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SEZAYİ HIZLIYEL

MEHMET ÇAĞLIYAN

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A boundary value problem for Bitsadze equation in matrix form

Sezayi Hızlıyel and Mehmet Çağlıyan

Abstract

In this work, we investigate the solvability of the problem

$$\frac{\partial^2 w}{\partial \overline{\phi}^2} = f$$

$$Re\{i\phi(z)w(z)\} = \gamma_1(z), Rew_{\overline{\phi}}(z) = \gamma_2(z) \quad z \in \partial\mathbb{D}$$

in the unit disk of complex plane. Here f , γ_1 and γ_2 are given $m \times s$-complex matrix-valued functions; $f \in L^p(\overline{\mathbb{D}})$, $\gamma_1, \gamma_2 \in C(\partial\mathbb{D})$ and ϕ is a generating solution for Q-holomorphic functions.

Key Words: Q-Holomorphic; Bitsadze equation

1. Introduction

Let Q be an $m \times m$ complex matrix valued function. If $Q(z)$ is self-commuting, i.e.,

$$Q(z_1)\,Q(z_2) = Q(z_2)\,Q(z_1)$$

for any two points z_1, z_2 in the domain Ω_0 of $\mathbb{C}$ and if $Q(z)$ has no eigenvalues of magnitude 1 for each z in Ω_0, then the solutions of the matrix equation

$$Dw(z) := w_{\overline{z}}(z) - Q(z)w_z(z) = 0, \tag{1}$$

have similar properties as the complex analytic functions. This equation was first considered by Hile [10]. The solutions of this equation are called Q-holomorphic functions.

We recall next a few elementary properties associated with the operator $D := \frac{\partial}{\partial \overline{z}} - Q\frac{\partial}{\partial z}$. First, there exits so-called generating solution $\phi(z) := \phi_0(z)\,I + N(z)$ which satisfies equation (1), where N is the nilpotent part of ϕ and ϕ_0 is the main diagonal term of ϕ satisfying the complex Beltrami equation

$$\frac{\partial \phi_0}{\partial \overline{z}} - \lambda\frac{\partial \phi_0}{\partial z} = 0, \quad |\lambda(z)| \neq 1.$$

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If a function $\Phi(z)$ is Q-holomorphic, then it can be written solely as an analytic function of a generating solution, i.e. $\Phi(z) \equiv f(\phi(z))$ [10]. This suggests formally defining differentiation with respect to ϕ as

$$\frac{\partial}{\partial \phi} = \left(\phi_z \overline{\phi}_{\overline{z}} - \phi_{\overline{z}} \overline{\phi}_z\right)^{-1} \left[\overline{\phi}_z \frac{\partial}{\partial \overline{z}} - \overline{\phi}_{\overline{z}} \frac{\partial}{\partial z}\right]$$

and differentiation with respect to $\overline{\phi}$ conjugate to ϕ as

$$\frac{\partial}{\partial \overline{\phi}} = \left(\phi_z \overline{\phi}_{\overline{z}} - \phi_{\overline{z}} \overline{\phi}_z\right)^{-1} \phi_z D \tag{2}$$

weakly. From (2) we may rewrite (1) as

$$\frac{\partial w}{\partial \overline{\phi}} = 0.$$

Also, the generalized Pompeiu operator defined as

$$\underset{\sim}{T} f(z) = -2iP^{-1} \iint_{\Omega} \phi_{\zeta}(\zeta) \left(\phi(\zeta) - \phi(z)\right)^{-1} f(\zeta)\, d\xi d\eta$$

satisfies

$$\frac{\partial \underset{\sim}{T} f}{\partial \overline{\phi}} = f, \quad \frac{\partial \underset{\sim}{T}}{\partial \phi} = \underset{\sim}{\Pi} f,$$

where $\underset{\sim}{\Pi}$ is

$$\underset{\sim}{\Pi} f \;= -2iP^{-1} \iint_{\Omega} \phi_{\zeta}(\zeta) \left(\phi(\zeta) - \phi(z)\right)^{-2} f(\zeta)\, d\xi d\eta.$$

$\underset{\sim}{T}$ and $\underset{\sim}{\Pi}$ have similar properties as T- and Π-operators of I. N. Vekua's theory (see [11]). Iterating $\underset{\sim}{T}$ with itself as well as its complex conjugate leads to the representation formulas. These representation formulas can be used to solve some boundary value problems as in complex and hyperanalytic case see [2, 3, 4, 5, 6, 7, 8, 9], also see [1]. In this work we consider the solvability of the problem

$$\frac{\partial^2 w}{\partial \overline{\phi}^2} = f$$

$$Re\{i\phi(z)w(z)\} = \gamma_1(z), Rew_{\overline{\phi}}(z) = \gamma_2(z) \quad z \in \partial\mathbb{D} \tag{3}$$

in the unit disk $\mathbb{D}$ of the complex plane and give a representation formula for solutions. The method used is the one given in [8] in hyperanalytic case. Here, f is an $m \times s$ complex matrix in $L^p(\mathbb{D})$, $p > 2$, γ_1 and γ_2 are $m \times s$ matrices in $C(\partial\mathbb{D})$ and ϕ is a generating solution to (1). Further, we assume that Q is commuting with $\overline{Q}$

2. First and second order representation formulas

For a solution $v \in C^1(\Omega)$ of (1) commuting with Q and any $u \in C^1(\Omega) \cap C(\overline{\Omega})$ $m \times s$ complex matrix-valued function, Hile [10] proved that

$$\int_{\partial\Omega} (dv)u = 2i \iint_{\Omega} v_z(u_{\overline{z}} - Qu_z)dxdy. \tag{4}$$

From (4), taking ϕ and w instead of v and u, respectively gives

$$\int_{\partial\Omega} d\phi\,(z)\,w\,(z) + \iint_{\Omega} d\phi\,(z)\,d\overline{\phi\,(z)}\frac{\partial w\,(z)}{\partial\overline{\phi}} = 0. \tag{5}$$

Similarly

$$\int_{\partial\Omega} d\overline{\phi\,(z)}w\,(z) - \iint_{\Omega} d\phi\,(z)\,d\overline{\phi\,(z)}\frac{\partial w\,(z)}{\partial\phi} = 0.$$

Also, Hile gave a representation formula called the generalized Cauchy-Pompieu representation for an $m \times s$ complex matrix-valued function:

Theorem 1 *Let Ω be a regular subdomain of Ω_0, $\partial\Omega$ its boundary and w be a $m\times s$-matrix in $C^1(\Omega)\cap C\left(\overline{\Omega}\right)$ with bounded first derivatives in Ω. Then for z in Ω*

$$w\,(z) = P^{-1}\int_{\partial\Omega} (\phi(\zeta) - \phi(z))^{-1}\,d\phi\,(\zeta)\,w\,(\zeta) + P^{-1}\iint_{\Omega} d\phi\,(\zeta)\,d\overline{\phi\,(\zeta)}\,(\phi(\zeta) - \phi(z))^{-1}\,\frac{\partial w(\zeta)}{\partial\overline{\phi(\zeta)}}. \tag{6}$$

In (6) P is a constant matrix defined by

$$P\,(z) = \int_{|z|=1} (zI + \overline{z}Q)^{-1}\,(Idz + Qd\overline{z})\,.$$

It is called the P -value for (1).

Now, we give a second order representation formula for complex-matrix valued function.

