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Some properties of Associate and Presimplifiable rings

Manal Ghanem

Abstract

In this paper we study some properties of associate and presimplifiable rings. We give a characterization of the associate (resp., domainlike) pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$, where R_1 and R_2 are two presimplifiable (resp., domainlike) rings. We prove that R is presimplifiable ring if and only if the factor ring $R/\text{nil}(R)$ is presimplifiable and the ideal $\text{nil}(R)$ is presimplifiable. Then we investigate the associate and presimplifiable property of the dual rings $R[x]/\langle x^2 \rangle$ and its modules through the base ring R and its modules.

Key Words: Associate ring, presimplifiable ring, domainlike, pullback and dual ring.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with unity and all modules are unitary. If R is a ring, the Jacobson radical of R , the nilradical of R , the set of zero divisors of R and the set of units of R are denoted by $J(R)$, $\text{nil}(R)$, $Z(R)$ and $U(R)$, respectively. And the annihilator of a subset X of a module over R is denoted by $\text{ann}_R(X)$. Any unexplained terminology will be standard as in Hungerford [10].

Definition 1.1 *A ring R is called presimplifiable if, whenever for any $a, b \in R$ with $a=ab$, we have that $a = 0$ or $b \in U(R)$.*

One can easily verify that a ring R is presimplifiable if and only if $Z(R) \subseteq J(R)$. And every presimplifiable ring is indecomposable while the converse of this statement need not necessarily be true. Domainlike rings (i.e., $Z(R) \subseteq \text{nil}(R)$) and local rings are examples of presimplifiable rings.

Definition 1.2 *A ring R is called an associate ring if whenever any two elements a and b generates the same principal ideal of R there is a unit u such that $a = ub$.*

The class of associate rings contains a large class of rings such as presimplifiable rings, principal ideal rings, artinian rings, von Neumann regular rings and PP rings. This class of rings was originally studied by Kaplansky [12]. Then Bouvier studied presimplifiable rings in a series of papers [5]–[8]. Recently, the class of associate rings was studied more extensively by Anderson and Valdes Leon[1],[2], Spelman et al [13] and Anderson et al

[3]. Our aim of this paper is to study some properties of presimplifiable and associate rings and to investigate the associate and presimplifiable properties of the dual ring $R[x]/\langle x^2 \rangle$ and its modules.

2. Some properties of associate and presimplifiable rings

Definition 2.1 A ring R is said to be superpresimplifiable (resp., superassociate) if every subring of R is presimplifiable (resp., associate).

Remark 2.1 (Anderson et al [3]).

- (1) Domainlike is superpresimplifiable.
- (2) A subring of a presimplifiable (resp., associate) ring need not be presimplifiable (resp., associate).
- (3) A superassociate ring need not be presimplifiable.
- (4) A direct product of superassociate rings need not be superassociate.

Anderson et al in [3] gave $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ as an examples of superassociate rings. It is easy to generalize this result to a direct product of any two presimplifiable rings with the set of units $\{\pm 1\}$.

Theorem 2.1 A direct product of any two presimplifiable rings with the set of units $\{\pm 1\}$ is superassociate.

Proof.

Suppose that $R = R_1 \times R_2$ where R_1 and R_2 are two presimplifiable rings. Let S be a subring of R . Let (a, b) and (c, d) be two nonzero elements of S that generate the same principal ideal of S . Then $(a, b) = (x, y)(c, d)$ and $(c, d) = (m, n)(a, b)$ for some $(x, y), (m, n) \in S$. Then if $a \neq 0$ and $b \neq 0$ we have $(xm, yn) \in U(R)$ because the rings R_1 and R_2 are presimplifiable and $(a, b) = (xm, yn)(a, b)$. So $(x, y) \in U(R)$ and $(x, y)^2 = (1, 1)$. Hence $(x, y) \in U(S)$. And if $a = 0$ then $(y, y) \in U(S)$ and $(a, b) = (y, y)(c, d)$ because R_2 is presimplifiable ring with the set of units $\{\pm 1\}$ and $b = ynb$. If $b = 0$ then, likewise, $(x, x) \in U(S)$ with $(a, b) = (x, x)(c, d)$.

