Turkish Journal of Mathematics

Volume 35 | Number 1 Article 2

1-1-2011

Central simple superalgebras with superantiautomorphism of order two of the second kind

AMEER JABER

Turk J Math
35 (2011) , 11 – 21.

doi:10.3906/mat-0904-20

Central simple superalgebras with superantiautomorphism of order two of the second kind

Ameer Jaber

Abstract

Our main purpose is to develop the theory of existence of superantiautomorphisms of order two of the second kind (which are caled superinvolutions of the second kind) on finite dimensional central simple superalgebras $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D}$ is a finite dimensional division superalgebra with nontrivial grading over K, where K is a field of any characteristic. We determine which finite dimensional central simple superalgebras posses a superinvolution of the second kind and put these results in the context of the *Albert-Reihm* Theorem on the existence of involutions of the second kind.

Key word and phrases: Central simple superalgebras, Superantiautomorphisms, Superinvolutions, Brauer-Wall Groups.

1. Introduction

An *associative* super-ring $R = R_0 + R_1$ is nothing but a $\mathbb{Z}_2$-graded *associative* ring. A $\mathbb{Z}_2$-graded ideal $I = I_0 + I_1$ of an associative super-ring R is called a *superideal* of R. An associative *super-ring* R is *simple* if it has no non-trivial superideals. Let R be an associative super-ring with $1 \in R_0$ then R is said to be a *division* super-ring if all nonzero homogeneous elements are invertible, i.e., every $0 \neq r_\alpha \in R_\alpha$ has an inverse r_α^{-1}, necessarily in R_α.

An *associative* $\mathbb{Z}_2$-graded K-algebra $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ is a *finite dimensional central simple* superalgebra over a field K, if $Z(\mathcal{A}) \cap \mathcal{A}_0 = K$, where $Z(\mathcal{A}) = \{a \in \mathcal{A} \mid ab = ba \ \forall\, b \in \mathcal{A}\}$ is the *center* of $\mathcal{A}$, and the only superideals of $\mathcal{A}$ are $\mathcal{A}$ and (0).

Finite dimensional central simple associative superalgebras over a field K are isomorphic to $\mathrm{End}V \cong M_n(\mathcal{D})$, where $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1$ is a finite dimensional associative division superalgebra over K, i.e., all nonzero elements of $\mathcal{D}_\alpha$, $\alpha = 0, 1$, are invertible, and $V = V_0 + V_1$ is an n-dimensional $\mathcal{D}$-superspace.

If $\mathcal{D}_1 = \{0\}$, the grading of $M_n(\mathcal{D})$ is induced by that of $V = V_0 + V_1$, $\mathcal{A} = M_{p+q}(\mathcal{D})$, $p = \dim_{\mathcal{D}} V_0$, $q = \dim_{\mathcal{D}} V_1$, so $p + q$ is a nontrivial decomposition of n. While if $\mathcal{D}_1 \neq \{0\}$ then the grading of $M_n(\mathcal{D})$ is given by $(M_n(\mathcal{D}))_\alpha = M_n(\mathcal{D}_\alpha)$, $\alpha = 0, 1$.

For completeness, we recall the structure theorem for Central simple associative division superalgebras.

2000 *AMS Mathematics Subject Classification:* 16W10, 16W20, 16W50.

Theorem 1.1 (Division Superalgebra Theorem [8],[7]) *If $\mathcal{D}=\mathcal{D}_0+\mathcal{D}_1$ is a finite dimensional associative division superalgebra over a field K then exactly one of the following holds where throughout $\mathcal{E}$ denotes a finite dimensional associative division algebra over K.*

(i) $\mathcal{D}=\mathcal{D}_0=\mathcal{E}$, and $\mathcal{D}_1=\{0\}$.

(ii) $\mathcal{D}=\mathcal{E}\otimes_K K[u]$, $u^2=\lambda\in K^\times$, $\mathcal{D}_0=\mathcal{E}\otimes K1$, $\mathcal{D}_1=\mathcal{E}\otimes Ku$,

(iii) $\mathcal{D}=\mathcal{E}$ or $M_2(\mathcal{E})$, $u\in\mathcal{D}$ such that $u^2=\lambda\in K/K^2$ ($\lambda\in K/\{\alpha+\alpha^2\mid\alpha\in K\}$ if $Char(K)=2$) $\mathcal{D}_0=C_{\mathcal{D}}(u)$, $\mathcal{D}_1=S_{\mathcal{D}}(u)$, where $C_{\mathcal{D}}(u)=\{d\in\mathcal{D}\mid du=ud\}$, $S_{\mathcal{D}}(u)=\{d\in\mathcal{D}\mid du=u^\sigma d\}$, for some quadratic Galois extension $K[u]\subseteq\mathcal{D}$ with Galois automorphism σ. Moreover, in the second case, $u=\left(\begin{smallmatrix}0&1\\\lambda&0\end{smallmatrix}\right)$ ($u=\left(\begin{smallmatrix}0&1\\\lambda&1\end{smallmatrix}\right)$ if $Char(K)=2$) and $K[u]$ does not embed in $\mathcal{E}$. □

Following [5] we say that a division superalgebra $\mathcal{D}$ is an *even* if $Z(\mathcal{D})\cap\mathcal{D}_1=\{0\}$, i.e., $\mathcal{D}$ is even if its form is (i) or (iii), and that $\mathcal{D}$ is an *odd* otherwise, i.e., $\mathcal{D}$ is odd if its form is (ii). Also, if $\mathcal{A}=M_n(\mathcal{D})$ is a finite dimensional central simple superalgebra over a field K, then we say that $\mathcal{A}$ is an even K-superalgebra if $\mathcal{D}$ is an even division superalgebra and $\mathcal{A}$ is an odd K-superalgebra if $\mathcal{D}$ is an odd division superalgebra.

