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A fixed point theorem for a compact and connected set in Hilbert space

Hülya Duru

Abstract

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and let K be a compact and connected subset of H . We show that every continuous mapping $T : K \rightarrow K$ satisfying a mild condition has a fixed point.

Key Words: Fixed point, nonexpansive mapping, Hilbert space a fixed point theorem for a compact and connected set in Hilbert space

1. Introduction

Let K be a nonempty, close, convex and bounded subset of a real Hilbert space H . Let $T : K \rightarrow K$ be a continuous mapping.

If K is compact, then by The Schauder Fixed Point Theorem [8] (a generalization of [1]), T has a fixed point. If $K = B$ is the closed unit ball of H and the dimension of H is finite, then by The Brouwer Fixed Point Theorem [1], T has a fixed point. In the case where the dimension of H is infinite this is no longer the case [4, p.198 and 207]. In this case, it is necessary to impose some extra conditions to assure the existence of a fixed point of T . The conditions imposed are usually compactness or monotonicity or nonexpansiveness of T [6]. For instance, Browder [2], Browder [3]–Göhde [5] and Kirk [7] discovered in 1965 independently that the nonexpansiveness of T is a guarantee the existence of a fixed point of T .

In the present paper, we impose an extra condition on T to obtain the same result. The extra condition imposed is this: For a certain number $r > 0$, the inclusion $T(\partial B_r) \subseteq B_r$ holds. Here, B_r denotes the closed ball, $B_r = \{x \in H : \|x\| \leq r\}$ and ∂B_r is the boundary of the ball B_r .

Moreover we impose some new conditions for The Schauder Fixed Point Theorem [8], and for Theorem 1 in [2]. In addition, we obtain some results related to the imposed conditions. To explain these conditions let us define a subset $A^T(x_0)$ of K , for $x_0 \in K$, as

$$A^T(x_0) = \{x \in K : \|x - x_0\| \leq \|T(x) - x_0\|\}.$$

The main theorem of this paper obtains The Schauder Fixed Point Theorem, which states that every continuous mapping on a compact and convex subset of a Banach space has a fixed point, for compact and

connected subset K of H . To get this we replace the convexity of K in this theorem with the condition $K = A^T(x_0)$ for some $x_0 \neq 0$. To obtain Theorem 1 in [2] for continuous mapping T , we replace the nonexpansiveness of T with the following conditions:

(a) The existence of a point a in K satisfying $\|a\| = \|T(a)\|$;

and

(b) $((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset$,

where f is the mapping defined by, $f(x) = (x + T(x)) / 2$, $r = \sup\{\|x\| : x \in K\}$ and $P : B_r \rightarrow K$ is the nearest point projection, that is, for $x \in B_r$,

$$\|x - P(x)\| = \inf\{\|x - u\| : u \in K\}.$$

We remark that the mapping f also applies K into itself.

Proposition 1 *Let K be a nonempty and convex subset of H . Let $T : K \rightarrow K$ be a continuous mapping. Suppose that there exists a point $a \in K$ such that*

$$\|a\| = \|T(a)\| = \|f(a)\|.$$

Then a is a fixed point of T .

Proof. Let $\|a\| = \|T(a)\| = \|f(a)\|$. The equality $\|a\| = \|f(a)\|$ is equivalent to

$$\langle a, a \rangle = \frac{1}{4} \langle a + T(a), a + T(a) \rangle.$$

Developing this product, and taking into account the condition $\|T(a)\| = \|a\|$, we obtain the equality

$$\|a\|^2 = \langle a, T(a) \rangle.$$

This equality in turn implies that $\|a - T(a)\|^2 = \langle a - T(a), a - T(a) \rangle = 0$ so that $T(a) = a$. □

The following corollary is now obvious. Since the inequality $\|a\| \leq \|f(a)\|$ in turn implies that $\|a\| \leq \|T(a)\|$.