Theorem 2 *Any $m \times s$ complex matrix-valued function $w \in C^2(\overline{\Omega})$ is representable by*

$$\begin{aligned} w(z) \;&=\; P^{-1}\int_{\partial\Omega} d\phi(\zeta)\left\{(\phi(\zeta) - \phi(z))^{-1}\,w(\zeta) - \overline{(\phi(\zeta) - \phi(z))}\,(\phi(\zeta) - \phi(z))^{-1}\,\frac{\partial w(\zeta)}{\partial\overline{\phi(\zeta)}}\right\} \\ &\quad -P^{-1}\iint_{\Omega} d\phi(\zeta)d\overline{\phi(\zeta)(\phi(\zeta) - \phi(z))}\,(\phi(\zeta) - \phi(z))^{-1}\,\frac{\partial^2 w(\zeta)}{\partial\overline{\phi(\zeta)}^2} \end{aligned} \tag{7}$$

Proof. Applying formula (6) to $\partial w(z)/\partial\overline{\phi(z)}$ and denoting $\partial^2 w(z)/\partial\overline{\phi(z)}^2 = f(z)$ gives

$$\frac{\partial w\phi(z)}{\partial\overline{\phi(z)}} = P^{-1}\int_{\partial\Omega} d\phi(\zeta)\,(\phi(\zeta)-\phi(z))^{-1}\,\frac{\partial w(\zeta)}{\partial\overline{\phi(\zeta)}} + \underset{\sim}{T}f(z).$$

Substituting this in (6) we obtain

$$w(z) = P^{-1}\int_{\partial\Omega} d\phi(\zeta)\,(\phi(\zeta)-\phi(z))^{-1}\,w(\zeta) + P^{-1}\int_{\partial\Omega} d\phi(\zeta) I(\zeta,z)\frac{\partial w(\zeta)}{\partial\overline{\phi(\zeta)}} + \underset{\sim}{T}^2 f(z)$$

where

$$\begin{aligned} I(\zeta,z) &= P^{-1}\iint_{\Omega} d\phi(\widetilde{\zeta})d\overline{\phi(\widetilde{\zeta})}\left(\phi(\zeta)-\phi(\widetilde{\zeta})\right)^{-1}\left(\phi(\widetilde{\zeta})-\phi(z)\right)^{-1} \\ &= (\phi(\zeta)-\phi(z))^{-1}\,P^{-1}\iint_{\Omega} d\phi(\widetilde{\zeta})d\overline{\phi(\widetilde{\zeta})}\left[\left(\phi(\widetilde{\zeta})-\phi(z)\right)^{-1}-\left(\phi(\widetilde{\zeta})-\phi(\zeta)\right)^{-1}\right] \end{aligned}$$

and

$$\underset{\sim}{T}^2 f(z) = P^{-1}\iint_{\Omega} d\phi(\zeta)d\overline{\phi(\zeta)} I(\zeta,z) f(\zeta).$$

To evaluate $I(\zeta,z)$, apply (6) with $\overline{\phi(z)}$ to obtain

$$\overline{\phi(z)} - \vartheta(z) = P^{-1}\iint_{\Omega} d\phi(\zeta)d\overline{\phi(\zeta)}\,(\phi(\zeta)-\phi(z))^{-1}$$

$$\vartheta(z) = P^{-1}\int_{\partial\Omega} d\phi(\zeta)\,(\phi(\zeta)-\phi(z))^{-1}\,\overline{\phi(\zeta)}.$$

Thus

$$I(\zeta,z) = (\phi(\zeta)-\phi(z))^{-1}\left[\vartheta(\zeta)-\vartheta(z)-\overline{\phi(\zeta)}+\overline{\phi(z)}\right].$$

Since ϑ is Q-holomorphic from (5),

$$\begin{aligned} &P^{-1}\int_{\partial\Omega} d\phi(\zeta)\,(\phi(\zeta)-\phi(z))^{-1}\,(\vartheta(\zeta)-\vartheta(z))\,\frac{\partial w(\zeta)}{\partial\overline{\phi(\zeta)}} \\ &+P^{-1}\iint_{\Omega} d\phi(\zeta)d\overline{\phi(\zeta)}\,(\phi(\zeta)-\phi(z))^{-1}\,(\vartheta(\zeta)-\vartheta(z))\,f(\zeta) = 0. \end{aligned}$$

This proves the representation formula (7). □

Remark 3 *Since ϕ_0 is a solution of the Beltrami equation, this solution can be found by $\rho(z) = \phi_0(z)$ in the case of $|\lambda| \le q_0 < 1$ and by $\rho(z) = \phi_0(\overline{z})$ in the case of $|\lambda| \ge q_0 > 1$ with nonnegative constant q_0 (see [13], Chapter II). Hence, under the coordinate transformation $\rho = \rho(z)$, (1) becomes*

$$w_{\overline{z}} - Qw_z = \left[\overline{\rho_z}\left(\overline{\lambda}Q - I\right)\right]\left(w_{\overline{\rho}} - \left[\overline{\rho_z}\left(\overline{\lambda}Q - I\right)\right]^{-1}\left[\rho_z\left(\lambda I - Q\right)\right]w_\rho\right).$$

The coefficient of w_ρ is a self-commuting matrix whose main diagonal terms are zero (see [11], pp. 431). If we denote the coefficient of w_ρ again by Q and return to using z as the independent variable, we have

$$w_{\overline{z}} = Q w_z.$$

Note that in this normal form the generating solution is $\phi(z) = zI + N(z)$. Especially choosing $m = 1$, $s = 1$ in (6), we obtain

$$w(z) = P^{-1} \int_{\partial\Omega} \frac{w(\zeta)}{\zeta - z} d\zeta - 2iP^{-1} \iint_{\Omega} \frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} d\xi d\eta. \tag{8}$$

where

$$P := \begin{cases} 2\pi i, & |\lambda| \le q_0 < 1 \\ -2\pi i, & |\lambda| \ge q_0 > 1 \end{cases}$$

(see [12], pp. 581).

Also, using (8) and the complex Green formulas, we obtain the following theorem.

Theorem 4 *$\varphi \in C^1(\mathbb{D};\mathbb{C})$. Then*

$$\varphi(z) = -2iP^{-1} \int_{\partial\mathbb{D}} Re\left[i\zeta\varphi(\zeta)\right] \frac{d\zeta}{\zeta(\zeta - z)} + P^{-1} \iint_{\mathbb{D}} \left(\frac{\varphi_{\overline{\zeta}}}{\zeta - z} + \frac{\overline{\zeta\varphi_{\overline{\zeta}}}}{1 - z\overline{\zeta}} \right) d\zeta d\overline{\zeta}.$$

3. A boundary value problem for the matrix Bitsadze equation

Note that (7) is in the form

$$w(z) = \varphi(z) + \overline{\phi(z)}\psi(z) + \underset{\sim}{T_{0,2}} f(z),$$

where φ and ψ are arbitrary Q-holomorphic functions and

$$\underset{\sim}{T_{0,2}} f(z) = -P^{-1} \iint_{\Omega} d\phi(\zeta) d\overline{\phi(\zeta)(\phi(\zeta) - \phi(z))} \left(\phi(\zeta) - \phi(z)\right)^{-1} f(\zeta).$$

Now we show that the arbitrary Q-holomorphic functions φ and ψ can be determined by prescribing boundary values for w in (3). The first boundary condition in (3) gives

$$\begin{aligned} Re\{i\phi(z)\varphi\} &= \gamma_1 - Re\{i\phi(z)\overline{\phi(z)}\psi(z) + i\phi(z)\underset{\sim}{T_{0,2}} f(z)\} \\ &=: \widetilde{\gamma}(z). \end{aligned}$$

In component form this is

$$\begin{aligned} Re\{iz\varphi_{k\ell}(z)\} &= \widetilde{\gamma}_{k\ell}(z) - \sum_{j=1}^{k-1} Re\{i\phi_{kj}(z)\varphi_{j\ell}(z)\} \\ &=: \widehat{\gamma}_{k\ell}(z), \quad 1 < k \le m, \quad 1 \le \ell \le s. \end{aligned}$$