In my work on the existence of superinvolutions of the first kind, which has yet to appear, I prove that finite dimensional central simple division superalgebras of odd or even type with nontrivial grading over a field K of characteristic not 2 have no superinvolutions of the first kind; these results were introduced in [7, Proposition 9], [8]. Moreover we introduce an example of a central simple superalgebra $\mathcal{A}=M_n(\mathcal{D})$ over a field K of characteristic not 2, where $\mathcal{D}_1\neq\{0\}$, such that $\mathcal{A}$ has no superinvolution of the first kind, but it is of order 2 in the Brauer-Wall group $\mathrm{BW}(K)$, which means that Albert's Theorem does not hold for superinvolutions and this is one of the reasons why one introduces a generalization for which it does. Therefore, if $\mathcal{A}$ is a finite dimensional central simple associative superalgebra over a field K of characteristic not 2 such that $\mathcal{A}$ has a superinvolution of the first kind, then $\mathcal{A}=M_{p+q}(\mathcal{D})$, where $\mathcal{D}$ is a division algebra over K.

In [7] M. Racine described all types of superinvolutions on $\mathcal{A}=M_{p+q}(\mathcal{D})$. It appears that if $*$ is a superinvolution on $\mathcal{A}$ such that $(\mathcal{A}_0,*)$ is simple algebra, then $p=q$ and $*$ is conjugate to the transpose involution. Otherwise, $*$ is conjugate to the orthosymplectic involution.

In [2] we proved that $\mathcal{A}=M_n(\mathcal{D})$, where $\mathcal{D}_1\neq\{0\}$ has a pseudo-superinvolution of the first kind if and only if $\mathcal{A}$ is of order 2 in the Brauer-Wall group $\mathrm{BW}(K)$, where K is a field of characteristic not 2. But if K is a field of characteristic 2, and $\mathcal{A}$ is a central simple associative superalgebra over K, then a superinvolution (which is a pseudo-superinvolution) on $\mathcal{A}$ is just an involution on $\mathcal{A}$ respecting the grading. Moreover, if $\mathcal{A}$ is of order 2 in the Brauer-Wall group $\mathrm{BW}(K)$, then the supercenter of $\mathcal{A}$ equals the center of $\mathcal{A}$ and $\hat{\otimes}_K=\otimes_K$, which means that $\mathcal{A}$ is of order 2 in the Brauer group $\mathrm{Br}(K)$. Thus, by theorem of Albert $\mathcal{A}$ has an involution of the first kind, but since $\mathcal{A}$ is of order 2 in the Brauer-Wall group $\mathrm{BW}(K)$, $\mathcal{A}$ has an superantiautomorphism of the first kind respecting the grading, therefore by [10, Chapter 8, Theorem 8.2] $\mathcal{A}$ has an involution of the first kind respecting the grading, which means that $\mathcal{A}$ has a superinvolution (which is a pseudo-superinvolution) of the first kind if and only if $\mathcal{A}$ is of order 2 in the Brauer-Wall group $\mathrm{BW}(K)$.

Let K/k be a separable quadratic field extension over k with Galois group $\mathrm{Gal}(K/k)$ where $\mathrm{Gal}(K/k)=\{1,\sigma\}$ and $\theta^2\in k-k^2$, $\sigma(\theta)=-\theta$ ($\theta^2\in k/\{\alpha+\alpha^2\mid\alpha\in k\}$, $\sigma(\theta)=\theta+1$ if $\mathrm{Char}(k)=2$). We recall a theorem of *Albert-Reihm* on the existence of K/k-involution (involution of the second kind) which states that finite dimensional central simple algebra $\mathcal{A}$ over K has a K/k-involution if and only if the corestriction of $\mathcal{A}$ splits over k. In [1] A. Elduque and O. Villa gave a much better exposition and motivation for the whole theory of the existence of superinvolutions.

Throughout this paper we classify the existence of superinvolution of the second kind on $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D}$ is a division superalgebra over K with $\mathcal{D}_1 \neq \{0\}$, and we show that $\mathcal{D}$ has a K/k-superinvolution (superinvolution of the second kind) if and only if the corestriction of $\mathcal{D}$ splits over k, which implies that the natural graded version of the classical AlbertRiehm Theorem holds.

2. Basic facts and definitions

Definition 1 A *superantiautomorphism* of an associative superalgebra $\mathcal{A}$ is a graded additive map $J: \mathcal{A} \to \mathcal{A}$ such that for $a_\alpha \in \mathcal{A}_\alpha$, $b_\beta \in \mathcal{A}_\beta$

$$J(a_\alpha b_\beta) = (-1)^{\alpha\beta} J(b_\beta) J(a_\alpha).$$

A *superinvolution* of an associative superalgebra $\mathcal{A}$ is a superantiautomorphism $* : \mathcal{A} \to \mathcal{A}$ such that $a^{**} = a$ for any $a \in \mathcal{A}$. If $\mathcal{A}$ is a finite dimensional central simple superalgebra over a field K, and $*$ is a superinvolution on $\mathcal{A}$, then $K^* = K$, so $*$ is called a *superinvolution of the second kind* if $*|_K = \sigma$ is a Galois automorphism of order 2 on K.

Let $\mathcal{A}$ be any K-superalgebra, we define the map $\varphi : \mathcal{A} \to \mathcal{A}$ by

$$\varphi(a_\alpha) = (-1)^\alpha a_\alpha \quad \forall a_\alpha \in \mathcal{A}_\alpha, \quad \text{and} \quad \forall \alpha = 0, 1.$$

This map φ is a superalgebra automorphism, called the sign automorphism, since

$$(a_\alpha b_\beta)^\varphi = (-1)^{\alpha+\beta} a_\alpha b_\beta = a_\alpha^\varphi b_\beta^\varphi$$

for all $a_\alpha \in \mathcal{A}_\alpha$ and $b_\beta \in \mathcal{A}_\beta$. The automorphism φ has order 2, if Char$(K) \neq 2$ (unless $\mathcal{A}_1 = 0$), and $\varphi = id_{\mathcal{A}}$ if Char$(K) = 2$.

