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Generalized derivations on Lie ideals in prime rings

Öznur Gölbaşı and Emine Koç

Abstract

Let R be a prime ring with characteristic different from two, U a nonzero Lie ideal of R and f be a generalized derivation associated with d . We prove the following results: (i) If $[u, f(u)] \in Z$, for all $u \in U$, then $U \subset Z$. (ii) (f, d) and (g, h) be two generalized derivations of R such that $f(u)v = ug(v)$, for all $u, v \in U$, then $U \subset Z$. (iii) $f([u, v]) = \pm[u, v]$, for all $u, v \in U$, then $U \subset Z$.

Key Words: Derivations, Lie ideals, generalized derivations, centralizing mappings, prime rings.

1. Introduction

Throughout R will represent an associative ring with center Z . Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U, r \in R$. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \rightarrow R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. Let S be a nonempty subset of R . A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$, for all $x \in S$. In [11], Posner showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative. In [3], Awtar considered centralizing derivations on Lie and Jordan ideals. For prime rings Awtar showed that a nontrivial derivation which is centralizing on Lie ideal implies that the ideal is contained in the center if the ring is not of characteristic two or three. In [10], Lee and Lee obtained the same result while removing the restriction of characteristic not three.

In the year 1991, Bresar [5], defined the following concept. An additive mapping $f : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that

$$f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R.$$

One may observe that the concept of generalized derivation includes the concept of derivations, also of left multipliers when $d = 0$. Hence it should be interesting to extend some results concerning these notions to generalized derivations. In [2], Argaç and Albaş extended a well known result of Posner for generalized

derivations of prime rings. Our first objective in this paper is to prove corresponding results for generalized derivations on Lie ideals.

On the other hand, in [6] Daif and Bell showed that if a semiprime ring R has a derivation d satisfying the following condition, then I is a central ideal; there exists a nonzero ideal I of R such that either $d([x, y]) = [x, y]$ for all $x, y \in I$ or $d([x, y]) = -[x, y]$ for all $x, y \in I$.

These results are extended for semiprime rings in [1]. Our second objective of this note is to show the same conditions imposed on Lie ideals of a prime ring with generalized derivation.

Throughout the present paper, R will denote a prime ring of characteristic not two and U will denote a nonzero Lie ideal of R . We make some extensive use of the basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z$$

$$[xy, z] = [x, z]y + x[y, z]$$

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

We denote a generalized derivation $f : R \rightarrow R$ determined by derivation d of R by (f, d) . If $d = 0$ then $f(xy) = f(x)y$ for all $x, y \in R$ and there exists $q \in Q_r(R_C)$ (a right Martindale ring of quotients) such that $f(x) = qx$, for all $x \in R$ by [9, Lemma 2]. So, we assume that $d \neq 0$.

2. Preliminaries

We shall require the following lemmas.

Lemma 2.1 [10, Theorem 5] *Let R be a prime ring with $\text{char}R \neq 2$, d be a nonzero derivation of R and U be a Lie ideal of R . If $[u, d(u)] \in Z$ for all $u \in U$, then $U \subset Z$.*

Lemma 2.2 [4, Theorem 1] *Let R be a prime ring with $\text{char}R \neq 2$, d be a nonzero derivation of R and U be Lie ideal of R . If $d^2(U) = 0$, then $U \subset Z$.*

Lemma 2.3 [4, Lemma 6] *Let R be a prime ring with $\text{char}R \neq 2$, d be a nonzero derivation of R and U be Lie ideal of R . If $d(U) \subseteq Z$, then $U \subset Z$.*

Lemma 2.4 [4, Lemma 1] *Let R be a prime ring with $\text{char}R \neq 2$. If $U \not\subseteq Z$ is a Lie ideal of R , then there exists an ideal M of R such that $[M, R] \subset U$, but $[M, R] \not\subseteq Z$.*

Lemma 2.5 [8, Lemma 1] *Let R be a semiprime 2-torsion free ring and U be a Lie ideal of R . Suppose that $[U, U] \subset Z$, then $U \subset Z$.*

Lemma 2.6 [10, Theorem 2] *Let R be a prime ring with $\text{char}R \neq 2$, d be a nonzero derivation of R , U be a Lie ideal of R and $a \in R$ such that $[a, d(U)] \subset Z$. Then either $a \in Z$ or $U \subset Z$.*

3. Results

Definition 3.1 [7, Definition] *Let R be a ring, d a derivation of R . An additive mapping $f : R \rightarrow R$ is said to be right generalized derivation of R associated with d if*

$$f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R$$

and f is said to be left generalized derivation of R associated with d if

$$f(xy) = d(x)y + xf(y) \text{ for all } x, y \in R.$$

f is said to be a generalized derivation of R associated with d if it is both a left and right generalized derivation of R associated with d .