Hence, the solution of

$$\begin{aligned} \frac{\partial \varphi_{k\ell}}{\partial \overline{z}} &= \sum_{j=1}^{k-1} q_{kj} (\varphi_{j\ell})_z \\ Re\{iz\varphi_{k\ell}(z)\} &= \widehat{\gamma}_{k\ell}(z) \end{aligned}$$

is

$$\begin{aligned} \varphi_{k\ell} &= C_0\widetilde{\gamma}_{k\ell} + \sum_{j=1}^{k-1} C_0\left(\phi_{kj}\varphi_{j\ell}\right) + \sum_{j=1}^{k-1} T_{kj}\left(\varphi_{j\ell}\right)_\zeta \\ &= : C_0\widetilde{\gamma}_{k\ell} + \sum_{j=1}^{k-1}\left(A_{kj}\varphi_{j\ell} + T_{kj}\left(\varphi_{j\ell}\right)_\zeta\right) \end{aligned} \tag{9}$$

provided that

$$P^{-1}\int_{\partial\mathbb{D}} \widehat{\gamma}_{k\ell}(\zeta)\frac{d\zeta}{\zeta} - P^{-1}\iint_{\mathbb{D}} Im\left\{\sum_{j=1}^{k-1} q_{kj}(\zeta)\left(\varphi_{j\ell}\right)_\zeta(\zeta)\right\} d\zeta d\overline{\zeta} = 0. \tag{10}$$

where C_0, A_{kj} and T_{kj} are the operators defined respectively by

$$\begin{aligned} C_0\rho\,(z) &= -2iP^{-1}\int_{\partial\mathbb{D}} \frac{\rho\,(\zeta)}{\zeta\,(\zeta - z)} d\zeta, \\ A_{kj}\rho(z) &= C_0 Im\left(\phi_{kj}(z)\rho(z)\right) \\ T_{kj}\rho(z) &= P^{-1}\iint_{\mathbb{D}}\left(\frac{q_{kj}(\zeta)\rho(\zeta)}{\zeta - z} + \frac{\overline{\zeta q_{kj}(\zeta)\rho(\zeta)}}{1 - z\overline{\zeta}}\right) d\zeta d\overline{\zeta}. \end{aligned}$$

A_{kj} can be transformed into area integrals

$$\begin{aligned} A_{kj}\rho(z) &= P^{-1}\int_{\partial\mathbb{D}} \frac{\left(\phi_{kj}(\zeta)\rho(\zeta) - \overline{\phi_{kj}(\zeta)\rho(\zeta)}\right)}{1 - z\overline{\zeta}} d\overline{\zeta} \\ &= P^{-1}\iint_{\mathbb{D}} \frac{\partial}{\partial\zeta}\left(\phi_{kj}(\zeta)\rho(\zeta) - \overline{\phi_{kj}(\zeta)\rho(\zeta)}\right)\frac{d\zeta d\overline{\zeta}}{1 - z\overline{\zeta}}\ . \end{aligned}$$

$T_{kj}\rho$ operators have similar properties as T, especially

$$\frac{\partial}{\partial\overline{z}}T_{kj}\rho(z) = q_{kj}(z)\rho(z), \quad \frac{\partial}{\partial z}T_{kj}\rho(z) := \Pi_{kj}\rho(z)$$

in the weak sense with

$$\Pi_{kj}\rho(z) = P^{-1}\iint_{\mathbb{D}}\left(\frac{q_{kj}(\zeta)\rho(\zeta)}{(\zeta - z)^2} + \frac{\overline{\zeta^2 q_{kj}(\zeta)\rho(\zeta)}}{\left(1 - z\overline{\zeta}\right)^2}\right) d\zeta d\overline{\zeta}.$$

Note that, since $q_{kj} = 0$ for $k < j$, $A_{kj} = 0$ and $T_{kj} = 0$. For iterating (9), denote by A'_{kj}, T'_{kj} the z-derivatives of the operator A_{kj}, T_{kj} respectively.

Definition 5 *Let* A_{kj}, T_{kj} $, 1 \leq k \leq m, 1 \leq j \leq s$ *be operators given above and* A'_{kj}, T'_{kj} *related ones. Assume that all these operators are composite with one another and denote composition as product. We define an operation* B *on* k*-tuples of these operators by*

$$\begin{aligned}
B(A) &= A, \\
B(A_{k,\sigma_1}, T_{\sigma_1,\sigma_2}) &= A_{k,\sigma_1} T_{\sigma_1,\sigma_2} + T_{k,\sigma_1} T'_{\sigma_1,\sigma_2} \\
B\left(A'_{k,\sigma_1}, T_{\sigma_1,\sigma_2}\right) &= A'_{k,\sigma_1} T_{\sigma_1,\sigma_2} + T'_{k,\sigma_1} T'_{\sigma_1,\sigma_2} \\
B(A_{k,\sigma_1}, A_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-1},\sigma_p}) &= A_{k,\sigma_1} B(A_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-1},\sigma_p}) \\
&\quad + T_{k,\sigma_1} B\left(A'_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-1},\sigma_p}\right) \\
B(A_{k,\sigma_1}, \cdots, A_{\sigma_{p-2},\sigma_{p-1}}, T_{\sigma_{p-1},\sigma_p}) &= A_{k,\sigma_1} B(A_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-2},\sigma_{p-1}}, T_{\sigma_{p-1},\sigma_p}) \\
&\quad + T_{k,\sigma_1} B\left(A'_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-2},\sigma_{p-1}}, T_{\sigma_{p-1},\sigma_p}\right) \\
B\left(A'_{k,\sigma_1}, A_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-1},\sigma_p}\right) &= A_{k,\sigma_1} B(A_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-1},\sigma_p}) \\
&\quad + T'_{k,\sigma_1} B\left(A'_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-1},\sigma_p}\right) \\
B\left(A'_{k,\sigma_1}, \cdots, A_{\sigma_{p-2},\sigma_{p-1}}, T_{\sigma_{p-1},\sigma_p}\right) &= A'_{k,\sigma_1} B(A_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-2},\sigma_{p-1}}, T_{\sigma_{p-1},\sigma_p}) \\
&\quad + T'_{k,\sigma_1} B\left(A'_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-2},\sigma_{p-1}}, T_{\sigma_{p-1},\sigma_p}\right),
\end{aligned}$$

where $A \in \{A_{kj},\ T_{kj},\ A'_{kj},\ T'_{kj} : 1 \leq k \leq m,\ 1 \leq j \leq k\}$.

Lemma 6 *The solution to the system*

$$\varphi_{k\ell} = C_0 \widetilde{\gamma}_{k\ell} + \sum_{j=1}^{k-1} \left(A_{kj} \varphi_{j\ell} + T_{kj} (\varphi_{j\ell})_\zeta \right)$$

is uniquely given by

$$\begin{aligned}
\varphi_{k\ell} &= C_0 \widetilde{\gamma}_{k\ell} + \sum_{\sigma_1=1}^{k-1} B(T_{k,\sigma_1}) C'_0 \widetilde{\gamma}_{\sigma_1,\ell} + \sum_{p=1}^{k-1} \sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} B\left(A_{k,\sigma_1}, A_{\sigma_1,\sigma_2}, \cdots, A_{\sigma_{p-1},\sigma_p}\right) C_0 \widetilde{\gamma}_{\sigma_p,\ell} \\
&\quad + \sum_{p=2}^{k-1} \sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} B\left(A_{k,\sigma_1}, \cdots, A_{\sigma_{p-2},\sigma_{p-1}}, T_{\sigma_{p-1},\sigma_p}\right) C'_0 \widetilde{\gamma}_{\sigma_p,\ell}. \qquad (11)
\end{aligned}$$

Proof. Obviously, for $k = 1,\ 1 \leq \ell \leq s$ the solution is $\varphi_{1\ell} = C_0 \widetilde{\gamma}_{1\ell}$. We also have

$$\begin{aligned}
\varphi_{2\ell} &= C_0 \widetilde{\gamma}_{2\ell} + A_{21} \varphi_{1\ell} + T_{21} (\varphi_{1\ell})_\zeta \\
&= C_0 \widetilde{\gamma}_{2\ell} + A_{21} C_0 \widetilde{\gamma}_{1\ell} + T_{21} C'_0 \widetilde{\gamma}_{1\ell}.
\end{aligned}$$