If $R = R_0 + R_1$ is an associative super-ring, a (right) R-supermodule M is a right R-module with a grading $M = M_0 + M_1$ as R_0-modules such that $m_\alpha r_\beta \in M_{\alpha+\beta}$ for any $m_\alpha \in M_\alpha$, $r_\beta \in R_\beta$, $\alpha, \beta \in \mathbf{Z}_2$.

An R-supermodule M is *irreducible* if $MR \neq \{0\}$ and M has no proper subsupermodule.

Following [5] we have the following definition of R-supermodule homomorphism.

Definition 2 An R-supermodule homomorphism from M to N, where M and N are R-supermodules, is an R_0-module homomorphism $h_\gamma : M \to N$, $\gamma \in \mathbf{Z}_2$, such that $M_\alpha h_\gamma \subseteq N_{\alpha+\gamma}$ and

$$(m_\alpha r_\beta) h_\gamma = (m_\alpha h_\gamma) r_\beta, \ \forall\, m_\alpha \in M_\alpha,\ r_\beta \in R_\beta,\ \alpha, \beta \in \mathbf{Z}_2.$$

Following [7] the *commuting super-ring* of R on M is defined to be $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$

$$\text{where} \quad \mathcal{C}_\gamma := \{c_\gamma \in \text{End}_\gamma M \mid c_\gamma r_\alpha = (-1)^{\alpha\gamma} r_\alpha c_\gamma \ \forall r_\alpha \in R_\alpha,\ \alpha \in \mathbb{Z}_2\}.$$

Definition 3 Let $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$, $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1$ be associative superalgebras. Then we define the graded tensor product by

$$\mathcal{A}\hat{\otimes}_K\mathcal{B} = [(\mathcal{A}_0 \otimes \mathcal{B}_0) \oplus (\mathcal{A}_1 \otimes \mathcal{B}_1)] \oplus [(\mathcal{A}_0 \otimes \mathcal{B}_1) \oplus (\mathcal{A}_1 \otimes \mathcal{B}_0)]$$

where the multiplication on $\mathcal{A}\hat{\otimes}_K\mathcal{B}$ is induced by

$$(a_\alpha \otimes b_\beta)(c_\gamma \otimes d_\delta) = (-1)^{\beta\gamma} a_\alpha c_\gamma \otimes b_\beta d_\delta, \quad a_\alpha \in \mathcal{A}_\alpha,\ c_\gamma \in \mathcal{A}_\gamma,\ b_\beta \in \mathcal{B}_\beta,\ d_\delta \in \mathcal{B}_\delta.$$

If $\mathcal{A}$ and $\mathcal{B}$ are associative superalgebras, then $\mathcal{A}\hat{\otimes}_K\mathcal{B}$ is an associative superalgebra.

The opposite super-ring R° of the super-ring R is defined to be $R^\circ = R$ as an additive group, with the multiplication given by

$$b_\beta \circ c_\gamma := (-1)^{\beta\gamma} c_\gamma b_\beta \quad b_\beta \in R_\beta\ ,\ c_\gamma \in R_\gamma.$$

So, if $\mathcal{A}$ is a central simple associative superalgebra over a field K, then one can easily show that $\mathcal{A}^\circ$ is also a central simple associative superalgebra over K, and by [5] $\mathcal{A}\hat{\otimes}\mathcal{A}^\circ \cong M_n(K)$, where $n = \dim_K(\mathcal{A})$.

Definition 4 Two finite dimensional central simple. superalgebras $\mathcal{A}$ and $\mathcal{B}$ over a field K are called similar ($\mathcal{A} \sim \mathcal{B}$) if there exist graded K-vector spaces $V = V_0 \oplus V_1$, $W = W_0 \oplus W_1$, such that $\mathcal{A}\hat{\otimes}_K \mathrm{End}_K V \cong \mathcal{B}\hat{\otimes}_K \mathrm{End}_K W$ as a K-superalgebras.

Similarity is obviously an equivalence relation. The set of similarity classes will be denoted by $\mathrm{BW}(K)$ (the Brauer-Wall group of K). If $[\mathcal{A}]$ denotes the class of A in $\mathrm{BW}(K)$ by using [5, Chap. 4, Theorem 2.3 (3)] the operation $[\mathcal{A}][\mathcal{B}] = [\mathcal{A}\hat{\otimes}_K\mathcal{B}]$ is well-defined, and makes the set of similarity classes of finite dimensional central simple superalgebras over K into a commutative group, $\mathrm{BW}(K)$, where the class of the matrix algebras $M_{p+q}(K)$ is a neutral element for this product.

We say that a central simple superalgebra $\mathcal{A}$ over K is similar to 1 ($\mathcal{A} \sim 1$) or splits, if $[\mathcal{A}] = [K]$ in $\mathrm{BW}(K)$.

3. The corestriction

In [7, Theorem 3] Michel Racine proved that finite dimensional associative central simple superalgebras $\mathcal{A} = M_n(\mathcal{D})$ over a field K are primitive superalgebras, and then he proved in [7, Theorem 7] that primitive superalgebra $\mathcal{A} = M_n(\mathcal{D})$ has a superinvolution if and only if D has. Thus we have the following result.