Corollary 2 *Let K be a nonempty and convex subset of H and let $T : K \rightarrow K$ be a continuous mapping. If there exists a point $a \in K$ such that $\|T(a)\| \leq \|a\| \leq \|f(a)\|$, then a is a fixed point of T .*

Proposition 3 *Let K be a nonempty and convex subset of H and let $T : K \rightarrow K$ be a continuous mapping. If there exists a point $x_0 \in K$ such that both sets $A^T(x_0)$ and $A^f(x_0)$ are at most countable then x_0 is a fixed point of the mapping T .*

Proof. Remark that $A^f(x_0) = \{x \in K : \|x - x_0\| \leq \| \|f(x)\| - \|x_0\| \}$. Let both $A^T(x_0)$ and $A^f(x_0)$ be at most countable. We shall show that $\|x_0\| = \|T(x_0)\|$ and $\|x_0\| = \|f(x_0)\|$. Let show that $\|x_0\| = \|T(x_0)\|$. On the contrary, assume that $\|x_0\| \neq \|T(x_0)\|$. Let $\varepsilon = (\|x_0\| - \|T(x_0)\|) / 2$. Since T is continuous, there exists $\delta > 0$ such that if $x \in K$ and $\|x - x_0\| < \delta$ then $|\|T(x)\| - \|T(x_0)\|| < \varepsilon$. Let $\delta_0 = \min\{\varepsilon, \delta\}$. We claim that $B_{\delta_0}(x_0) \cap K \subset A^T(x_0)$. Indeed, let $x \in K$ and $\|x - x_0\| < \delta_0$. Then ,

$$\|x - x_0\| < 2\varepsilon - \varepsilon \leq \|x_0\| - \|T(x_0)\| - \|T(x) - T(x_0)\|.$$

From here we conclude that

$$\|x - x_0\| \leq \|T(x) - T(x_0)\|.$$

That means $x \in A^T(x_0)$. This implies that $B_{\delta_0}(x_0) \cap K \subset A^T(x_0)$. But this is impossible, since K is convex and $A^T(x_0)$ is at most countable. To prove that $\|x_0\| = \|f(x_0)\|$, it is enough to replace T with f and repeat the proof. By Proposition 1, x_0 is a fixed point of T . □

Next is the main theorem of this paper.

Theorem 4 *Let K be a nonempty, compact and connected subset of H and let $T : K \rightarrow K$ be a continuous mapping. Assume that there exists a point $x_0 \in K$, $x_0 \neq 0$, such that $A^T(x_0) = K$. Then T has a fixed point.*

Proof. Let $a \in K$ be fixed. Now we define the sequences $\alpha_n = \|T^n(a) - x_0\|$ and $\beta_n = \|T^n(a)\| - \|x_0\|$. Since $A^T(x_0) = K$ for some $x_0 \in K$, $x_0 \neq 0$, the sequence $(T^n(a))_{n \in \mathbb{N}}$ is in the set $A^T(x_0)$. Hence we have

$$\begin{aligned} \|a - x_0\| &\leq \|T(a) - x_0\| \leq \|T(a) - x_0\| \\ \|T(a) - x_0\| &\leq \|T^2(a) - x_0\| \leq \|T^2(a) - x_0\|, \end{aligned}$$

and so on. In this way we get,

$$\|T^n(a) - x_0\| \leq \|T^{n+1}(a) - x_0\| \leq \|T^{n+1}(a) - x_0\|,$$

for all $n = 0, 1, 2, \dots$ (Here $T^n = T \circ T \circ \dots \circ T$ n -times and $a = T^0(a)$)

Hence, for all n , we get

$$\alpha_n \leq \beta_{n+1} \leq \alpha_{n+1}.$$

The last inequalities show that $(\alpha_n)_{n \in \mathbb{N}}$ and $(\beta_n)_{n \in \mathbb{N}}$ are increasing sequences. Moreover, since K is bounded, they converge and approach the same limit. On the other hand, since K is compact, the sequence $(T^n(a))_{n \in \mathbb{N}}$ has a convergent subsequence. We show this subsequence $(T^{n_k}(a))_{k \in \mathbb{N}}$. Let $\lim_{n \rightarrow \infty} T^{n_k}(a) = b \in K$. Since T is continuous, we get

$$\lim_{n \rightarrow \infty} \|T^{n_k+p}(a)\| = \|T^p(b)\|,$$

for all $p = 0, 1, 2, \dots$ Now $\alpha_{n_k+p} = \|T^{n_k+p}(a) - x_0\|$ and $\beta_{n_k+p} = \|T^{n_k+p}(a)\| - \|x_0\|$ are all subsequences of $(\alpha_n)_{n \in \mathbb{N}}$ and $(\beta_n)_{n \in \mathbb{N}}$. Since they have the same limit, we get

$$\|b - x_0\| = \|b\| - \|x_0\| = \|T(b) - x_0\| = \|T(b)\| - \|x_0\| = \dots \tag{1}$$