□

However, the product of two presimplifiable rings with torsion units groups need not be superassociate. For example, $R = \{(n, f(x)) \in \mathbb{Z} \times \mathbb{Z}_5[x] : f(0) \equiv n \pmod{5}\}$ is a subring of $\mathbb{Z} \times \mathbb{Z}_5[x]$. And \mathbb{Z} , $\mathbb{Z}_5[x]$ are presimplifiable rings with the torsion units groups $U(\mathbb{Z}) = \{\pm 1\}$ and $U(\mathbb{Z}_5[x]) = \{1, 2, 3, 4\}$. But R is not associate because $(0, x)$ and $(0, 2x)$ generate the same principal ideal of R and there is no unit (a, b) satisfies $(0, x) = (a, b)(0, 2x)$ (see Anderson et al [3], page 1).

We consider now the pullback.

Definition 2.2 Let R_1 , R_2 and R_3 be any three rings with homomorphisms $p_i : R_i \rightarrow R_3$, $i = 1, 2$, which preserve the unity. The subring $P = \{(r_1, r_2) \in R_1 \times R_2 : p_1(r_1) = p_2(r_2)\}$ of the ring $R_1 \times R_2$ is called the pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$ with the set of units $U(P) = \{(u_1, u_2) \in P : u_1 \in U(R_1) \text{ and } u_2 \in U(R_2)\}$.

Anderson et al in [3] determined when the pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$, where R_1 and R_2 are integral domains, is presimplifiable or associate.

Theorem 2.2 Let R_1, R_2 and R_3 be any three rings with epimorphisms $p_i : R_i \rightarrow R_3, i = 1, 2$, which are not isomorphisms. Suppose that R_1 and R_2 are integral domains. Then the pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$ is presimplifiable (resp., associate) if and only if $p_i^{-1}(1) \subseteq U(R_i), i = 1, 2$ (resp., $p_1(U(R_1)) = p_2(U(R_2))$).

In the same manner we can prove that the pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$ where R_1 and R_2 are presimplifiable rings.

Theorem 2.3 Let R_1, R_2 and R_3 be any three rings with epimorphisms $p_i : R_i \rightarrow R_3, i = 1, 2$, which are not isomorphisms. Suppose that R_1 and R_2 are presimplifiable rings. Then the pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$ is presimplifiable if and only if $p_i^{-1}(1) \subseteq U(R_i), i = 1, 2$.

Proof. Let $a \in p_1^{-1}(1)$ and $x \in Ker(p_2) - \{0\}$. Then $(a, 1), (0, x) \in P$ with $(0, x)(a, 1) = (0, x)$. Since P is presimplifiable we have $(a, 1) \in U(P)$. Thus $a \in U(P)$.

Conversely, let $(a, b) = (x, y)(a, b)$ where $(a, b), (x, y) \in P$. Assume that $(a, b) \neq (0, 0)$, say $a \neq 0$. Since $a = xa$ we have $x \in U(R_1)$. So there exists $t \in R_2$ such that $p_1(x^{-1}) = p_2(t)$. Hence $(x^{-1}, t) \in P$. Therefore, $(x, y)(x^{-1}, t) = (1, ty) \in P$ and hence $ty \in p_2^{-1}(1)$. But $p_2^{-1}(1) \subseteq U(R_2)$. So $(x, y) \in U(P)$ and hence P is presimplifiable. \square

Theorem 2.4 Let R_1, R_2 and R_3 be any three rings with epimorphisms $p_i : R_i \rightarrow R_3, i = 1, 2$, which are not isomorphisms. Suppose that R_1 and R_2 are two presimplifiable rings. Then the following statements are equivalent:

(1) The pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$ is associate; and

(2) $p_1(U(R_1)) = p_2(U(R_2))$ or $p_i(U(R_i)) \not\subseteq p_j(U(R_j))$ implies that $Ker(p_i) \subsetneq Z(R_i)$ and for any $u_i \in U(R_i)$ with $p_i(u_i) \in p_i(U(R_i)) - p_j(U(R_j))$ and $a \in Ker(p_i)$ there exists $v_i \in U(R_i)$ with $p_i(v_i) \in p_1(U(R_1)) \cap p_2(U(R_2))$ and $u_i - v_i \in ann_{R_i}(a)$.