Theorem 3.1 *A finite dimensional associative central simple superalgebra $\mathcal{A} = M_n(\mathcal{D})$ over a field K has a superinvolution $*$ if and only if $\mathcal{D}$ has.* □

If $\mathcal{A} = M_n(\mathcal{D})$ is a finite dimensional central simple super algebra over a field K, where $\mathcal{D}$ is a finite dimensional division superalgebra with nontrivial grading over K, that is $\mathcal{D}_1 \neq \{0\}$, then by Theorem 3.1, it is enough to classify the existence of superinvolutions of the second kind on $\mathcal{D}$ to ascertain the existence of superinvolutions of the second kind on $\mathcal{A}$.

The situation for superinvolution of the second kind is analogous to the ungraded case in the sense that we can define a corestriction and prove that a superinvolution of the second kind exists if and only if the corestriction is trivial (graded *Albert-Riehm* Theorem).

Let $\mathcal{D}$ be a central division superalgebra of any type with nontrivial grading over K. Let $K = k(\theta)$ be a separable quadratic field extension over k with Galois group $\mathrm{Gal}(K/k) = \{1, \sigma\}$ and $\theta^2 \in k - k^2$, $\sigma(\theta) = -\theta$ ($\theta^2 \in k/\{\alpha + \alpha^2 \mid \alpha \in k\}$, $\sigma(\theta) = \theta + 1$ if $\mathrm{Char}(k) = 2$).

A K/k-superantiautomorphism J of $\mathcal{D}$ is a superantiautomorphism on $\mathcal{D}$ which is K/k-smilinear, i.e., $J(\lambda x) = \sigma(\lambda)J(x)$ for all $\lambda \in K$, $x \in \mathcal{D}$. Accordingly, a K/k-superantiautomorphism J is called K/k-superinvolution (superinvolution of the second kind) if $J^2(x) = x$ for all $x \in \mathcal{D}$.

Let $\overline{\mathcal{D}}$ be the superalgebra which is identical with $\mathcal{D}^\circ$ as a super-ring but with the action of K twisted by σ, then $\overline{\mathcal{D}}$ is called the conjugate K-division superalgebra of $\mathcal{D}$.

Let $\mathcal{A} := \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}$, then $\mathcal{A}$ is an even central simple superalgebra over K. Let $\pi : \mathcal{A} \to \mathcal{A}$ be a map defined as follows for all $a_\alpha \in \mathcal{D}_\alpha$, $b_\beta \in \overline{\mathcal{D}}_\beta$ we have $a_\alpha \otimes b_\beta \mapsto (-1)^{\alpha\beta} b_\beta \otimes a_\alpha$. We now define

$$\mathcal{A}^+ = \{x \in \mathcal{A} \ : \ \pi(x) = x\}.$$

Definition 5 The k-superalgebra $\mathcal{A}^+$ is called the Corestriction of the superalgebra $\mathcal{D}$, where $\mathcal{A} := \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}$.

Since $\mathcal{A} = \mathcal{A}^+ + \theta\mathcal{A}^+ \cong \mathcal{A}^+ \otimes_k K$, we conclude that $\mathcal{A}^+$ is an even central simple superalgebra over k. Of course, if $\mathcal{D} = \mathcal{D}_0$, the corestriction $\mathcal{A}^+$ coincides with the usual ungraded corestriction (see [10, P. 308])

Example 1 Consider the quadratic superalgebra $\mathcal{D} = K[\sqrt{\mu}] = K \oplus Ku$, with the relation $u^2 = \mu \in K$, and the grading $\mathcal{D}_0 = K$, $\mathcal{D}_1 = Ku$. Then $\mathcal{A} = \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}} = \mathcal{A}_0 + \mathcal{A}_1$, where $\mathcal{A}_0 = \mathrm{Span}\{1 \otimes 1, \ \theta u \otimes u\}$ and $\mathcal{A}_1 = \mathrm{Span}\{u \otimes 1 + 1 \otimes u, \ \theta u \otimes 1 - 1 \otimes \theta u\}$ (over K), is a graded quaternion algebra over K and since $\mathcal{A}^+ \otimes_k K \cong \mathcal{A}$, which is a graded quaternion algebra over K, then $\mathcal{A}^+$ is a graded quaternion algebra over k, with $(\mathcal{A}^+)_0 = \mathrm{Span}\{1 \otimes 1, \ \theta u \otimes u\}$, $(\mathcal{A}^+)_1 = \mathrm{Span}\{u \otimes 1 + 1 \otimes u, \ \theta u \otimes 1 - 1 \otimes \theta u\}$.

Next, we define a (right) $\mathcal{A}^+$-module structure on $\mathcal{D}$ when $\mathcal{D}$ possesses a K/k-superantiautomorphism. In particular, if J is a K/k-superantiautomorphism, then we define a right action of $\mathcal{A} = \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}$ on $\mathcal{D}$:

$$e : \mathcal{A} \to \mathrm{End}_K\mathcal{D}$$

given by

$$x_\gamma(a_\alpha \otimes b_\beta) = x_\gamma(a_\alpha \otimes b_\beta)^e := (-1)^{\gamma\alpha} J(a_\alpha) x_\gamma b_\beta$$

for all $a_\alpha \in \mathcal{D}_\alpha$, $b_\beta \in \overline{\mathcal{D}}_\beta$, $x_\gamma \in \mathcal{D}_\gamma$. The map e is an isomorphism (because $\mathcal{A}$ is a central simple superalgebra and by dimension count).

We will often use the equivalences $\mathcal{D}$ has a K/k-superantiautomorphism if and only if K-central simple division superalgebras $\overline{\mathcal{D}}$ and $\mathcal{D}^\circ$ are isomorphic if and only if $\mathcal{A} \cong \mathrm{End}_K\mathcal{D} \sim 1$ in the Brauer-Wall group $\mathrm{BW}(K)$. Since

$$\mathcal{A}^+ \hookrightarrow \mathcal{A} \cong \mathrm{End}_K\mathcal{D} \hookrightarrow \mathrm{End}_k\mathcal{D}$$

we have a (right) $\mathcal{A}^+$-module structure on $\mathcal{D}$.

