  $D_w(K, E_n^\sim)$  denotes the set of all mappings  $\beta = \beta(k)$  from  $K$  into  $E_n^\sim$  satisfying

$$\sup_{\|f\| \leq 1} \sum_k |\beta(k)|(|f(k)|) < \infty$$

for each  $f \in CD_w(K, E)$  and  $\sum_k |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$  whenever  $f_\alpha \downarrow 0$ .

As usual,  $\sum_k |\beta(k)|(|f(k)|)$  is the supremum of the sums

$$\sum_S |\beta(k)|(|f(k)|),$$

where  $S \subset K$  and is finite.  $D_w(K, E_n^\sim)$  is a normed Riesz space under pointwise operations and supremum norm.

**Theorem 7** Let  $K$  and  $E$  be as above. Then  $CD_w(K, E)_n^\sim$  and  $D_w(K, E_n^\sim)$  are isometrically lattice isomorphic spaces.

## 2. Main results

Throughout this section, the symbol  $\chi_k \otimes f$  denotes the vector-valued function which takes the value  $f(k)$  at  $k$  and 0 otherwise.

We start with the following definition which is not very commonly known.

**Definition 8** Let  $E$  and  $F$  be two Banach lattices. The regular norm, denoted by  $\|\cdot\|_r$  of a linear operator  $T : E \rightarrow F$  with modulus  $|T|$  is defined by

$$\|T\|_r := \| |T| \| := \sup_{\|x\| \leq 1} \| |T|(x) \|$$

It is useful to note that  $L_n(E, F)$  under the norm  $\|\cdot\|_r$  is a Dedekind complete Banach lattice whenever  $F$  is Dedekind complete.

In this section, we give a generalization of Theorem 5 and Theorem 7 in two directions. Firstly, we replace  $CD_0(K, E)_n^\sim$  (or  $CD_w(K, E)_n^\sim$ ) by  $L_n(CD_0(K, E), F)$  (or  $L_n(CD_w(K, E), F)$ ) where  $E$  and  $F$  are Banach lattices with  $F$  Dedekind complete. We take  $F$  as a Dedekind complete Banach lattice to ensure that  $L_n(CD_0(K, E), F)$  (or  $L_n(CD_w(K, E), F)$ ) is a Dedekind complete Banach lattice under the regular norm  $\|\cdot\|_r$ . Secondly, we replace  $E_n^\sim$  by  $L_n(E, F)$ . We now give the following definition which is similar to Definition 4.

**Definition 9** Let  $K$  be a compact Hausdorff space without isolated points,  $E$  and  $F$  be two Banach lattices with  $F$  Dedekind complete. We define  $l^1(K, L_n(E, F))$  as the set of all mappings  $\varphi = \varphi(k)$  from  $K$  into  $L_n(E, F)$  satisfying

$$\sum_k |\varphi(k)|(|f(k)|) \in F$$

for each  $f \in CD_0(K, E)$  and  $\sum_k |\varphi(k)|(f_\alpha(k)) \downarrow_\alpha 0$  whenever  $f_\alpha \downarrow 0$  in  $CD_0(K, E)$ .

As usual,  $\sum_k |\varphi(k)|(|f(k)|)$  is the supremum of the sums

$$\sum_S |\varphi(k)|(|f(k)|)$$

where  $S \subset K$  and is finite.

$l^1(K, L_n(E, F))$  is a Banach lattice under pointwise operations and supremum norm.

We now give the following theorem which is the main result of this note.

**Theorem 10** *Let  $K$  be a compact Hausdorff space without isolated points,  $E$  and  $F$  be two Banach lattices with  $F$  Dedekind complete. Then  $L_n(CD_0(K, E), F)$  is isometrically lattice isomorphic to  $l^1(K, L_n(E, F))$ .*

**Proof.** Let us define a map

$$\phi : L_n(CD_0(K, E), F) \rightarrow l^1(K, L_n(E, F))$$

at  $e \in E$  by the formula

$$\phi(G)(k)(e) = G(\chi_k \otimes e)$$

for each  $G \in L_n(CD_0(K, E), F)$  and  $k \in K$ . It is clear that  $\phi$  is a linear map. Using the linearity of  $\phi$  and the fact that  $\phi(G^+)(k)$  and  $\phi(G^-)(k)$  are order bounded  $F$ -valued operators for each  $G$  on  $CD_0(K, E)$ ,  $\phi(G)(k)$  is order bounded.