Proof. (1) \implies (2): Suppose that $p_1(U(R_1)) \neq p_2(U(R_2))$. WLOG assume that $p_1(U(R_1)) \not\subseteq p_2(U(R_2))$. Then by (Proposition 5, [3]), $Ker(p_1) \not\subseteq Z(R_1)$ implies that P is not associate. So we must assume that $Ker(p_1) \subseteq Z(R_1)$. Now let, $a \in Ker(p_1) - \{0\}$ and $u \in U(R_1)$ with $p_1(u) \in p_1(U(R_1)) - p_2(U(R_2))$. Let $t, r \in R_2$ with $p_1(u) = p_2(r)$ and $p_1(u^{-1}) = p_2(t)$. Therefore $(u, r), (u^{-1}, t) \in P - U(P)$. Consequently, $(a, 0) = (u^{-1}, t)(au, 0)$ and $(au, 0) = (u, r)(a, 0)$, so, $(a, 0)$ and $(au, 0)$ generate the same ideal of P . Since P is associate, there exists a unit $(m, n) \in P$ with $(au, 0) = (m, n)(a, 0)$, so, $a(u - m) = 0$ and hence, $u - m \in ann_{R_1}(a) \subseteq Z(R_1)$. Also, $u - m \in R_1 - Ker(p_1)$. Otherwise, $p_1(u) = p_1(m) = p_2(n)$, a contradiction. So $Ker(p_1) \subsetneq Z(R_1)$.

(2) \implies (1): Let (a, b) and (c, d) be two nonzero elements of P with $(c, d) = (r, s)(a, b)$ and $(a, b) = (m, n)(c, d)$ for some $(m, n), (r, s) \in P$. If $a \neq 0$ and $b \neq 0$ then $(r, s), (m, n) \in U(P)$ because R_1 and R_2 are presimplifiable. So WLOG assume that $b = 0$ and $(r, s), (m, n) \in P - U(P)$. Then $r, m \in U(R_1)$ and $s, n \in R_2 - U(R_2)$. If there exists $x \in U(R_2)$ such that $p_1(m) = p_2(x)$, then $(m, x) \in U(P)$ with $(a, b) = (m, x)(c, d)$. Otherwise, $p_1(m) \in p_1(U(R_1)) - p_2(U(R_2))$. But $c \in Ker(p_1)$ because $c(1 - rm) = 0$. So there exists $v \in U(R_1)$ such that $p_1(v) \in p_1(U(R_1)) \cap p_2(U(R_2))$ and $m - v \in ann_{R_1}(c)$. Then $p_1(v) = p_2(t)$ for some $t \in U(R_2)$. Hence $(a, b) = (v, t)(c, d)$, $(v, t) \in U(P)$. Thus P is associate. \square

Example 2.1 The pullback P of $\mathbb{Z} \rightarrow \mathbb{Z}_8 \leftarrow \mathbb{Z}_{16}$ with natural maps $p_i, i = 1, 2$, is associate eventhough $p_1(U(\mathbb{Z})) = \{1, 7\} \neq \{1, 3, 5, 7\} = p_2(U(\mathbb{Z}_{16}))$ since $\text{Ker}(p_2) \subsetneq Z(\mathbb{Z}_{16})$ and $5 - 7, 3 - 1 \in \text{ann}_{R_2}(8)$.

Now, we determine when the pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$ where R_1 and R_2 are domainlike is again domainlike.