Lemma 3.2 *Let J be a K/k-superantiautomorphism on $\mathcal{D}$, where $\mathcal{D}$ is an even or odd central division superalgebra over K with $\mathcal{D}_1 \neq \{0\}$, such that*

$$J^2(x_\gamma) = (-1)^{\gamma\alpha} b_\alpha x_\gamma b_\alpha^{-1},$$

where $b_\alpha \in \mathcal{D}_\alpha$, then $\mathcal{A}^+ \sim \mathrm{End}_{\mathcal{A}^+}\mathcal{D}$ in the Brauer-Wall group $BW(k)$ and

$$\dim_k(\mathrm{End}_{\mathcal{A}^+}\mathcal{D}) = 4$$

and $K \subseteq (\mathrm{End}_{\mathcal{A}^+}\mathcal{D})_0$, where $\mathcal{A} := \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}$.

Proof. Using the supermodule isomorphism $J : \mathcal{D}^\circ \cong \overline{\mathcal{D}}$, we get the supermodule isomorphism

$$\mathrm{End}_K\mathcal{D} \cong \mathcal{D}\hat{\otimes}_K\mathcal{D}^\circ \cong \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}.$$

Since $\mathcal{A}^+ \subseteq \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}$ we consider $\mathcal{D}$ as a $\mathcal{A}^+$-supermodule.

Now, by [7, Theorem 3] $\mathcal{A}^+ \cong M_n(\mathcal{C})$, where $\mathcal{C}$ is the commuting super-ring of $\mathcal{A}^+$ on I, where I is a minimal right superideal in $\mathcal{A}^+$. Since $\mathcal{D}$ is a finite dimensional $\mathcal{A}^+$-supermodule, $\mathrm{End}_{\mathcal{A}^+}\mathcal{D} \cong \mathrm{End}_{\mathcal{A}^+}M^r$, where M is an irreducible $\mathcal{A}^+$-supermodule of $\mathcal{D}$. Since I is a minimal (right) superideal in $\mathcal{A}^+$, then I is irreducible $\mathcal{A}^+$-supermodule and by [7, Proposition 4]

$$\mathrm{End}_{\mathcal{A}^+}M \cong \mathrm{End}_{\mathcal{A}^+}I \cong \mathcal{C},$$

and hence

$$\mathrm{End}_{\mathcal{A}^+}\mathcal{D} \cong \mathrm{End}_{\mathcal{A}^+}M^r \cong M_r(\mathcal{C})$$

but $M_r(\mathcal{C}) \sim M_n(\mathcal{C}) \cong \mathcal{A}^+$ in the Brauer-Wall group $\mathrm{BW}(k)$, therefore

$$\mathrm{End}_{\mathcal{A}^+}\mathcal{D} \sim \mathcal{A}^+$$

in the Brauer-Wall group $\mathrm{BW}(k)$.

Now, $\dim_k \mathrm{End}_{\mathcal{A}^+}\mathcal{D} = \dim_k M_r(\mathcal{C}) = r^2 \dim_k \mathcal{C}$, and

$$\begin{aligned}
\dim_k \mathcal{A}^+ = n^2 \dim_k \mathcal{C} &= \dim_K \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}} \\
&= [\frac{1}{2}\dim_k M^r]^2 \\
&= \frac{1}{4}r^2(\dim_k M)^2, \quad \text{since } M \cong I \\
&= \frac{1}{4}r^2(\dim_k I)^2 \\
&= \frac{1}{4}r^2(n\dim_k \mathcal{C})^2 \\
&= \frac{1}{4}r^2n^2(\dim_k \mathcal{C})^2.
\end{aligned}$$

Therefore, $n^2 \dim_k \mathcal{C} = \frac{1}{4}r^2n^2(\dim_k \mathcal{C})^2$ which implies that $4 = r^2 \dim_k \mathcal{C}$, and hence $\dim_k \mathrm{End}_{\mathcal{A}^+}\mathcal{D} = 4$, thus $\mathrm{End}_{\mathcal{A}^+}\mathcal{D}$ has dimension 4 over k.

For $\xi \in K$ multiplication by ξ is a $\mathcal{A}^+$-supermodule endomorphism on $\mathcal{D}$. This gives an inclusion $K \subseteq \text{End}_{\mathcal{A}^+}\mathcal{D}$. □

4. Existence of superinvolutions of the second kind

In this section we investigate the existence of superinvolutions of the second kind on the division superalgebra $\mathcal{D}$, where $\mathcal{D}_1 \neq \{0\}$. Then we generalize the method due to Riehm (see [4], [6]and [9]) which improves on Albert's original approach to the superalgebra case, to formulate the existence criterion of superinvolutions of the second kind.

Remark 1 (1) For $\gamma \in k^\times$ a 4-dimensional k-superalgebra with a trivial grading $Q = Q_0 = K + Ku$ with $u^2 = \gamma \in k^\times$ and $uxu^{-1} = \overline{x}$ for all $x \in K$, where $\overline{\alpha + \beta\lambda} = \alpha - \beta\lambda$ $\forall \alpha, \beta \in k$ (if $\text{Char}(k) \neq 2$) and $\overline{\alpha + \beta\lambda} = \alpha + \beta(\lambda + 1)$ $\forall \alpha, \beta \in k$ (if $\text{Char}(k) = 2$), is a quaternion algebra over k which splits over k if and only if $\gamma \in N(K^\times)$. See [6, Theorem 8].

(2) $H = K + Ku$, with $H_0 = K$, $H_1 = Ku$ and $u^2 = \gamma \in k^\times$ and with $uxu^{-1} = \overline{x}$ for all $x \in K$ is a quaternion division superalgebra over k, which does not split over k.