Moreover, if  $e_\alpha \downarrow 0$  in  $E$ , then  $\chi_k \otimes e_\alpha \downarrow 0$  in  $CD_0(K, E)$  for each  $k \in K$ . Using the order continuity of  $G$ , we have that  $G(\chi_k \otimes e)$  is order convergent to 0 so that  $\phi(G)(k) \in L_n(E, F)$  for each  $G \in L_n(CD_0(K, E), F)$ . We thus have a map  $\phi(G)$  from  $K$  into  $L_n(E, F)$ .

Now we will show that

$$\sum_{k \in K} |\phi(G)(k)|(|f(k)|) \in F, \quad (f \in CD_0(K, E)).$$

Let  $S$  be a finite subset of  $K$  and  $G \in L_n(CD_0(K, E), F)$ . Then

$$\begin{aligned} \sum_{k \in S} |\phi(G)(k)|(|f(k)|) &= \sum_{k \in S} |\phi(G^+ - G^-)(k)|(|f(k)|) \\ &\leq \sum_{k \in S} \phi(G^+)(k)(|f(k)|) \\ &\quad + \sum_{k \in S} \phi(G^-)(k)(|f(k)|) \\ &= \sum_{k \in S} G^+(\chi_k \otimes |f|) + \sum_{k \in S} G^-(\chi_k \otimes |f|) \\ &= G^+(\sum_{k \in S} \chi_k \otimes |f|) + G^-(\sum_{k \in S} \chi_k \otimes |f|) \end{aligned}$$

for each  $f \in CD_0(K, E)$ . As  $\sum_{k \in S} \chi_k \otimes |f| \uparrow_S |f|$ ,  $G^+$  and  $G^-$  are order continuous, we obtain

$$\sum_{k \in S} |\phi(G)(k)|(|f(k)|) \leq G^+(|f|) + G^-(|f|) = |G|(|f|).$$

Hence

$$\sum_{k \in K} |\phi(G)(k)|(f(k)) \in F,$$

since  $F$  is Dedekind complete. We also have to show that

$$\sum_k |\phi(G)(k)|(f_\alpha(k)) \downarrow_\alpha 0 \text{ in } F$$

for each  $f_\alpha \in CD_0(K, E)$  such that  $f_\alpha \downarrow 0$ . It is enough to show this for positive elements in  $L_n(CD_0(K, E), F)$ . Let us take  $0 \leq G \in L_n(CD_0(K, E), F)$  and  $f_\alpha \downarrow 0$  in  $CD_0(K, E)$ . For a fixed  $\alpha$ , we have  $\sum_{k \in S} \chi_k \otimes f_\alpha \uparrow_S f_\alpha$ . As  $G$  is order continuous and positive,

$$G\left(\sum_{k \in S} \chi_k \otimes f_\alpha\right) = \sum_{k \in S} G(\chi_k \otimes f_\alpha) \uparrow G(f_\alpha),$$

so that

$$\begin{aligned} \sum_{k \in K} |\phi(G)(k)|(f_\alpha(k)) &= \sum_{k \in K} \phi(G)(k)(f_\alpha(k)) \\ &= \sum_{k \in K} G(\chi_k \otimes f_\alpha) = G(f_\alpha) \downarrow 0. \end{aligned}$$

Hence the map  $\phi(G)$  is an element of  $l^1(K, L_n(E, F))$ .