We assume (11) holds for $1 \le t < k \le m-1,\ 1 \le \ell \le s$. Then applying (11) to system (9) for $3 \le k$ gives

$$\begin{aligned}
\varphi_{k\ell} &= C_0\widetilde{\gamma}_{k\ell} + \sum_{j=1}^{k-1}\left(A_{kj}\varphi_{j\ell} + T_{kj}\left(\varphi_{j\ell}\right)_{\zeta}\right) \\
&= C_0\widetilde{\gamma}_{k\ell} + A_{k1}C_0\widetilde{\gamma}_{1\ell} + T_{k1}C_0^{'}\widetilde{\gamma}_{1,\ell} \\
&\quad + \sum_{j=2}^{k-1}\sum_{\sigma_1=1}^{j-1} A_{kj}B\left(T_{j,\sigma_1}\right)C_0^{'}\widetilde{\gamma}_{\sigma_1,\ell} + \sum_{j=2}^{k-1}\sum_{\sigma_1=1}^{j-1} T_{kj}B\left(T_{j,\sigma_1}^{'}\right)C_0^{'}\widetilde{\gamma}_{\sigma_1,\ell} \\
&\quad + \sum_{j=2}^{k-1}A_{kj}\left\{C_0\widetilde{\gamma}_{j\ell} + \sum_{p=1}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{j-1} B\left(A_{j,\sigma_1}, A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\widetilde{\gamma}_{\sigma_p,\ell}\right\} \\
&\quad + \sum_{j=2}^{k-1}A_{kj}\left\{\sum_{p=2}^{j-1}\sum_{\sigma_1,\cdots,\sigma_{p=1}}^{j-1} B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\widetilde{\gamma}_{\sigma_p,\ell}\right\} \\
&\quad + \sum_{j=2}^{k-1}T_{kj}\left\{C_0^{'}\widetilde{\gamma}_{j\ell} + \sum_{p=1}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{j-1} B\left(A_{j,\sigma_1}^{'}, A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\widetilde{\gamma}_{\sigma_p,\ell}\right\} \\
&\quad + \sum_{j=2}^{k-1}T_{kj}\left\{\sum_{p=2}^{j-1}\sum_{\sigma_1,\cdots,\sigma_{p=1}}^{j-1} B\left(A_{j,\sigma_1}^{'},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\widetilde{\gamma}_{\sigma_p,\ell}\right\} \\
&= C_0\widetilde{\gamma}_{k\ell} + \sum_{j=1}^{k-1}\left[B\left(A_{kj}\right)C_0 + B\left(T_{kj}\right)C_0^{'}\right]\widetilde{\gamma}_{j\ell} + \sum_{j=2}^{k-1}\sum_{\sigma_1=1}^{j-1} B\left(A_{kj},T_{j,\sigma_1}\right)C_0^{'}\widetilde{\gamma}_{\sigma_1,\ell} \\
&\quad + \sum_{j=2}^{k-1}\sum_{p=1}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{j-1} B\left(A_{kj},A_{j,\sigma_1},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\widetilde{\gamma}_{\sigma_p,\ell} \\
&\quad + \sum_{j=2}^{k-1}\sum_{p=2}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{j-1} B\left(A_{kj},A_{j,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\widetilde{\gamma}_{\sigma_p,\ell} \\
&= C_0\widetilde{\gamma}_{k\ell} + \sum_{\sigma_1=1}^{k-1} B\left(T_{k,\sigma_1}\right)C_0^{'}\widetilde{\gamma}_{j\ell} + \sum_{p=1}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} B\left(A_{k,\sigma_1},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\widetilde{\gamma}_{\sigma_p,\ell} \\
&\quad + \sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} B\left(A_{k,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\widetilde{\gamma}_{\sigma_p,\ell}.
\end{aligned}$$

This proves Lemma 6 □

Using matrix notation we can rewrite (11) as

$$\varphi = C_0\widetilde{\gamma} + \underset{\sim}{A_Q}\widetilde{\gamma}, \quad \widetilde{\gamma} = \gamma_1 + Im\left(\phi\overline{\phi}\psi + \phi \underset{\sim}{T_{0,2}} f\right) \tag{12}$$

where

$$
\begin{aligned}
\underset{\sim}{A_Q} : &= \sum_{k=2}^{m}\sum_{\sigma_1=1}^{k-1}\left[B\left(A_{k,\sigma_1}\right)C_0 + B\left(T_{k,\sigma_1}\right)C_0^{'}\right]e^{k\sigma_1} \\
&+\sum_{k=3}^{m}\sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} B\left(A_{k,\sigma_1},A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0 e^{k\sigma_p} \\
&+\sum_{k=3}^{m}\sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} B\left(A_{k,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'} e^{k\sigma_p}
\end{aligned}
$$

is a matrix integral operator and e^{k,σ_p} denotes constant matrix in which the $k-th$ row and σ_p-th column term is 1 and the other terms are 0. In order to check that (12) gives a Q-holomorphic function we observe that for any $m\times s$ complex matrix-valued function μ whose the components are analytic functions,

$$
\begin{aligned}
I_1 &= \frac{\partial}{\partial\overline{z}}\underset{\sim}{A_Q}\mu \\
&= \frac{\partial}{\partial\overline{z}}\sum_{k=2}^{m}\sum_{\ell=1}^{s}\sum_{\sigma_1=1}^{k-1}\left[B\left(A_{k,\sigma_1}\right)\mu_{1\ell} + B\left(T_{k,\sigma_1}\right)\mu_{1\ell}^{'}\right]e^{k\ell} \\
&+\frac{\partial}{\partial\overline{z}}\sum_{k=3}^{m}\sum_{\ell=1}^{s}\sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} B\left(A_{k,\sigma_1},A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-1},\sigma_p}\right)\mu_{\sigma_p,\ell}e^{k\ell} \\
&+\frac{\partial}{\partial\overline{z}}\sum_{k=3}^{m}\sum_{\ell=1}^{s}\sum_{\sigma_1,\sigma_2=1}^{k-1} B\left(A_{k,\sigma_1},T_{\sigma_1,\sigma,2}\right)\mu_{\sigma_2\ell}^{'}e^{k\ell} \\
&+\frac{\partial}{\partial\overline{z}}\sum_{k=4}^{m}\sum_{\ell=1}^{s}\sum_{p=3}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} B\left(A_{k,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)\mu_{\sigma_p,\ell}^{'}e^{k\ell} \\
&= \sum_{k=2}^{m}\sum_{\ell=1}^{s}\sum_{\sigma_1=1}^{k-1} q_{k,\sigma_1}\mu_{\sigma_1\ell}^{'}e^{k\ell} + \sum_{k=3}^{m}\sum_{\ell=1}^{s}\sum_{\sigma_1,\sigma_2=1}^{k-1} q_{k\sigma_1}\left(A_{\sigma_1,\sigma_2}^{'}\mu_{\sigma_2\ell} + \Pi_{\sigma_1,\sigma_2}\mu_{\sigma_2\ell}^{'}\right)e^{k\ell} \\
&+\sum_{k=4}^{m}\sum_{\ell=1}^{s}\sum_{p=3}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} q_{k\sigma_1}B\left(A_{k,\sigma_1}^{'},A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-1},\sigma_p}\right)\mu_{\sigma_p\ell}e^{k\ell} \\
&+\sum_{k=4}^{m}\sum_{\ell=1}^{s}\sum_{p=3}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} q_{k\sigma_1}B\left(A_{k,\sigma_1}^{'},A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-2},\sigma_{p-1}}T_{\sigma_{p-1}\sigma,p}\right)\mu_{\sigma_p\ell}^{'}e^{k\ell}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= Q\frac{\partial}{\partial z}\underset{\sim}{A_Q}\mu \\
&= Q\frac{\partial}{\partial z}\left[\sum_{k=2}^{m}\sum_{\ell=1}^{s}\sum_{\sigma_1=1}^{k-1}\left(A_{k\sigma_1}\mu_{1\ell}+T_{k\sigma_1}\mu_{1\ell}^{'}\right)\right]e^{k\ell} \\
&\quad +Q\frac{\partial}{\partial z}\sum_{k=3}^{m}\sum_{\ell=1}^{s}\sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}B\left(A_{k,\mu_1},\cdots,A_{\sigma_{p-1},\sigma_p}\right)\mu_{\sigma_p,\ell}e^{k\ell} \\
&\quad +Q\frac{\partial}{\partial z}\sum_{k=3}^{m}\sum_{\ell=1}^{s}\sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}B\left(A_{k,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)\mu_{\sigma_p,\ell}^{'}e^{k\ell} \\
&= \sum_{i=3}^{m}\sum_{\ell=1}^{s}\sum_{j=2}^{i-1}\sum_{\sigma_1=1}^{j-1}\left[q_{ij}\left(A_{j,\sigma_1}^{'}\mu_{\sigma_1,\ell}+\Pi_{j,\sigma_1}\mu_{\sigma_1,\ell}^{'}\right)\right]e^{i\ell} \\
&\quad +\sum_{i=4}^{m}\sum_{\ell=1}^{s}\sum_{j=3}^{i-1}\sum_{p=2}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{j-1}q_{ij}B\left(A_{j,\sigma_1}^{'},\cdots,A_{\sigma_{p-1},\sigma_p}\right)\mu_{\sigma_p,\ell}e^{i\ell} \\
&\quad +\sum_{i=4}^{m}\sum_{\ell=1}^{s}\sum_{j=3}^{i-1}\sum_{p=2}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{j-1}q_{ij}B\left(A_{j,\sigma_1}^{'},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)\mu_{\sigma_p,\ell}^{'}e^{i\ell}.
\end{aligned}
$$