(3) By Lemma 3.2 $\text{End}_{\mathcal{A}^+}\mathcal{D}$ is either a quaternion central division superalgebra with nontrivial grading or a quaternion central division superalgebra with a trivial grading over k.

If $\mathcal{B}$ a superalgebra and J is a superantiautomorphism on $\mathcal{B}$, then $J^2 = \psi_{b_\delta}$ for some $b_\delta \in \mathcal{B}_\delta^\times$ if for any $x_\alpha \in \mathcal{B}_\alpha$

$$J^2(x_\alpha) = (-1)^{\alpha\delta} b_\delta x_\alpha b_\delta^{-1}.$$

Lemma 4.1 *Let $\mathcal{D}$ be a division superalgebra over K of even or odd type with $\mathcal{D}_1 \neq \{0\}$, and let J be a K/k-superantiautomorphism on $\mathcal{D}$ such that*

$$J^2(x_\alpha) = (-1)^{\alpha\delta} b_\delta x_\alpha b_\delta^{-1},$$

where $b_\delta \in \mathcal{D}_\delta$, and let $\mathcal{A} := \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}$. Then one and only one of the following cases occurs:

(1) Either $J^2 = \psi_b$, with $b \in \mathcal{D}_0$. In this case $\mathcal{A}^+ \sim Q$ in the Brauer-Wall group $BW(k)$, where $Q = Q_0 = K + Ku$ is a quaternion algebra over k with $u^2 = J(b)b \in k^\times$ and $uxu^{-1} = \overline{x}$ for any $x \in K$.

(2) Or $J^2 = \psi_b$, with $b \in \mathcal{D}_1$. In this case $\mathcal{A}^+ \sim H$ in the Brauer-Wall group $BW(k)$, where $H = K + Ku$, $H_0 = K$, $H_1 = Ku$ is a quaternion central division superalgebra over k.

Proof. We already know that $\text{End}_{\mathcal{A}^+}\mathcal{D}$ is either a quaternion central division superalgebra or a quaternion algebra with trivial grading over k. Let $J^2 = \psi_{b_\delta}$, with $b_\delta \in \mathcal{D}_0 \cup \mathcal{D}_1$. For any $x_\alpha \in \mathcal{D}_\alpha$,

$$\begin{aligned} J^3(x_\alpha) = J(J^2(x_\alpha)) &= J((-1)^{\alpha\delta} b_\delta x_\alpha b_\delta^{-1}) \\ &= (-1)^{\alpha\delta}(J(b_\delta))^{-1} J(x_\alpha) J(b_\delta), \end{aligned}$$

but also $J^3(x_\alpha) = J^2(J(x_\alpha)) = (-1)^{\alpha\delta} b_\delta J(x_\alpha) b_\delta^{-1}$, so

$$J(x_\alpha) = J(b_\delta) b_\delta J(x_\alpha) (J(b_\delta) b_\delta)^{-1}.$$

Thus $J(b_\delta)b_\delta \in Z(\mathcal{D}) \cap \mathcal{D}_0 = K$, and it is fixed by J, therefore, $J(b_\delta)b_\delta \in k^\times$.

Now consider the k-linear map $f : \mathcal{D} \to \mathcal{D}$ given by $f(x_\alpha) = (-1)^{\alpha\delta} J(x_\alpha) b_\delta$ for any $x_\alpha \in \mathcal{D}_\alpha$. The map f is even or odd according to b_δ being even or odd.

For any $a \in K,\ x_\alpha \in \mathcal{D}_\alpha$,

$$f(ax_\alpha) = (-1)^{\alpha\delta}\ J(a)J(x_\alpha)b_\delta = (-1)^{\alpha\delta}\ \overline{a}J(x_\alpha)b_\delta = \overline{a}f(x_\alpha)$$

which implies that $fa = \overline{a}f$ for all $a \in K$, and

$$\begin{aligned} f^2(x_\alpha) = f(f(x_\alpha)) &= f((-1)^{\alpha\delta} J(x_\alpha) b_\delta) \\ &= (-1)^{\alpha\delta} f(J(x_\alpha) b_\delta) \\ &= (-1)^{\delta} J(J(x_\alpha) b_\delta) b_\delta \\ &= (-1)^{\delta+\alpha\delta} J(b_\delta) J^2(x_\alpha) b_\delta \\ &= (-1)^{\delta} J(b_\delta) b_\delta x_\alpha b_\delta^{-1} b_\delta \\ &= (-1)^{\delta} J(b_\delta) b_\delta x_\alpha \end{aligned}$$

so $f^2 = l_{(-1)^\delta J(b_\delta) b_\delta}$ and the algebra over k generated by K and f is $K + Kf$, which is a quaternion ungraded algebra if b_δ is even ($\delta = 0$), or a quaternion superalgebra with even part K if b_δ is odd ($\delta = 1$). It remains to show that $f \in \mathrm{End}_{\mathcal{A}^+}\mathcal{D}$. For any homogeneous elements x_γ, a_α, c_β in $\mathcal{D}$,