Theorem 2.5 Let R_1, R_2 and R_3 be rings with epimorphisms $p_i : R_i \rightarrow R_3, i = 1, 2$, which are not isomorphisms. Suppose that R_1 and R_2 are domainlike. Then the pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$ is domainlike if and only if $\text{Ker}(p_i) \subseteq Z(R_i), i = 1, 2$.

Proof. Let $a \in \text{Ker}(p_1) - \{0\}$. Then $(a, 0) \in Z(P)$ because $(a, 0)(0, b) = (0, 0)$ for any $b \in \text{Ker}(p_2)$. But P is domainlike so $(a, 0)^n = (0, 0)$ and hence $a^n = 0$. Thus $a \in Z(R_1)$.

Conversely, let $(a, b) \in Z(P)$. If $a \neq 0, b \neq 0$, then $a \in Z(R_1)$ and $b \in Z(R_2)$. But R_1 and R_2 are domainlike, so $a^n = 0$ and $b^m = 0$ for some $m, n \in \mathbb{Z}$. Therefore, $(a, b)^{mn} = (0, 0)$ and hence $(a, b) \in \text{nil}(P)$. Also, if $b = 0$ then $a \in \text{Ker}(p_1)$. But $\text{Ker}(p_1) \subseteq Z(R_1)$ and R_1 is domainlike. So, we have $a \in \text{nil}(R_1)$. Thus $(a, b) \in \text{nil}(P)$. \square

Example 2.2 The pullback P of $\mathbb{Z}_{p^\alpha} \rightarrow \mathbb{Z}_{p^\beta} \leftarrow \mathbb{Z}_{p^\gamma}$ where $\beta < \alpha$ and $\beta < \gamma$ and p is prime number is domainlike because \mathbb{Z}_{p^α} and \mathbb{Z}_{p^γ} are domainlike and $\text{Ker}(p_i) \subseteq Z(R_i), i = 1, 2$.

The following is a consequence of Theorem 2.6 and Theorem 2.7.

Corollary 2.1 If R_1 and R_2 are domainlike, then the pullback P of $R_1 \rightarrow R_3 \leftarrow R_2$ is associate but not domainlike if and only if $\text{Ker}(p_i) \not\subseteq Z(R_i), i = 1$ or 2 and $p_1(U(R_1)) = p_2(U(R_2))$.

Example 2.3 Let P be the pullback of $\mathbb{Z}_9 \rightarrow \mathbb{Z}_3 \leftarrow \mathbb{Z}_9[x]$ with the epimorphisms mapping $p_1 : \mathbb{Z}_9 \rightarrow \mathbb{Z}_3$ defined by $p_1(x) = \bar{x}$ and $p_2 : \mathbb{Z}_9[x] \rightarrow \mathbb{Z}_3$ defined by $p_2(f(x)) = f(0)$. Note that, $p_1(U(\mathbb{Z}_9)) = p_2(U(\mathbb{Z}_9[x])) = \{1, 2\}$ and $\text{Ker}(p_2) \not\subseteq Z(\mathbb{Z}_9[x])$ since $3 + x \in \text{Ker}(p_2) - Z(\mathbb{Z}_9[x])$. So P is associate but not domainlike. Moreover, P is not presimplifiable because $(3, 0) = (1, 1 + x)(3, 0)$ and $1 + x \in \mathbb{Z}_9[x] - U(\mathbb{Z}_9[x])$.

Next we give a new characterization of presimplifiable rings. But first we introduce the following definition.

Definition 2.3 Let R be a ring and I be an ideal of R . Then I is called a presimplifiable ideal if for every $a \in I - \{0\}$ and $b \in R$ with the property $a = ba$ implies that $b \in U(R)$.

Clearly, every ideal of a presimplifiable ring is presimplifiable.

Lemma 2.1 Let R be a ring and I be an ideal of R . Then I is a presimplifiable ideal if and only if $\text{ann}_R(a) \subseteq J(R)$ for every $a \in I - \{0\}$.