That is,

$$\|T^n(b) - x_0\| = \|T^n(b)\| - \|x_0\| = \|T^{n+1}(b) - x_0\|.$$

From here we conclude that, for all $n \in \mathbb{N}$,

$$\langle x_0, T^n(b) \rangle = \|x_0\| \|T^n(b)\|.$$

The last equality implies that there exist positive real numbers t_n such that all equalities below,

$$x_0 = t_0 b = t_1 T(b) = t_2 T^2(b) = \dots = t_n T^n(b) = \dots \tag{2}$$

hold. Now there exist two cases:

Case 1. Assume that the equality $\|T^n(b)\| = \|T^{n+1}(b)\|$ holds for some n . Then the point $T^n(b)$ is a fixed point of T . Indeed by (2),

$$\|x_0\| = t_n \|T^n(b)\| = t_{n+1} \|T^{n+1}(b)\|.$$

Hence $t_n = t_{n+1}$. Again by (2), we get $T^n(b) = T^{n+1}(b) = T(T^n(b))$.

Case 2. Suppose that $\|T^n(b)\| \neq \|T^{n+1}(b)\|$ for all n . In this case x_0 is a fixed point of T . Indeed, by taking square of the equalities in (1), we obtain,

$$\|x_0\|^2 = (\|b\|^2 + \|T(b)\|^2)/2 = (\|T(b)\|^2 + \|T^2(b)\|^2)/2 = \dots \tag{3}$$

From here, we get for all n

$$\|b\| = \|T^2(b)\| = \|T^{2n}(b)\| \text{ and } \|T(b)\| = \|T^3(b)\| = \|T^{2n+1}(b)\|. \tag{4}$$

Together with (2), the equalities in (4) imply that both

$$\|x_0\| = t_0 \|b\| = t_2 \|T^2(b)\| = t_{2n} \|T^{2n}(b)\| \tag{5}$$

and

$$\|x_0\| = t_1 \|T(b)\| = t_3 \|T^3(b)\| = t_{2n+1} \|T^{2n+1}(b)\|. \tag{6}$$

Hence relations (4), (5) and (6) give us for all n

$$t_0 = t_2 = t_{2n} \text{ and } t_1 = t_3 = t_{2n+1}.$$

By (2), $b = T^2(b) = T^{2n}(b)$. Let $g : K \rightarrow R$ be a function defined by

$$g(x) = |\|b\| - \|T^2(x)\|| - |\|T(b)\| - \|T^2(x)\||.$$

Clearly g is continuous and $g(b) < 0$ and $0 < g(T(b))$ hold. Since K is connected, by The Intermediate Value Theorem, there exists a point c in K such that

$$g(c) = |\|b\| - \|T^2(c)\|| - |\|T(b)\| - \|T^2(c)\|| = 0.$$

That is, $\|T^2(c)\| = (\|b\| + \|T(b)\|) / 2$. By (3), $\|x_0\| = \|T^2(c)\|$. On the other hand since $T(c) \in A^T(x_0) = K$, we get $\|x_0 - T(c)\| \leq |\|x_0\| - \|T^2(c)\|| = 0$. Hence $x = T(c)$. Similarly since $c \in A^T(x_0)$, we get $\|x_0 - c\| \leq |\|x_0\| - \|T(c)\|| = 0$. Hence $x_0 = c$. From here we conclude that x_0 is a fixed point of T . \square

We use the symbol \overline{A} to denote the closure of a set A .

Corollary 5 *Let K and T be as in Theorem 4. If T has no fixed point then*

$$A^T(x) \cap \overline{K/A^T(x)} \neq \emptyset$$

for all $x \in K$, $x \neq 0$. That is, for all $x \in K$, there exists a point y in K such that

$$\|x - y\| = \left| \|x\| - \|T(y)\| \right|.$$

Proof. On the contrary, suppose that $A^T(x) \cap \overline{K/A^T(x)} = \emptyset$ for some $x \in K$. Then the set $A^T(x)$ is both open and closed in K . Since K is connected and $x \in A^T(x) \neq \emptyset$ we must have $A^T(x) = K$. By Theorem 4, T has a fixed point, which is not the case. \square

Lemma 6 *Let K be a nonempty, closed, convex and bounded subset of H and $0 \in K$. Let $\dot{T} : K \rightarrow K$ be a continuous mapping. Assume that there exists at least one point $a \in K$ such that the equality $\|T(a)\| = \|a\|$ holds. Then,*