$$\begin{aligned} f(x_\gamma \cdot (a_\alpha \otimes c_\beta + (-1)^{\alpha\beta} c_\beta \otimes a_\alpha)) &= f((-1)^{\alpha\gamma} J(a_\alpha) x_\gamma c_\beta + (-1)^{(\alpha+\gamma)\beta} J(c_\beta) x_\gamma a_\alpha) \\ &= (-1)^{\delta(\alpha+\gamma+\beta)} ((-1)^{\alpha\gamma} J(J(a_\alpha) x_\gamma c_\beta) b_\delta \\ &+ (-1)^{(\alpha+\gamma)\beta} J(J(c_\beta) x_\gamma a_\alpha) b_\delta) \\ &= (-1)^{\delta(\alpha+\gamma+\beta)} ((-1)^{\beta(\alpha+\gamma)} J(c_\beta) J(x_\gamma) J^2(a_\alpha) b_\delta \\ &+ (-1)^{\alpha\gamma} J(a_\alpha) J(x_\gamma) J^2(c_\beta) b_\delta) \\ &= (-1)^{\delta(\alpha+\gamma+\beta)} ((-1)^{\beta(\alpha+\gamma)+\delta\alpha} J(c_\beta) J(x_\gamma) b_\delta a_\alpha \\ &+ (-1)^{\alpha\gamma+\beta\delta} J(a_\alpha) J(x_\gamma) b_\delta c_\beta) \\ &= (-1)^{\beta(\alpha+\gamma+\delta)} (J(c_\beta)((-1)^{\delta\gamma} J(x_\gamma) b_\delta) a_\alpha) \\ &+ (-1)^{\alpha(\delta+\gamma)} (J(a_\alpha)((-1)^{\delta\gamma} J(x_\gamma) b_\delta) c_\beta) \\ &= f(x_\gamma) \cdot ((-1)^{\alpha\beta} c_\beta \otimes a_\alpha) + f(x_\gamma) \cdot (a_\alpha \otimes c_\beta) \\ &= f(x_\gamma) \cdot (a_\alpha \otimes c_\beta\ +\ (-1)^{\alpha\beta} c_\beta \otimes a_\alpha). \end{aligned}$$

Since the elements $a_\alpha \otimes c_\beta\ +\ (-1)^{\alpha\beta} c_\beta \otimes a_\alpha$ span $\mathcal{A}^+$, we conclude f is in $\mathrm{End}_{\mathcal{A}^+}\mathcal{D}$, which implies that $K + Kf \cong \mathrm{End}_{\mathcal{A}^+}\mathcal{D}$; and, since by Lemma 3.2, $\mathrm{End}_{\mathcal{A}^+}\mathcal{D} \sim \mathcal{A}^+$, then $K + Kf \sim \mathcal{A}^+$. □

Next theorem is the generalization of Albert-Riehm Theorem to the supercase. Let $\mathcal{D}$ be a central division superalgebra of any type with nontrivial grading over K. Let $K = k(\theta)$ be a separable quadratic field extension over k with Galois group $\mathrm{Gal}(K/k) = \{1, \sigma\}$ and $\theta^2 \in k - k^2$, $\sigma(\theta) = -\theta$ ($\theta^2 \in k/\{\alpha+\alpha^2 \mid \alpha \in k\}$, $\sigma(\theta) = \theta+1$ if $\mathrm{Char}(k) = 2$). Then we get the following theorem.

Theorem 4.2 *Let $\mathcal{D}$ be a central division superalgebra over K with $\mathcal{D}_1 \neq \{0\}$. Then $\mathcal{D}$ has a K/k-superinvolution (a superinvolution of the second kind) if and only if $\mathcal{A}^+ \sim 1$ in the Brauer-Wall group $BW(k)$, where $\mathcal{A} := \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}$.*

Proof. First we assume that J is a K/k-superinvolution on $\mathcal{D}$. Clearly $J^2 = \psi_1$. By Lemma 4.1, $\mathcal{A}^+ \sim Q$ in the Brauer-Wall group $\mathrm{BW}(k)$, where $Q = Q_0 = K + Ku$, with $u^2 = J(1)1 = 1$, but Q splits over k, therefore $\mathcal{A}^+ \sim 1$ in the Brauer-Wall group $\mathrm{BW}(k)$.

Conversely, suppose that $\mathcal{A}^+ \sim 1$ in the Brauer-Wall group $\mathrm{BW}(k)$. Then $K \otimes_k \mathcal{A}^+ \cong \mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}}$ splits over K, since $\mathcal{A}^+ \sim 1$ in the Brauer-Wall group $\mathrm{BW}(k)$. Hence $\mathcal{D}^\circ \cong \overline{\mathcal{D}}$. Let J be a K/k-superantiautomorphism on $\mathcal{D}$, since $\mathcal{A}^+ \sim 1$ in the Brauer-Wall group $\mathrm{BW}(k)$, then by Lemma 4.1, $J^2 = \psi_b$, where $b \in \mathcal{D}_0$. Moreover, we may assume that $J(b)b = 1$ (because the fact that $\mathcal{A}^+ \sim 1$ in the Brauer-Wall group $\mathrm{BW}(k)$ implies that there is an element $\lambda \in K$ such that $J(b)b = \lambda\overline{\lambda}$ and we may change b with $\lambda^{-1}b$). If $b = -1$, we have finished. If $b \neq -1$, then the map $I = \psi_{(1+b)^{-1}} \circ J$ is a K/k-superinvolution on $\mathcal{D}$. □

For central division superalgebra $\mathcal{D}$ of odd type we have the following easier criterion for the existence of superinvolutions of the second kind than the general one in terms of corestriction.

Theorem 4.3 *Let $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_0 v$ be central division superalgebra of odd type over K, where $v \in Z(\mathcal{D}) \cap \mathcal{D}_1$. Then $\mathcal{D}$ has a K/k-superinvolution if and only if both $Z(\mathcal{D}) = K + Kv$ and $\mathcal{D}_0$ has K/k-superinvolutions.*

Proof. If $Z(\mathcal{D}) = K + Kv$ has a K/k-superinvolution J_1 and $\mathcal{D}_0$ has a K/k-superinvolution (K/k-involution) J_2, then $J_1 \otimes J_2$ is a K/k-superinvolution on $\mathcal{D} = Z(\mathcal{D}) \otimes_K \mathcal{D}_0$.