Proof. Let $a \in I - \{0\}$ and $x \in \text{ann}_R(a)$. Then $x \cdot a = 0$. Hence $a(tx + 1) = a$ for any $t \in R$. Therefore $tx + 1 \in U(R)$ and hence $x \in J(R)$.

Conversely, let $a = ba$ where $0 \neq a \in I$ and $b \in R$. Then $b - 1 \in \text{ann}_R(a)$, $\text{ann}_R(a) \subseteq J(R)$. Consequently, $b \in U(R)$. □

Theorem 2.6 *A ring R is presimplifiable if and only if the factor ring $R/\text{nil}(R)$ is presimplifiable and the ideal $\text{nil}(R)$ is presimplifiable.*

Proof. Let $\bar{a} \in Z(R/\text{nil}(R)) - \{\bar{0}\}$. Then $\bar{a}\bar{b} = \bar{0}$ for some $\bar{b} \in Z(R/\text{nil}(R)) - \{\bar{0}\}$. So there exists $m \in \mathbb{N}$ such that $a^m b^m = 0$ where $b^m \neq 0$. Therefore, $a \in Z(R)$. Since R is presimplifiable ring we have $a \in J(R)$. So $ax + 1 \in U(R)$ for every $x \in R$. Consequently, $\bar{a}\bar{x} + \bar{1} \in U(R/\text{nil}(R))$ for every $\bar{x} \in R/\text{nil}(R)$ i.e. $\bar{a} \in J(R/\text{nil}(R))$.

Conversely, let $ab = a$ where a and b are nonzeros belong to R . Then $a \in \text{nil}(R)$ or $\bar{0} \neq \bar{a} \in R/\text{nil}(R)$. Since $\text{nil}(R)$ is presimplifiable ideal and $R/\text{nil}(R)$ is presimplifiable ring, we have $b \in U(R)$. □

Anderson and Valdes Leon [3] extended the presimplifiable and associate property to modules. Recall that a module M is presimplifiable if for every $m \in M$ and $u \in R$, $m = um$ implies that $u \in U(R)$. While a module M is associate if for every $m, n \in M$ with $mR = nR$ implies that there exists $u \in U(R)$ such that $m = un$. It is easy to verify that every presimplifiable module is associate. And a module M is presimplifiable if and only if $\text{ann}_R(m) \subseteq J(R)$ for every $m \in M - \{0\}$. Recall that, a module M is torsion-free as an R -module if $\text{ann}_R(m) \subseteq Z(R)$ for every $m \in M - \{0\}$. Clearly, for any presimplifiable ring, a torsion-free R -module is presimplifiable. However the converse need not be true as the following example shows.

Theorem 2.7 *Let R be a ring and T be a maximal ideal of R with $T \not\subseteq Z(R)$. Let $M = R/T$ and $S = R - T$. Then M is presimplifiable module over a ring of fractions $S^{-1}R$ even though it is not torsion-free.*

Proof. Note that, $S^{-1}R$ is local ring so it is presimplifiable. And M is $S^{-1}R$ module under the multiplication $\frac{a}{b} \cdot \bar{m} = \overline{acm}$ where $\frac{a}{b} \in S^{-1}R$, $\bar{m} \in M$ and $\bar{bc} = \bar{1}$. Now, let $\bar{m} = \frac{a}{b} \cdot \bar{m}$ and $\frac{a}{b} \in S^{-1}R - U(S^{-1}R)$. Then $\bar{a} = \bar{0}$ since $a \in T$. Thus, $\bar{m} = \bar{0}$ and hence M is presimplifiable as an $S^{-1}R$ module. However, M is not torsion-free. To show this let $m \in T - Z(R)$. Therefore, $\frac{m}{1} \cdot \bar{1} = \bar{0}$. Consequently, $\text{ann}_{S^{-1}R}(\bar{1}) \not\subseteq Z(S^{-1}R)$. So the result holds. □

3. The ring of dual numbers

Let R be a commutative ring with unity. Then the ring $R_0 = R[x]/(x^2)$ is called the ring of dual numbers. This section is devoted to study the associate and presimplifiable properties of the dual ring R_0 and its modules via the basic ring R and its modules.