- (a) *The set $F = \{x \in K : \|x\| = \|f(x)\|\} = A^f(0) \cap \overline{K/A^f(0)}$ is nonempty.*
- (b) *The quantity $\delta(f) = \inf\{\|x\| : x \in F\}$ is zero iff $T(0) = 0$.*

Proof. (a) If $f(0) = 0$, then there is nothing to prove. If

$$a \in A^f(0) = F \cup \{x \in K : \|x\| < \|f(x)\|\}$$

then, $\|a\| \leq \|f(a)\|$. By Corollary 2, a is a fixed point of both T and f . Hence $a \in F \neq \emptyset$. Hence we suppose that $0 < \|f(0)\|$ and $a \notin A^f(0)$. Let $g : K \rightarrow R$ be a continuous function defined by, $g(x) = \|x\| - \|f(x)\|$. Since $g(0) < 0$ and $g(a) > 0$ and since K is connected, by the intermediate value theorem, there is a point $b \in K$ such that $g(b) = 0$. Hence $b \in F \neq \emptyset$.

b) Since $F \neq \emptyset$, the quantity $\delta(f) = \inf\{\|x\| : x \in F\}$ exists. This quantity is zero iff $T(0) = 0$. Indeed, if $T(0) = 0$ then $0 \in F$ so that $\delta(f) = 0$. Conversely, if $\delta(f) = 0$ then there is a sequence $(x_n)_{n \in N}$ in F such that $\|x_n\| \rightarrow 0$, as $n \rightarrow \infty$. Since f is continuous on K and since $\|x_n\| = \|f(x_n)\|$, we see that $f(0) = 0$. This implies that $T(0) = 0$, too \square

Let K be a nonempty, closed, convex and bounded subset of H . In this case the quantity $\sup\{\|x\| : x \in K\} = r$ exists and $K \subset B_r$. Let $P : B_r \rightarrow K$ be the nearest point projection, that is, for $x \in B_r$, $\|x - P(x)\| = \inf\{\|x - u\| : u \in K\}$. Now we give the next corollary below.

Corollary 7 *Let K , T and a be as in lemma 6. Suppose that*

$$((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset.$$

Then T has a fixed point.

Proof. Let $((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset$. In this case, there is a $x \in \partial B_r$ such that $\|((f \circ P)(x))\| = \|x\| = r$. The equality $\|((f \circ P)(x))\| = r$ implies that both

$$\|P(x)\| \leq \|((f \circ P)(x))\| \text{ and } \|T(P(x))\| \leq \|((f \circ P)(x))\|.$$

As $(f \circ P)(x) = (P(x) + T(P(x))) / 2$, we have $\|P(x)\| = \|T(P(x))\| = \|((f \circ P)(x))\|$. By Proposition 1, $P(x)$ is a fixed point of \dot{T} . □

In the previous corollary the nearest point projection can be replaced with the radial retraction, which uniquely defined for a ball in any strictly convex normed space.

Theorem 8 *Let $K = B$ be the closed unit ball of H and let $T : K \rightarrow K$ be a continuous mapping. Suppose that for each $r \geq \delta(f)$, the inclusion $T(\partial B_r) \subseteq B_r$ holds. Then T has a fixed point in B .*

Proof. If we take $a \in K$ with $\|a\| = 1$ and repeat the proof of Lemma 6(a), we see that the set $F \neq \emptyset$. By Lemma 6(b), If $\delta(f) = 0$ then zero is a fixed point of T so that there is nothing to prove in this case. Hence we suppose that $\delta(f) > 0$.

Case 1. $f(\partial B) \cap \partial B \neq \emptyset$. In this case, there is a point $x \in \partial B$ such that $\|f(x)\| = 1 = \|x\|$. As $f(x) = (x + T(x)) / 2$, the equality $\|f(x)\| = \|x\|$ implies that $\|T(x)\| \geq \|x\|$. Since $\|x\| = 1$, this is possible only if $\|T(x)\| = \|x\|$. By Proposition 1, $T(x) = x$.