Remark 3.2 *For all $x, y \in R$,*

$$f([x, y]) = f(xy - yx) = f(x)y + xd(y) - d(y)x - yf(x) = [f(x), y] + [x, d(y)].$$

Theorem 3.3 *If $[u, f(u)] \in Z$ for all $u \in U$, then $U \subset Z$.*

Proof. Writing u by $u + v, v \in U$ in the hypothesis, we have

$$[u, f(v)] + [v, f(u)] \in Z, \text{ for all } u, v \in U.$$

Replacing v by $[u, r], r \in R$ in this equation, we get

$$[u, [f(u), r]] + [u, [u, d(r)]] + [[u, r], f(u)] \in Z, \text{ for all } u \in U, r \in R.$$

Using Jacobi identity and the hypothesis in this equation, we obtain

$$[u, [u, d(r)]] \in Z, \text{ for all } u \in U, r \in R.$$

This yields that $[u, I_{d(r)}(u)] \in Z$, for all $u \in U$, where $I_{d(r)} : R \rightarrow R, I_{d(r)} = [x, d(r)]$ is an inner derivation of R . We have $d(R) \subset Z$ or $U \subset Z$ by Lemma 2.1. If $d(R) \subset Z$, then R is commutative and so, $U \subset Z$. \square

Theorem 3.4 *Let (f, d) and (g, h) be two generalized derivations of R . If $f(u)v = ug(v)$ for all $u, v \in U$, then $U \subset Z$.*

Proof. Assume that $U \not\subset Z$. Then there exists a nonzero ideal M of R such that $[R, M] \not\subset Z$, but $[R, M] \subset U$ by Lemma 2.4. For any $x \in R$ and $m \in M, m[x, m] = [mx, m] \in U$. If we take $m[x, m]$ instead of u in the hypothesis, we have

$$\begin{aligned} f(m[x, m])v &= m[x, m]g(v) \\ d(m)[x, m]v + mf([x, m])v &= m[x, m]g(v). \end{aligned}$$

Using the hypothesis in the above relation, we get

$$d(m)[x, m]v + m[x, m]g(v) = m[x, m]g(v)$$

and so

$$d(m)[x, m]v = 0, \text{ for all } m \in M, v \in U, x \in R.$$

Replacing v by $[v, r]$, $r \in R$ in above equation and using this, we have

$$d(m)[x, m]rv = 0, \text{ for all } m \in M, v \in U, x, r \in R.$$

and so

$$d(m)[x, m]RU = \{0\}, \text{ for all } m \in M, x \in R.$$

Since R is prime ring and $U \neq \{0\}$, it follows that

$$d(m)[x, m] = 0, \text{ for all } m \in M, x \in R.$$

Writing x by xy , $y \in R$ in the last equation and using this, we obtain that

$$d(m)R[y, m] = \{0\}, \text{ for all } m \in M, y \in R.$$

Primeness of R yields that for a fixed $m \in M$,

$$m \in Z \text{ or } d(m) = 0.$$

Let $L = \{m \in M \mid m \in Z\}$ and $K = \{m \in M \mid d(m) = 0\}$. Clearly each of L and K is additive subgroup of M such that $M = L \cup K$. But, a group can not be the set-theoretic union of its two proper subgroups. Hence $L = M$ or $K = M$. In the former case, $M \subset Z$, which forces R to be commutative. This is impossible because of $U \not\subseteq Z$. In the latter case, $d(M) = 0$. Since R is prime ring M a nonzero ideal of R , we get $d = 0$, which is a contradiction. This completes the proof. \square

Corollary 3.5 *Let (f, d) and (g, h) be two generalized derivations of R . If $f(u)u = ug(u)$, for all $u \in U$, then $U \subset Z$.*

Theorem 3.6 *If (f, d) satisfies one of the following conditions then $U \subset Z$.*

- (i) $f([u, v]) = [u, v]$, for all $u, v \in U$.
- (ii) $f([u, v]) = -[u, v]$, for all $u, v \in U$.
- (iii) For each $u, v \in U$, either $f([u, v]) = [u, v]$ or $f([u, v]) = -[u, v]$.

Proof. (i) For any $u, v \in U$, we have $f([u, v]) = [u, v]$, which gives

$$f([u, v]) = [f(u), v] + [u, d(v)] = [u, v].$$

Replacing u by $[u, w]$, $w \in U$, we get

$$f([u, w]), v + [[u, w], d(v)] = [[u, w], v].$$

Using the hypothesis, we obtain

$$[[u, w], v] + [[u, w], d(v)] = [[u, w], v]$$

and so

$$[[u, w], d(v)] = 0, \text{ for all } u, v, w \in U.$$

That is

$$[[U, U], d(U)] = 0.$$

By Lemma 2.6, we have $[U, U] \subset Z$ or $U \subset Z$. If $[U, U] \subset Z$, then again $U \subset Z$ by Lemma 2.5. This completes the proof.

(ii) can be proved by using the same techniques.

(iii) For each $w \in U$, we put

$$U_w = \{v \in U \mid f([w, v]) = [w, v]\} \text{ and } U_w^* = \{v \in U \mid f([w, v]) = -[w, v]\}.$$

Then $(U, +) = U_w \cup U_w^*$, but a group cannot be the union of its two proper subgroups, hence $U = U_w$ or $U = U_w^*$. By the same method in (i) or (ii), we complete the proof. \square

Corollary 3.7 *If (f, d) satisfies one of the following conditions then $U \subset Z$.*

(i) $f(uv) = uv$, for all $u, v \in U$.

(ii) $f(uv) = -uv$, for all $u, v \in U$.

(iii) For each $u, v \in U$, either $f(uv) = uv$ or $f(uv) = -uv$.

Proof. (i) Assume that $f(uv) = uv$ for all $u, v \in U$. Then we have

$$f(uv - vu) = f(uv) - f(vu) = uv - vu.$$

Hence $f([u, v]) = [u, v]$, for all $u, v \in U$. By Theorem 3.6 (i), we obtain that $U \subset Z$.

(ii) can be proved similarly.

(iii) can be proved by using the similar arguments in Theorem 3.6 (iii). \square

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