First, we need to prove the following important lemma.

Lemma 3.1 *Let $R_0 = R[x]/(x^2)$ be a ring of dual numbers. Then*

1. $U(R_0) = \{a + bx : a, b \in R \text{ and } a \in U(R)\}$
2. $J(R_0) = \{a + bx : a, b \in R \text{ and } a \in J(R)\}$
3. $Z(R_0) = \{a + bx : a, b \in R \text{ and } a \in Z(R)\}$
4. $nil(R_0) = \{a + bx : a, b \in R \text{ and } a \in nil(R)\}$.

Proof.

(1) Note that, $a + bx \in R_0$ and $a \in U(R)$ implies that $(a + bx)(a^{-1} + (-a^{-2}b)x) = 1$.

(2) $a + bx \in J(R_0) \iff 1 + (a + bx)(c + dx) \in U(R_0)$ for every $c + dx \in R_0 \iff 1 + ac \in U(R)$ for every $c \in R \iff a \in J(R)$.

(3) Let $a \in Z(R)$. Then $a = 0$ gives $(a + bx)(a + bx) = 0$ for any $b \in R$. And $a \neq 0$ implies that there exists $0 \neq c \in Z(R)$ with $(a + bx)(cx) = 0$ for every $b \in R$. So, $a + bx \in Z(R_0)$.

(4) Let $a \in nil(R)$. Then $a^n = 0$ for some $n \in \mathbb{N}$. But $(a + bx)^{n+1} = a^{n+1} + (n + 1)a^n bx$. Thus $a + bx \in nil(R_0)$. □

As a simple consequence of Lemma 3.1 we obtain the following results.

Theorem 3.1 *If $R_0 = R[x]/(x^2)$ is the ring of dual numbers then*

1. R_0 is presimplifiable if and only if R is presimplifiable.
2. R_0 is domainlike if and only if R is domainlike.

Proof. Note that, $Z(R_0) \subseteq J(R_0)$ (resp., $Z(R_0) \subseteq nil(R_0)$) if and only if $Z(R) \subseteq J(R)$ (resp., $Z(R) \subseteq nil(R)$). □

Theorem 3.2 *Let $R_0 = R[x]/(x^2)$ be the ring of dual numbers. Then R_0 is associate implies that R is associate.*

Proof. Suppose that a and b are two nonzero elements belonging to R with $aR = bR$. Then $aR_0 = bR_0$ and hence $a = (c + dx)b$ for some $c + dx \in U(R_0)$. So, $a = cb, c \in U(R)$. Therefore, R is associate. □

This raises the question: if the ring R is associate, is it true that the ring of the dual numbers R_0 is also associate? However, next we will give some examples of associate rings R for which the rings of dual numbers R_0 are also associate.

Recall that a ring R is a stable-range1 if for any $a, x, b \in R$ satisfying $ax + b = 1$, there exists $y \in R$ such that $a + by \in U(R)$. This class of rings is studied extensively in the literature, see, for example, Bass [4], Mental [11] and Chen [9]. It is easy to prove that a ring of stable range1 is associate.

Theorem 3.3 *Let $R_0 = R[x]/(x^2)$ be a ring of stable-range1. If R is stable-range1, then so is R_0 .*

Proof. Let $a + bx, c + dx$ and $m + nx \in R_0$ satisfying $(a + bx) + (m + nx)(c + dx) = 1$. Then $a + mc = 1$. Since R is a stable-range1 ring, there exist $v \in R$ and $u \in U(R)$ with $av + c = u$. So, $(a + bx)v + (c + dx) = u + (bv + d)x \in U(R_0)$. □

Theorem 3.4 *Let $\{R_\lambda : \lambda \in \Gamma\}$ be a nonempty family of rings then $(\prod R_\lambda)_0 \cong \prod (R_\lambda)_0$*

Proof. Define $\Psi : (\prod R_\lambda)_0 \rightarrow \prod (R_\lambda)_0$ by the relation

$$\Psi((a_\lambda) + (b_\lambda)x) = (a_\lambda + b_\lambda x).$$

Then it is easy to verify that Ψ is a ring homomorphism. □

Since the direct product of associate rings is associate ([13], Theorem 6) using Theorems 3.1 and 3.3, we obtain the following results.