Case 2. $f(\partial B) \cap \partial B = \emptyset$. In this case, for all $x \in \partial B$, $\|f(x)\| < 1$ so that $\|f(x)\| < \|x\|$. We fix a $y \in \partial B$. Since $\|f(y)\| < \|y\|$ and $\|f(0)\| > 0$, as in the proof of Lemma 6(a), the function $g(x) = \|x\| - \|f(x)\|$ vanishes at some point $a \in B$. That is, $\|a\| = \|f(a)\|$. This point a belongs to F so that $\|a\| \geq \delta(f)$. As $a \in \partial B_r$, where $r = \|a\|$ and since $T(\partial B_r) \subseteq B_r$, we have $\|T(a)\| \leq \|a\|$. On the other hand, since $\|a\| = \|f(a)\|$, we also have $\|T(a)\| \geq \|a\|$. Hence $\|T(a)\| = \|a\|$. By Proposition 1, $T(a) = a$. Hence T has a fixed point in B . □

The next corollaries are now obvious.

Corollary 9 *Suppose that for each $x \in B$ with $\|x\| \geq \delta(f)$, we have $\|T(x)\| \leq \|x\|$. Then T has a fixed point in B .* □

Corollary 10 *Let K be a closed, convex and bounded subset of H with $0 \in K$, and let $T : K \rightarrow K$ be a continuous mapping. Suppose that the inclusion $T(\partial C) \subseteq C$ holds for all convex and closed subsets of K . Then T has a fixed point in K .*

Theorem 11 *Let H be infinite dimensional and let $K = B$ be the closed unit ball of H . Let $T : K \rightarrow K$ be a continuous mapping. Set*

$$M_i = \{x \in B : x = x_i e_i\},$$

where $e_i = (0, 0, \dots, y_i, 0, \dots)$, $y_i = 1$. Then $F \cap M_i \neq \emptyset$, for all i .

Proof. Let $H = \ell_2$. We give the proofs without loss of generality for $M = M_1$.

(a) If $f(0) = 0$ then $0 \in F \cap M$. If $a = (x_1, 0, 0, \dots) \in F$ where $|x_1| = 1$ then $a \in F \cap M$. So we suppose that $\{0, e_1, -e_1\} \cap F = \emptyset$. Now define the function $g : M \rightarrow \mathbb{R}$, $g(x) = \|x\| - \|f(x)\|$. Then since $g(0) < 0$ and $g(e_1) > 0$ and since M is convex, by the intermediate value theorem, there is a point $b \in M$ such that

$g(b) = 0$. That is $b \in F \cap M \neq \emptyset$. □

For the next corollary we put $\sup\{\|x\| : x \in F \cap M_i\} = \|x_i\| = r_i$ for some $x_i \in F \cap M_i$. If $r_i = 1$ for some i , then $\|x_i\| = \|f(x_i)\| = 1 \geq \|T(x_i)\|$. By Proposition 1, x_i is a fixed point of T .

Corollary 12 *Let B be the closed unit ball of H and $T : B \rightarrow B$ be a continuous mapping. If the inclusion $T(\partial B_{r_i}) \subseteq B_{r_i}$ for some i , then T has a fixed point .*

Proof. We remark that $r_i = \|x_i\|$ for some $x_i \in F \cap M_i$, for all i . If $r_i = 1$ for some i , then

$$\|x_i\| = \|f(x_i)\| = 1 \geq \|T(x_i)\| .$$

By Proposition 1, x_i is a fixed point of T . Let $0 < r_i < 1$ for all i . Then since $T(\partial B_{r_i}) \subseteq B_{r_i}$ and $x_i \in \partial B_{r_i}$, $\|T(x_i)\| \leq r_i = \|x_i\| = \|f(x_i)\|$. By Proposition 1, x_i is a fixed point of T . □

Example 13 *Let $H = \ell_2$ and B its closed unit ball.*

1- For $x = (x_1, x_2, ..) \in B$, let $T(x) = (1 - \|x\|, \|x\|, x_3, x_4, ...)$. Then clearly T takes B into itself. By a simple calculation, we have $r_1 = 1/\sqrt{3}$, $r_2 = 1$ and $r_i = \sqrt{2} - 1$, for all $i = 3, 4, ...$. It is clear that $T(\partial B_{r_2}) \subseteq B_{r_2}$. By Corollary 12, T has a fixed point. Moreover, T is not a nonexpansive mapping.

2- For $x = (x_1, x_2, ..) \in B$, let $T(x) = (1 - \|x\|, x_2, x_3, x_4, ...)$. Then we have, $r_1 = 1/2$ and $r_i = 1$ for all $i \geq 2$. It is clear that $T(\partial B_{r_1}) \subseteq B_{r_1}$. By Corollary 12, T has a fixed point.

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