Thus $I_1 - Q\mu^{'} = I_2$, and with $\mu = C_0\widetilde{\gamma}$, we obtain

$$
D\varphi = \frac{\partial}{\partial \overline{z}}\underset{\sim}{A_Q}\widetilde{\gamma} - Q\left(C_0^{'} + \frac{\partial}{\partial z}\underset{\sim}{A_Q}\right)\widetilde{\gamma} = 0.
$$

Further, from the second boundary condition in (3)

$$
Re\psi(z) = \gamma_2 - Re\left(\underset{\sim}{T}f(z)\right), \quad z \in \partial\mathbb{D}.
$$

Thus, for ψ we obtain a Schwarz problem. The solution of this problem is

$$
\psi - ic = (S + 2\underset{\sim}{T_Q}C')\widetilde{\gamma}_2, \quad \widetilde{\gamma}_2 = \gamma_2 - Re\underset{\sim}{T}f \tag{13}
$$

(see [12], pp 584). In order to determine c in (13) writing the value of $\widehat{\gamma}_{k\ell}$ in (10), we obtain in component form

$$
\begin{aligned}
0 &= P^{-1}\int_{\partial\mathbb{D}}(\gamma_1)_{k\ell}(\zeta)\frac{d\zeta}{\zeta} + ImP^{-1}\int_{\partial\mathbb{D}}\sum_{i=1}^{k}\sum_{j=1}^{i}\phi_{ki}(\zeta)\overline{\phi_{ij}(\zeta)}\psi_{j\ell}(\zeta)\frac{d\zeta}{\zeta} \\
&\quad +ImP^{-1}\int_{\partial\mathbb{D}}\left[\sum_{j=1}^{k}\phi_{kj}(\zeta)\left(\underset{\sim}{T_{0,2}}f(\zeta)\right)_{j\ell} + \sum_{j=1}^{k-1}\phi_{kj}(\zeta)\varphi_{j\ell}(\zeta)\right]\frac{d\zeta}{\zeta} \\
&\quad -ImP^{-1}\int_{\mathbb{D}}\sum_{j=1}^{k-1}q_{kj}\frac{\partial\varphi_{j\ell}(\zeta)}{\partial\zeta}d\zeta d\overline{\zeta}.
\end{aligned} \tag{14}
$$

where, $(\cdots)_{j\ell}$ means the i-th row and j-th column of $(\cdots)$. Writing ψ in (14) as $\psi + ic$ leads to system

$$\begin{aligned} c_{k\ell} &= -P^{-1}\int_{\partial\mathbb{D}} (\gamma_1)_{k\ell}(\zeta)\frac{d\zeta}{\zeta} \\ &-ImP^{-1}\int_{\partial\mathbb{D}} \sum_{j=1}^{k-1} i\left(\zeta\overline{\phi_{kj}(\zeta)}c_{j\ell} + \overline{\zeta}\phi_{kj}(\zeta)c_{j\ell}\right)\frac{d\zeta}{\zeta} \\ &-ImP^{-1}\int_{\partial\mathbb{D}} \sum_{t=2}^{k-1}\sum_{j=1}^{t-1}\phi_{kt}(\zeta)\overline{\phi_{tj}(\zeta)}ic_{j\ell}\frac{d\zeta}{\zeta} \\ &-ImP^{-1}\int_{\partial\mathbb{D}} \sum_{i=1}^{k}\sum_{j=1}^{k}\phi_{ki}(\zeta)\overline{\phi_{ij}(\zeta)}\psi_{j\ell}\frac{d\zeta}{\zeta} \\ &-ImP^{-1}\int_{\partial\mathbb{D}} \sum_{j=1}^{k-1}\phi_{kj}(\zeta)\left[\left(\underset{\sim}{T_{0,2}}f(\zeta)\right)_{j\ell} + \varphi_{j\ell}(\zeta)\right]\frac{d\zeta}{\zeta} \\ &+ImP^{-1}\iint_{\mathbb{D}} \sum_{j=1}^{k-1} q_{kj}\frac{\partial\varphi_{j\ell}(\zeta)}{\partial\zeta}d\zeta d\overline{\zeta}, \end{aligned} \tag{15}$$

where φ is given by (12). Since Q is commuting with $\overline{Q}$, $\phi\overline{\phi}$ is real (see [11], pp. 438). Hence, rewriting $\widetilde{\gamma}$ as

$$\widetilde{\gamma} = \gamma_1 + Im\left(\phi\overline{\phi}\psi + \phi\underset{\sim}{T_{0,2}}f\right) + \phi\overline{\phi}c,$$

(12) becomes

$$\begin{aligned} \varphi &= \Theta + \left(C_0 + \underset{\sim}{A_Q}\right)\phi\overline{\phi}c \\ \Theta &:= \left(C_0 + \underset{\sim}{A_Q}\right)\left[\gamma_1 + Im\left(\phi\overline{\phi}\psi + \phi\underset{\sim}{T_{0,2}}f\right)\right]. \end{aligned}$$

Since

$$C_0\phi\overline{\phi} = \sum_{k=2}^{m}\sum_{j=1}^{k-1} C_0\left(z\overline{\phi_{kj}(z)} + \overline{z}\phi_{kj}(z)\right)e^{kj} + \sum_{k=3}^{m}\sum_{i=2}^{k-2}\sum_{j=1}^{i-1} C_0\phi_{ki}(z)\overline{\phi_{ij}(z)}e^{kj}$$

is a nilpotent function the k-th row and ℓ-th column term of

$$C_0\phi\overline{\phi}c = \sum_{k=2}^{m}\sum_{\ell=1}^{s}\sum_{j=1}^{k-1} C_0\left(z\overline{\phi_{kj}(z)}\overline{z}\phi_{kj}\right)c_{j\ell}e^{k\ell} + \sum_{k=3}^{m}\sum_{\ell=1}^{s}\sum_{i=2}^{k-2}\sum_{j=1}^{i-1} C_0\phi_{ki}(z)\overline{\phi_{ij}(z)}c_{j\ell}e^{k\ell}$$

is independent of $c_{k\ell}$. Thus (15) can be written as

$$c_{k\ell} = -\beta_{k\ell} - \sum_{j=1}^{k-1}\alpha_{kj}c_{j\ell} \tag{16}$$

with

$$\beta_{k\ell} = P^{-1}\int_{\partial\mathbb{D}}(\gamma_1)_{k\ell}\frac{d\zeta}{\zeta} + P^{-1}\int_{\partial\mathbb{D}} Im\left\{\sum_{i=1}^{k}\sum_{j=1}^{i}\phi_{ki}\overline{\phi_{ij}}\psi_{j\ell} + \sum_{j=1}^{k}\phi_{kj}\left(\underset{\sim}{T_{0,2}f}\right)_{j\ell}\right\}\frac{d\zeta}{\zeta}$$
$$+P^{-1}\int_{\partial\mathbb{D}} Im\left(\sum_{j=1}^{k-1}\phi_{kj}\Theta_{j\ell}\right)\frac{d\zeta}{\zeta} - P^{-1}\iint_{\mathbb{D}} Im\left(\sum_{j=1}^{k-1}q_{kj}\Theta_{j\ell\zeta}\right)d\zeta d\overline{\zeta}.$$

$$\alpha_{kj} = \left\{P^{-1}\int_{\mathbb{D}}\left(\phi(\zeta)\overline{\phi(\zeta)} - I\right)\frac{d\zeta}{\zeta}\right\}_{kj} + \left\{P^{-1}\int_{\partial\mathbb{D}} Im\left[(\phi(\zeta)-\zeta)\left(C_0 + \underset{\sim}{A_Q}\right)\phi(\zeta)\overline{\phi(\zeta)}\right]\frac{d\zeta}{\zeta}\right\}_{kj}$$
$$-\left\{\iint_{\mathbb{D}} Im\left[Q(\zeta)\left(C_0^{'} + \underset{\sim}{A_Q^{'}}\right)\phi(\zeta)\overline{\phi(\zeta)}\right]d\zeta d\overline{\zeta}\right\}_{kj}.$$

Note that α is a nilpotent matrix whose main diagonal terms are zero.