Corollary 3.1 *Let $\{R_\lambda : \lambda \in \Gamma\}$ be a family of presimplifiable rings or stable-range 1 rings then $(\prod R_\lambda)_0$ is associate ring.*

Corollary 3.2 *For any principal ideal ring R , the ring of the dual numbers R_0 is associate.*

Proof. Note that, a principal ideal ring R is a direct product of principal ideal domains and special principal ideal rings ([14], Theorem 33, page 245), and special principal ideal rings are presimplifiable since they are local. So, R_0 is a direct product of presimplifiable rings. Hence R_0 is associate ring. □

Next, we investigate the relationship between presimplifiable (resp., associate) R_0 -module and presimplifiable (resp., associate) R -module. In the following theorem a characterization of R_0 -module is given.

Theorem 3.5 *If $R_0 = R[x]/(x^2)$ is the ring of dual numbers then M is an R_0 -module if and only if M is an R -module and there exists an R -homomorphism $\alpha : M \rightarrow M$ such that $\alpha^2 = 0$.*

Proof. Define $\alpha : M \rightarrow M$ by $\alpha(m) = x \cdot m$ then α is an R homomorphism with $\alpha^2 = 0$.

Conversely, let M be an R -module with R homomorphism $\alpha : M \rightarrow M$ with $\alpha^2 = 0$. Then M is an R_0 -module under multiplication $(a + bx) \cdot m = a \cdot m + b \cdot \alpha(m), m \in M$. □

Theorem 3.6 *Let M be an R_0 -module. Then M is presimplifiable R_0 -module if and only if M is presimplifiable R -module.*

Proof. Suppose that M is a presimplifiable R_0 module. Let $m \in M - \{0\}$ and $a \in R$ with $am = 0$ then, $a \in \text{ann}_{R_0}(m)$ and hence, $a \in J(R_0)$. Thus, $a \in J(R)$ and hence, $\text{ann}_R(m) \subseteq J(R)$.

Conversely, let $m \in M - \{0\}$ and $a + bx \in R_0$ with $(a + bx) \cdot m = 0$. Therefore, $ax \cdot m = 0$. Now, if $x \cdot m \neq 0$ then $a \in \text{ann}_R(x \cdot m) \subseteq J(R)$ and if $x \cdot m = 0$ then $a \cdot m = 0$ and hence $a \in \text{ann}_R(m) \subseteq J(R)$. So, $a + bx \in J(R_0)$. □

Theorem 3.7 *Let M be an R_0 -module. Then M is a torsion-free R -module if and only if M is a torsion-free R_0 -module.*

Proof. Suppose that M is torsion-free R -module. Let $a + bx \in R_0$ and $m \in M - \{0\}$ with $(a + bx)m = 0$. Then $a \cdot m = 0$ or $ax \cdot m = 0$. Hence $a \in \text{ann}_R(m)$ or $a \in \text{ann}_R(x \cdot m)$ and thus $a \in Z(R)$. Therefore, $a + bx \in Z(R_0)$. □

Anderson et al. [3] proved that for $R = \mathbb{Z}$, M is presimplifiable R -module if and only if M is torsion-free. So we can conclude the following.

Corollary 3.3 *Let $R = \mathbb{Z}$ and M be an R_0 module. Then M is presimplifiable if and only if M is torsion-free.*

However, if M is associate as R_0 module, it need not be the case that M is associate as R module.

Example 3.1 *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_9$ with $x \cdot \bar{1} = \bar{3}$. Then it is easy to see that M is associate R_0 module. However, M is nonassociate R -module otherwise $M = T \oplus F$ where $4T = 0$ or $6T = 0$ and F is torsion-free ([3], Theorem 12).*

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