We now show that  $\phi$  is bipositive. It is easy to show that  $\phi(G) \geq 0$  whenever  $G \geq 0$ . Conversely, assume that  $\phi(G) \geq 0$  for some  $G \in L_n(CD_0(K, E), F)$  and take  $0 \leq f \in CD_0(K, E)$ . We have  $\sum_{k \in S} G(\chi_k \otimes f) \rightarrow G(f)$ , since  $\sum_{k \in S} \chi_k \otimes f \uparrow_S f$  in  $CD_0(K, E)$ . As  $G(\chi_k \otimes f) = \phi(G)(k)(f) \geq 0$  and thus  $G(f) \geq 0$  for each  $0 \leq f \in CD_0(K, E)$ , i.e.,  $G \geq 0$ .

To show that  $\phi$  is one-to-one, let  $\phi(G) = 0$  for some  $G \in L_n(CD_0(K, E), F)$ . Then  $G(\chi_k \otimes f) = 0$  for each  $k \in K$  and  $0 \leq f \in CD_0(K, E)$ . As  $G$  is order continuous and  $\sum_{k \in S} \chi_k \otimes f \uparrow_S f$ , this gives that  $0 = \sum_{k \in S} G(\chi_k \otimes f) \rightarrow G(f)$  or  $G(f) = 0$ . As  $CD_0(K, E)$  is a vector lattice, we get  $G = 0$ .

To show that  $\phi$  is surjective, let us take an arbitrary  $0 \leq \alpha \in l^1(K, L_n(E, F))$  and define  $G : CD_0(K, E)_+ \rightarrow F_+$  by  $G(f) = \sum_{k \in K} \alpha(k)(f(k))$ . As  $G$  is additive on  $CD_0(K, E)$  and so  $G(f) = G(f^+) - G(f^-)$  extends  $G$  to  $CD_0(K, E)$ . We now verify that  $\phi(G) = \alpha$ . If  $0 \leq e \in E$ , then

$$\phi(G)(k_0)(e) = G(\chi_{k_0} \otimes e) = \sum_{k \in K} \alpha(k)(\chi_{k_0} \otimes e)(k) = \alpha(k_0)e.$$

Since  $e \in E$  is arbitrary, we conclude that  $\phi(G)(k_0) = \alpha(k_0)$  and  $k_0$  is arbitrary, we have  $\phi(G) = \alpha$ .

Finally we show that  $\phi$  is an isometry. Assume that  $G \in L_n(CD_0(K, E), F)$  and  $f \in CD_0(K, E)$ . Then

$$\begin{aligned} \|G\|_r &= \sup_{\|f\| \leq 1} \| |G|(f) \| = \sup_{\|f\| \leq 1} \| |G|(|f|) \| \\ &= \sup_{\|f\| \leq 1} \| |G| \left( \sum_{k \in K} \chi_k \otimes |f| \right) \| \\ &= \sup_{\|f\| \leq 1} \| \sum_{k \in K} |G|(\chi_k \otimes |f|) \| \\ &= \| \phi(|G|) \| = \| \phi(G) \|_r. \end{aligned}$$

This completes the proof. □

**Definition 11** Let  $K$  be a compact Hausdorff space without isolated points,  $E$  and  $F$  be two Banach lattices with  $F$  Dedekind complete. Then we define  $l_w^1(K, L_n(E, F))$  as the set of all maps  $\varphi = \varphi(k)$  from  $K$  into  $L_n(E, F)$  satisfying

$$\sum_k |\varphi(k)|(|f(k)|) \in F$$

for each  $f \in CD_w(K, E)$  and  $\sum_k |\varphi(k)|(f_\alpha(k)) \downarrow_\alpha 0$  whenever  $f_\alpha \downarrow 0$  in  $CD_w(K, E)$ .

$l_w^1(K, L_n(E, F))$  is a Banach lattice under pointwise operations and supremum norm. The following theorem is similar to Theorem 10 so we omit its proof.

**Theorem 12** Let  $K$  be a compact Hausdorff space without isolated points,  $E$  and  $F$  be two Banach lattices with  $F$  Dedekind complete. Then  $L_n(CD_w(K, E), F)$  is isometrically lattice isomorphic to  $l_w^1(K, L_n(E, F))$ .

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