Lemma 7 *The solution to (16) is*

$$c = -\beta - \sum_{p=1}^{m-1}(-1)^p\alpha^p\beta = -(I+\alpha)^{-1}\beta,$$

where

$$\alpha = \sum_{k=2}^{m}\sum_{\ell=1}^{k-1}\alpha_{k\ell}e^{k\ell}, \qquad \beta = \sum_{k=1}^{m}\sum_{\ell=1}^{s}\beta_{k\ell}e^{k\ell}.$$

Proof. For $k=1$ and any ℓ, $1\le\ell\le s$ the equation (16) is $c_{1\ell} = -\beta_{1\ell}$. Assume for $1\le j<k$ and any ℓ, $1\le\ell\le s$

$$c_{j\ell} = -\beta_{j\ell} - \sum_{p=1}^{j-1}(-1)^p\sum_{\sigma_1,\cdots,\sigma_p=1}^{j-1}\alpha_{j,\sigma_1}\cdots\alpha_{\sigma_{p-1},\sigma_p}\beta_{\sigma_p,\ell} \tag{17}$$

holds. Then from (16)

$$\begin{aligned}
c_{k\ell} &= -\beta_{k\ell} - \sum_{j=1}^{k-1}\alpha_{kj}c_{j\ell}\\
&= -\beta_{k\ell} - \sum_{j=1}^{k-1}\alpha_{kj}\left[-\beta_{j\ell} - \sum_{p=1}^{j-1}(-1)^p\sum_{\sigma_1=1}^{j-1}\cdots\sum_{\sigma_p=1}^{\sigma_p-1}\alpha_{j,\sigma_1}\cdots\alpha_{\sigma_{p-1},\sigma_p}\beta_{\sigma_p,\ell}\right]\\
&= -\beta_{k\ell} + \sum_{j=1}^{k-1}\alpha_{kj}\beta_{j\ell} + \sum_{j=2}^{k-1}\sum_{p=1}^{j-1}(-1)^p\sum_{\sigma_1=1}^{j-1}\cdots\sum_{\sigma_p=1}^{\sigma_p-1}\alpha_{kj}\alpha_{j,\sigma_1}\cdots\alpha_{\sigma_{p-1},\sigma_p}\beta_{\sigma_p,\ell}\\
&= -\beta_{k\ell} + \sum_{j=1}^{k-1}\alpha_{kj}\beta_{j\ell} + \sum_{p=1}^{k-2}(-1)^p\sum_{j=p+1}^{k-1}\sum_{\sigma_1=1}^{j-1}\cdots\sum_{\sigma_p=1}^{\sigma_p-1}\alpha_{kj}\alpha_{j,\sigma_1}\cdots\alpha_{\sigma_{p-1},\sigma_p}\beta_{\sigma_p,\ell}
\end{aligned}$$

$$\begin{aligned} &= -\beta_{k\ell} + \sum_{j=1}^{k-1} \alpha_{kj}\beta_{j\ell} + \sum_{p=2}^{k-1} (-1)^{p-1} \sum_{j=p}^{k-1} \sum_{\sigma_2=1}^{j-1} \cdots \sum_{\sigma_p=1}^{\sigma_p-1} \alpha_{kj}\alpha_{j,\sigma_2} \cdots \alpha_{\sigma_{p-1},\sigma_p}\beta_{\sigma_p,\ell} \\ &= -\beta_{k\ell} + \sum_{p=1}^{k-1} (-1)^{p-1} \sum_{\sigma_1=1}^{j-1} \cdots \sum_{\sigma_p=1}^{\sigma_p-1} \alpha_{k,\sigma_1} \cdots \alpha_{\sigma_{p-1},\sigma_p}\beta_{\sigma_p,\ell} \\ &= -\beta_{k\ell} - \sum_{p=1}^{k-1} (-1)^{p} \sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} \alpha_{k,\sigma_1} \cdots \alpha_{\sigma_{p-1},\sigma_p}\beta_{\sigma_p,\ell}. \end{aligned}$$

This proves (17) for any $1 \le k < m$ and any $\ell,\ 1 \le \ell \le s$. Hence, we obtain

$$\begin{aligned} c &= \sum_{k=1}^{m}\sum_{\ell=1}^{s} c_{k\ell}e^{k\ell} = -\beta - \sum_{k=2}^{m}\sum_{\ell=1}^{s}\sum_{p=1}^{k-1} (-1)^p \sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} \alpha_{k,\sigma_1}\cdots\alpha_{\sigma_{p-1},\sigma_p}\beta_{\sigma_p,\ell}e^{k\ell} \\ &= -\beta - \sum_{p=1}^{m-1} (-1)^p \sum_{k=p+1}^{m} \sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} \sum_{\ell=1}^{s} \alpha_{k,\sigma_1}\cdots\alpha_{\sigma_{p-1},\sigma_p} e^{k,\sigma_p}\beta_{\sigma_p,\ell}e^{\sigma_p,\ell} \\ &= -\beta - \sum_{p=1}^{m-1} (-1)^p \sum_{k=p+1}^{m} \sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} \sum_{\ell=1}^{s} (\alpha)^p_{k,\sigma_p} e^{k,\sigma_p}\beta_{\sigma_p,\ell}e^{\sigma_p,\ell} \\ &= -\beta - \sum_{p=1}^{m-1} (-1)^p \alpha^p\beta = -(I+\alpha)^{-1}\beta. \end{aligned} \tag{18}$$

This proves the lemma. □

On the other hand,

$$\begin{aligned} I+\alpha &= P^{-1}\int_{\partial\mathbb{D}} \left\{ \phi(\zeta)\overline{\phi(\zeta)} + Im\left[(\phi(\zeta)-\zeta)\left(C_0 + \underset{\sim}{A_Q}\right)\phi(\zeta)\overline{\phi(\zeta)} \right] \right\} \frac{d\zeta}{\zeta} \\ &\quad -P^{-1}\iint_{\mathbb{D}} Im\left[Q(\zeta)\left(C_0^{'} + \underset{\sim}{A_Q^{'}}\right)\phi(\zeta)\overline{\phi(\zeta)} \right] d\zeta d\overline{\zeta} \end{aligned}$$

and

$$\begin{aligned} \beta &= P^{-1}\int_{\partial\mathbb{D}} \left\{ \gamma_1(\zeta) + Im\left[\phi(\zeta)\overline{\phi(\zeta)}\psi(\zeta) + \phi(\zeta)\underset{\sim}{T_{0,2}}f(\zeta) + (\phi(\zeta)-\zeta)\Theta(\zeta) \right] \right\} \frac{d\zeta}{\zeta} \\ &\quad -P^{-1}\iint_{\mathbb{D}} Im\left(Q(\zeta)\frac{\partial\Theta(\zeta)}{\partial\zeta} \right) d\zeta d\overline{\zeta} \end{aligned}$$