Conversely, if J is a K/k-superinvolution on $\mathcal{D}$, then $J|_{\mathcal{D}_0}$ is a K/k-involution on $\mathcal{D}_0$ and $J|_{Z(\mathcal{D})}$ is a K/k-superinvolution on $Z(\mathcal{D})$. □

Finally, the following example shows that $\mathcal{D}\hat{\otimes}_K\overline{\mathcal{D}} \sim 1$ in the Brauer-Wall group $\mathrm{BW}(K)$ (or equivalently $\mathrm{End}_K\mathcal{D} \sim 1$ in the Brauer-Wall group $\mathrm{BW}(K)$), where $\mathcal{D}$ is a central division superalgebra of odd type over K, is not enough to say that $\mathcal{D}$ admits a K/k-superinvolution, but first we need the Skolem-Noether Theorem for the supercase, which has been proven in my Ph.D. thesis. (See [3, Theorem 2.1.5]).

Theorem 4.4 (Skolem-Noether Theorem) Let B be a central simple superalgebra over the field K, and let A be a finite dimensional simple subsuperalgebra over K and containing it. Then any superalgebra homomorphism f of A into B can be extended to an inner automorphism of B if B is even. If B is odd then f or $f\varphi$ can be extended to an inner automorphism but not both of them, where $\varphi : B \to B$ such that $\varphi(b_\beta) = (-1)^\beta b_\beta$. □

Example 2 Let $K = \mathbf{Q}[i](\sqrt{2})$ and let $\mathcal{D} = KI_2 \oplus Ku$, where $u = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so $u^2 = 1$. Let $J : \mathcal{D} \longrightarrow \mathcal{D}$ be the map defined as follows:

$$u \mapsto iu \quad \text{and} \quad \alpha + \beta\sqrt{2} \mapsto \alpha - \beta\sqrt{2} \quad \forall \alpha, \beta \in \mathbf{Q}[i].$$

Then a trivial computation shows that $\mathcal{D} \overset{J}{\approx} \overline{\mathcal{D}}^{\circ}$ but $\mathcal{D}$ does not have a superinvolution of the second kind, since if $\mathcal{D}$ had one (say $*$) then $\exists\ u' = \gamma u \in \mathcal{D}_1$ such that

$$(a + bu')^* = a^J + b^J u'.$$

Now, $*J$ is an automorphism on $\mathcal{D}$, since $\mathcal{D}$ is odd, we have by the Skolem-Noether Theorem that $*J$ or $\varphi * J$ is an inner automorphism but not both of them, where φ is the sign automorphism.

Case(1): $*J$ is inner . So $\exists v_\alpha \in \mathcal{D}_\alpha$ such that

$$\forall a, b \in K \ (a + bu')^{*J} = v_\alpha (a + bu') v_\alpha^{-1} = a + bu' = a + b(u')^J$$

and hence

$$(u')^J = u' = (\gamma u)^J = \gamma^J u^J = \gamma^J iu = \gamma u$$

thus, $\gamma^J i = \gamma$. If $\gamma = x + y\sqrt{2} \in \mathbf{Q}[i](\sqrt{2})$ then

$$(x - y\sqrt{2})i = x + y\sqrt{2} \quad \text{so} \quad x(1 - i) = -y\sqrt{2}(1 + i)$$

and therefore

$$\sqrt{2} = \frac{-x(1-i)}{y(1+i)} \in \mathbf{Q}[i]$$

($y \neq 0$, because $y = 0$ implies that $\gamma^J = \gamma \Rightarrow i = 1$), a contradiction.

Case(2): $\varphi * J$ is inner. So $\exists v_\alpha \in \mathcal{D}_\alpha$ such that

$$\forall a, b \in K \ \ (a + bu')^{\varphi * J} = v_\alpha (a + bu') v_\alpha^{-1} = a + bu' = a - b(u')^J$$

therefore

$$-(u')^J = u' = -(\gamma u)^J = -\gamma^J u^J = -\gamma^J iu = \gamma u$$

and so $-\gamma^J i = \gamma$, if $\gamma = x + y\sqrt{2} \in \mathbf{Q}[i](\sqrt{2})$ then $(-x + y\sqrt{2})i = x + y\sqrt{2}$. So $x(1 + i) = y\sqrt{2}(-1 + i)$ and therefore

$$\sqrt{2} = \frac{x(1+i)}{y(-1+i)} \in \mathbf{Q}[i]$$

($y \neq 0$, because $y = 0$ implies that $\gamma^J = \gamma \Rightarrow i = -1$), a contradiction.

References

[1] Elduque, A., and Villa, O. : The existence of superinvolutions, J. Algebra 319, 4338-4359(2008).

[2] Jaber, A. : Existence of Pseudo-Superinvolutions of The First Kind, International Journal of Mathematics and Mathematical Sciences, vol. 2008, Article ID 386468, 12 pages, 2008.

[3] Jaber, A. : Superinvolutions of Associative Superalgebras, P.h. D. thesis, University of Ottawa, Ottawa, Canada, 2003.

[4] Knus, M.-A., Merkurjev, A., Rost, M. and Tignol, J.-P. : The Book of Involutions, Amer. Math. Soc., 1998.

[5] Lam, T. Y. : The Algebraic Thoery of Quadratic Forms, the Benjamin/Cummings Publishing Company, 1973.

[6] Lewis, D. W. : Involutions and Anti-automorphisms of Algebras, London Math. Soc. 38, 529-545(2006).

[7] Racine, M. L. : Primitive superalgebras with superinvolution, J. Algebra 206, 588-614(1998).

[8] Racine, M. L. and Zelmanov, E.I. : Simple Jordan superalgebras with semisimple even part, J. Algebra 270, (2003)374-444.

[9] Riehm, C. : The Corestriction of Algebriac Structure, Invent. Math. 11, 73-98(1970).

[10] Scharlau, W. : Quadratic and Hermitian Forms, Springer-Verlag, Heidelberg, 1985.

Ameer JABER
Department of Mathematics,
The Hashemite University
Zarqa, 13115, JORDAN
e-mail: ameerj@hu.edu.jo

Received 09.04.2009