$$
\begin{aligned}
&= P^{-1}\int_{\partial\mathbb{D}}\left[\gamma_1(\zeta)+Im\left(\phi(\zeta)\overline{\phi(\zeta)}\psi(\zeta)+\phi(\zeta)\underset{\sim}{T_{0,2}}f(\zeta)\right)\right]\frac{d\zeta}{\zeta}\\
&\quad +P^{-1}\int_{\partial\mathbb{D}}Im\left\{(\phi(\zeta)-\zeta)\left(C_0+\underset{\sim}{A_Q}\right)\left[\gamma_1(\zeta)+Im\left(\phi(\zeta)\overline{\phi(\zeta)}\psi(\zeta)+\phi(\zeta)\underset{\sim}{T_{0,2}}f(\zeta)\right)\right]\right\}\frac{d\zeta}{\zeta}\\
&\quad -P^{-1}\iint_{\mathbb{D}}Im\left\{Q(\zeta)\left(C_0^{'}+\underset{\sim}{A_Q^{'}}\right)\left[\gamma_1(\zeta)+Im\left(\phi(\zeta)\overline{\phi(\zeta)}\psi(\zeta)+\phi(\zeta)\underset{\sim}{T_{0,2}}f(\zeta)\right)\right]\right\}d\zeta d\overline{\zeta}\\
&= P^{-1}\int_{\partial\mathbb{D}}\left\{\gamma_1(\zeta)+\phi(\zeta)\overline{\phi(\zeta)}\right.\\
&\quad \left.\times Im\left[\left(S+2\underset{\sim}{T_Q}C^{'}\right)\left(\gamma_2(\zeta)-Re\underset{\sim}{T}f(\zeta)\right)\right]+Im\left(\phi(\zeta)\underset{\sim}{T_{0,2}}f(\zeta)\right)\right\}\frac{d\zeta}{\zeta}\\
&\quad +P^{-1}\int_{\partial\mathbb{D}}Im\left[(\phi(\zeta)-\zeta)\left(C_0+\underset{\sim}{A_Q}\right)\right]\left\{\gamma_1(\zeta)+\phi(\zeta)\overline{\phi(\zeta)}\right.\\
&\qquad\qquad \left.\times\, Im\left[\left(S+2\underset{\sim}{T_Q}C^{'}\right)\left(\gamma_2(\zeta)-Re\underset{\sim}{T}f(\zeta)\right)\right]\right\}\frac{d\zeta}{\zeta}\\
&\quad +P^{-1}\int_{\partial\mathbb{D}}Im\left[(\phi(\zeta)-\zeta)\left(C_0+\underset{\sim}{A_Q}\right)Im\left(\phi(\zeta)\underset{\sim}{T_{0,2}}f(\zeta)\right)\right]\frac{d\zeta}{\zeta}\\
&\quad -\iint_{\mathbb{D}}Im\left\{Q(\zeta)\left(C_0^{'}+\underset{\sim}{A_Q^{'}}\right)\left(\gamma_1(\zeta)+\phi(\zeta)\overline{\phi(\zeta)}\right.\right.\\
&\qquad\qquad \left.\left.\times Im\left[\left(S+2\underset{\sim}{T_Q}C^{'}\right)\left(\gamma_2(\zeta)-Re\underset{\sim}{T}f(\zeta)\right)\right]\right\}d\zeta d\overline{\zeta}\\
&\quad -\iint_{\mathbb{D}}Im\left[Q(\zeta)\left(C_0^{'}+\underset{\sim}{A_Q^{'}}\right)\left(\phi(\zeta)\underset{\sim}{T_{0,2}}f(\zeta)\right)\right]d\zeta d\overline{\zeta}.
\end{aligned}
$$

Now, the solution (3) with the boundary conditions (3) is completely determined. From (13), (12) and (18)

$$
\begin{aligned}
w &= \varphi+\overline{\phi}\psi+\underset{\sim}{T_{0,2}}f\\
&= \left(C_0+\underset{\sim}{A_Q}\right)\left[\gamma_1+\phi\overline{\phi}\left(S+2\underset{\sim}{T_Q}C^{'}\right)\left(\gamma_2-Re\underset{\sim}{T}f\right)\right]+i\overline{\phi}c+\underset{\sim}{T_{0,2}}f\\
&\quad +\left(C_0+\underset{\sim}{A_Q}\right)\left[\phi\overline{\phi}c+Im\left(\phi\underset{\sim}{T_{0,2}}f\right)\right]+\overline{\phi}\left(S+2\underset{\sim}{T_Q}C^{'}\right)\left(\gamma_2-Re\underset{\sim}{T}f\right)\\
&= \left(C_0+\underset{\sim}{A_Q}\right)\left[\gamma_1+\phi\overline{\phi}Im\left(S+2\underset{\sim}{T_Q}C^{'}\right)\left(\gamma_2-Re\underset{\sim}{T}f\right)+Im\left(\phi\underset{\sim}{T_{0,2}}f\right)-\phi\overline{\phi}(I+\alpha)^{-1}\beta\right]\\
&\quad +\overline{\phi}\left[\left(S+2\underset{\sim}{T_Q}C^{'}\right)\left(\gamma_2-Re\underset{\sim}{T}f\right)-i(I+\alpha)^{-1}\beta\right]+\underset{\sim}{T_{0,2}}f. \qquad (19)
\end{aligned}
$$

Remark 8 *The matrix operators* $C_0+\underset{\sim}{A_Q}$ *and* $S+2\underset{\sim}{T_Q}C^{'}$ *acting on matrix-valued functions map integrable functions on* $\partial\mathbb{D}$ *onto the space of* Q*-holomorphic functions in* $\mathbb{D}$*. Hence it is obvious that (19) is a solution* *to equation* $\frac{\partial^2 w}{\partial \overline{\phi}^2} = f$.

Lemma 9 *For* $\gamma \in C(\partial\mathbb{D})$

$$Re\left[i\phi(z)\left(C_0 + \underset{\sim}{A_Q}\right)\gamma(z)\right] = \gamma(z) + d$$

with

$$\begin{aligned} d &= -P^{-1}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{d\zeta}{\zeta} - P^{-1}\int_{\partial D} Im\left[(\phi(\zeta)-\zeta)\left(C_0 + \underset{\sim}{A_Q}\right)\gamma(\zeta)\right]\frac{d\zeta}{\zeta} \\ &\quad + P^{-1}\iint_{\mathbb{D}} Im\left[Q(\zeta)\left(C_0' + \underset{\sim}{A_Q'}\right)\gamma(\zeta)\right]d\zeta d\overline{\zeta}. \end{aligned}$$

Proof. For proper real valued γ and $z \in \partial D$, we have

$$\begin{aligned} Re(izC_0\gamma) &= Re\left[iz(-2i)P^{-1}\int_{\partial\mathbb{D}} \frac{\gamma(\zeta)}{\zeta(\zeta - z)}d\zeta\right] \\ &= Re\left(2zP^{-1}\int_{\partial\mathbb{D}} \frac{\gamma(\zeta)}{\zeta(\zeta - z)}d\zeta\right) \\ &= Re\left(P^{-1}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} - P^{-1}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{d\zeta}{\zeta}\right) \\ &= \gamma(z) - P^{-1}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{d\zeta}{\zeta}, \end{aligned}$$

and

$$\begin{aligned} Re(izT_{kj}\rho) &= Re\left[iP^{-1}\iint_{\mathbb{D}}\left(\frac{\zeta(q_{kj}\rho)(\zeta)}{\zeta - z} - (q_{kj}\rho)(\zeta)\right)d\zeta d\overline{\zeta}\right] \\ &\quad - Re\left[iP^{-1}\iint_{\mathbb{D}} \frac{\overline{\zeta(q_{kj}\rho)(\zeta)}}{\overline{\zeta - z}}d\zeta d\overline{\zeta}\right] \\ &= P^{-1}\iint_{\mathbb{D}} Im\left[(q_{kj}\rho)(\zeta)\right]d\zeta d\overline{\zeta}. \end{aligned}$$

Hence, on $\partial\mathbb{D}$

$$\begin{aligned} Re\left[i\phi\left(C_0 + \underset{\sim}{A_Q}\right)\gamma\right] &= \sum_{k=1}^{m}\sum_{\ell=1}^{s} e^{k\ell} Re(izC_0\gamma_{k\ell}) + \sum_{k=2}^{m}\sum_{\ell=1}^{s} e^{k\ell}\left\{\sum_{\sigma_1=1}^{k-1} iz\left(A_{k,\sigma_1}C_0 + T_{k,\sigma_1}C_0'\right)\gamma_{\sigma_1,\ell}\right\} \\ &\quad + \sum_{k=3}^{m}\sum_{\ell=1}^{s} e^{k\ell} Re\left\{\sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1} iz\left[B\left(A_{k,\sigma_1},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\right.\right. \\ &\quad\quad \left.\left. + B\left(A_{k,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0'\right]\gamma_{\sigma_p,\ell}\right\} \end{aligned}$$

$$+\sum_{k=2}^{m}\sum_{\ell=1}^{s}e^{k\ell}Re\left(i\sum_{j=1}^{k-1}\phi_{kj}C_0\gamma_{j\ell}\right)$$

$$+\sum_{k=3}^{m}\sum_{\ell=1}^{s}e^{k\ell}Re\left[i\sum_{j=2}^{k-1}\sum_{\sigma_1=1}^{j-1}\phi_{kj}\left(A_{j,\sigma_1}C_0+T_{j,\sigma_1}C_0^{'}\right)\gamma_{\sigma_1,\ell}\right]$$

$$+\sum_{k=4}^{m}\sum_{\ell=1}^{s}e^{k\ell}Re\left\{i\sum_{j=3}^{k-1}\sum_{p=2}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}\phi_{kj}\left[B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\right.\right.$$

$$\left.\left.+B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\right]\gamma_{\sigma_p,\ell}\right\}$$

$$=\quad \gamma-P^{-1}\int_{\partial\mathbb{D}}\gamma\frac{d\zeta}{\zeta}-\sum_{k=2}^{m}\sum_{\ell=1}^{s}e^{k\ell}P^{-1}\int_{\partial\mathbb{D}}Im\left[\sum_{\sigma_1=1}^{k-1}\phi_{k,\sigma_1}C_0\gamma_{\sigma_1,\ell}\right]\frac{d\zeta}{\zeta}$$

$$-\sum_{k=3}^{m}\sum_{\ell=1}^{s}e^{k\ell}P^{-1}\int_{\partial\mathbb{D}}Im\left\{\sum_{p=2}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}\phi_{k,\sigma_1}\left[B\left(A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\right.\right.$$

$$\left.\left.+B\left(A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\right]\gamma_{\sigma_p,\ell}\right\}\frac{d\zeta}{\zeta}$$

$$+\sum_{k=2}^{m}\sum_{\ell=1}^{s}e^{k\ell}P^{-1}\iint_{\mathbb{D}}Im\left(\sum_{\sigma_1=1}^{k-1}q_{k,\sigma_1}C_0^{'}\gamma_{\sigma_1,\ell}\right)d\zeta d\overline{\zeta}$$

$$+\sum_{k=3}^{m}\sum_{\ell=1}^{s}e^{k\ell}P^{-1}\iint_{\mathbb{D}}Im\left\{\sum_{p=2}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}q_{k,\sigma_1}\left[B\left(A_{\sigma_1,\sigma_2}^{'},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\right.\right.$$

$$\left.\left.+B\left(A_{\sigma_1,\sigma_2}^{'},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\right]\gamma_{\sigma_p,\ell}\right\}d\zeta d\overline{\zeta}$$

$$+\sum_{k=3}^{m}\sum_{\ell=1}^{s}e^{k\ell}Im\left\{\sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}\phi_{k,\sigma_1}\right.$$

$$\left.\left[B\left(A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0+B\left(A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\right]\gamma\right\}_{\sigma_p,\ell}$$

$$-\sum_{k=3}^{m}\sum_{\ell=1}^{s}e^{k\ell}Im\left\{\sum_{j=2}^{k-1}\sum_{p=1}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}\phi_{kj}\left[B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\right.\right.$$

$$\left.\left.+B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\right]\gamma_{\sigma_p,\ell}\right\}.$$

Here the last sum is

$$\sum_{k=3}^{m}\sum_{\ell=1}^{s}e^{k\ell}Im\left\{\sum_{j=2}^{k-1}\sum_{p=1}^{j-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}\phi_{kj}\left[B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\right.\right.$$
$$\left.\left.+B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\right]\gamma_{\sigma_p,\ell}\right\}$$
$$=\sum_{k=3}^{m}\sum_{\ell=1}^{s}e^{k\ell}\left\{Im\sum_{p=1}^{k-2}\sum_{j=p+1}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}\phi_{kj}\left[B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\right.\right.$$
$$\left.\left.+B\left(A_{j,\sigma_1},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\right]\gamma_{\sigma_p,\ell}\right\}$$
$$=\sum_{k=3}^{m}\sum_{\ell=1}^{s}e^{k\ell}Im\left\{\sum_{p=2}^{k-1}\sum_{\sigma_1,\cdots,\sigma_p=1}^{k-1}\phi_{k,\sigma_1}\left[B\left(A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-1},\sigma_p}\right)C_0\right.\right.$$
$$\left.\left.+B\left(A_{\sigma_1,\sigma_2},\cdots,A_{\sigma_{p-2},\sigma_{p-1}},T_{\sigma_{p-1},\sigma_p}\right)C_0^{'}\right]\gamma\right\}_{\sigma_p,\ell}.$$

Therefore

$$Re\left[i\phi(z)\left(C_0+\underset{\sim}{A_Q}\right)\gamma(z)\right]=\gamma(z)-P^{-1}\int_{\partial\mathbb{D}}\gamma(\zeta)\frac{d\zeta}{\zeta}-P^{-1}\int_{\partial\mathbb{D}}Im\left[(\phi(\zeta)-\zeta)\left(C_0+\underset{\sim}{A_Q}\right)\gamma(\zeta)\right]\frac{d\zeta}{\zeta}$$
$$+\iint_{\mathbb{D}}Im\left[Q(\zeta)\left(C_0^{'}+\underset{\sim}{A_Q^{'}}\right)\gamma(\zeta)\right]d\zeta d\overline{\zeta}.$$

□

Theorem 10 *The boundary value problem (3) with* $\gamma_1,\gamma_2\in C(\partial\mathbb{D})$ *for the inhomogeneous matrix Bitsadze equation* $\frac{\partial^2 w}{\partial\overline{\phi}^2}=f,\ f\in L_p\left(\overline{\mathbb{D}}\right)$ *is uniquely solvable. The solution is given by (19).*

Proof. In order to verify (19) to satisfy boundary condition (3) applying previous lemma to (19) we obtain

$$Re\{i\phi w\}=\gamma_1-\beta$$
$$+\left\{P^{-1}\int_{\mathbb{D}}\left(\phi(\zeta)\overline{\phi(\zeta)}+Im\left[(\phi(\zeta)-\zeta)\left(C_0+\underset{\sim}{A_Q}\right)\phi(\zeta)\overline{\phi(\zeta)}\right]\right)\frac{d\zeta}{\zeta}\right.$$
$$\left.-\iint_{\mathbb{D}}Im\left[Q(\zeta)\left(C_0^{'}+\underset{\sim}{A_Q^{'}}\right)\phi(\zeta)\overline{\phi(\zeta)}\right]d\zeta d\overline{\zeta}\right\}(I+\alpha)^{-1}\beta$$
$$=\gamma_1.$$

For the second boundary condition in (3) differentiating (19) leads to

$$Re\left\{\frac{\partial w}{\partial\overline{\phi}}\right\}=\left(S+2\underset{\sim}{T_Q}\right)\left(\gamma_2-Re\underset{\sim}{T}f\right)-i(1+\alpha)^{-1}\beta+\underset{\sim}{T}f$$
$$=\gamma_2.$$

This follows from the properties of Schwarz operator S and from

$$ReT_{\underset{\sim}{Q}}C^{'}\rho = 0$$

(see [12], pp. 586). □

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Sezayi HIZLIYEL, Mehmet ÇAĞLIYAN
Uludağ University, Art and Science Faculty
Deparment of Mathematics
16059 Görükle, Bursa-TURKEY
e-mail : hizliyel@uludag.edu.tr, caglayan@uludag.edu.tr

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