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Coverings of Lie groupoids

İlhan İçen, M. Habil Gürsoy and A. Fatih Özcan

Abstract

In this work we constitute the category of coverings of the Lie fundamental groupoid associated with a connected smooth manifold. We show that this category is equivalent to the category of universal coverings of a connected smooth manifold. In addition, we prove the equivalence of the category of coverings of a Lie groupoid and the category of actions of this Lie groupoid on a connected smooth manifold. Also we present two side results related to actions of Lie groupoids on the manifolds and coverings of Lie groupoids.

Key Words: Lie groupoid, covering, action, lifting.

1. Introduction

The theory of covering space is one of the most important theories in algebraic topology. By studying categories and groupoids, the concept of covering is meaningful by investigation of relationships between fundamental groupoids of covering spaces and those of the base spaces. These relations are studied by Brown and Higgins in [1, 2, 8].

Brown defined fundamental groupoid $\pi_1 X$ for given a topological space X . Thus he defined the covering morphism $\pi_1 p : \pi_1 \tilde{X} \rightarrow \pi_1 X$ of groupoids for a covering map $p : \tilde{X} \rightarrow X$ of topological spaces. Later, he showed that the equivalence of the category $TCov(X)$ of coverings of X and the category $GdCov(\pi_1 X)$ of coverings of fundamental groupoid $\pi_1 X$, where X has universal covering space.

In this area, another algebraic study was studied by Gabriel and Zisman [5]. They showed the equivalence of the category $GdCov(G)$ of coverings of a groupoid G and the category $GdOp(G)$ of actions on the sets of G . The topological version of this paper is studied in [3].

For the smooth case, let M be a connected smooth manifold and let $p : \tilde{M} \rightarrow M$ be a topological covering map. Then \tilde{M} is a topological manifold, and it has a unique smooth structure such that p is a smooth covering map [9].

In this study, we proved that the equivalence of the category $SCov(M)$ of the coverings of a connected smooth manifold M and the category $LGdCov(\pi_1 M)$ of coverings of the fundamental groupoid $\pi_1 M$ associated to the connected smooth manifold M . Furthermore we proved the smooth version of the algebraic study in [5].

That is, we showed that the equivalence of the category $LGdCov(G)$ of coverings of a Lie groupoid G and the category $LGdOp(G)$ of actions of Lie groupoid G on a connected smooth manifold M .

Also we have been studying on the coverings of the structured Lie groupoids such as Lie group-groupoids and Lie ring-groupoids, and their actions on Lie groups M and Lie rings R , respectively [6, 7].

We consider C^r -manifolds for $r \geq -1$. Here, a C^{-1} -manifold is simply a topological space and for $r = -1$, a smooth map is simply a continuous map. This allow us to state results which include the topological case. Namely, the Lie groupoids in the C^{-1} case will simply be the topological groupoids. For $r = 0$, a C^0 -manifold is, as usual, a topological manifold, and a smooth map is just a continuous map. For $r \geq -1$, $r = \infty, \omega$ the definition of C^r -manifold and smooth map are as usual. We now fix $r \geq -1$.

One of the key differences between cases $r = -1$ or 0 and $r \geq -1$ is that, for $r \geq 1$, the pullback of C^r -maps need not be a smooth manifold of the product, and so differentiability of maps on the pullback cannot always be defined.

Throughout the paper, all manifolds that we consider are assumed to be smooth and second countable.

2. Lie groupoids

In this section, we shall recall the some basic concepts associated with Lie groupoids.

A groupoid is a category in which every arrow is invertible. More precisely, a groupoid consists of two sets G and G_0 called the set of morphisms or arrows and the set of objects of groupoid respectively, together with two maps $\alpha, \beta : G \rightarrow G_0$ called source and target maps respectively, a map $1_{\circ} : G_0 \rightarrow G$, $x \mapsto 1_x$ called the object map, an inverse map $i : G \rightarrow G$, $a \mapsto a^{-1}$ and a composition $G_2 = G \times_{\alpha \times \beta} G \rightarrow G$, $(b, a) \mapsto b \circ a$ defined on the pullback

$$G \times_{\alpha \times \beta} G = \{(b, a) \mid \alpha(b) = \beta(a)\}.$$

These maps should satisfy the following conditions.

1. $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$, for all $(b, a) \in G_2$;
2. $c \circ (b \circ a) = (c \circ b) \circ a$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$, for all $a, b, c \in G$;
3. $\alpha(1_x) = \beta(1_x) = x$, for all $x \in G_0$;
4. $a \circ 1_{\alpha(a)} = a$ and $1_{\beta(a)} \circ a = a$, for all $a \in G$ and
5. $\alpha(a^{-1}) = \beta(a)$ and $\beta(a^{-1}) = \alpha(a)$, $a^{-1} \circ a = 1_{\alpha(a)}$ and $a \circ a^{-1} = 1_{\beta(a)}$ [1, 10].

Let G be a groupoid. For all $x, y \in G_0$ we denote the set of all morphisms $a \in G$ such that $\alpha(a) = x$ and $\beta(a) = y$ by $G(x, y)$. For $x \in G_0$, we write $St_G x$ (or G_x) for the set of all morphisms started at x , and $CoSt_G x$ (or G^x) for the set of all morphisms ended at x . The *object* or *vertex group* at x is $G(x) = \{a \in G \mid \alpha(a) = \beta(a) = x\}$. We say that a groupoid G is transitive if $G(x, y) \neq \emptyset$ and is 1-transitive if $G(x, y)$ has only one element for all $x, y \in G_0$ [1].

During this work we shall assume that the manifold of objects G_0 and α -fibers $\alpha^{-1}(x) = St_G x$, $x \in G_0$, are Hausdorff.

Let G and H be two groupoids. A groupoid morphism from H to G is a pair (f, f_0) of maps $f : H \rightarrow G$ and $f_0 : H_0 \rightarrow G_0$ such that $\alpha_G \circ f = f_0 \circ \alpha_H$, $\beta_G \circ f = f_0 \circ \beta_H$ and $f(b \circ a) = f(b) \circ f(a)$ for all $(b, a) \in H_2$ [1].

Now we can give the definition of Lie groupoid.

Definition 1 *A groupoid G over G_0 is called Lie groupoid if G and G_0 are manifolds, α and β are surjective submersions and the composition map is smooth [10, 11].*

It follows that, 1_G is an immersion, the inverse map is a diffeomorphism, the sets $St_G x$, $CoSt_G x$ and $G(x, y)$ are closed submanifolds of G for all $x, y \in G_0$ and all vertex groups are Lie groups. Also, since α and β are submersions, G_2 is a closed submanifold of $G \times G$ [10].

Let $a \in G(x, y)$. Then the left-translation (right-translation) corresponding to a is the map $L_a : CoSt_G x \rightarrow CoSt_G y$, $b \mapsto a \circ b$ ($R_a : St_G y \rightarrow St_G x$, $b \mapsto b \circ a$). These maps are diffeomorphisms [10, 11].

Definition 2 *A morphism between Lie groupoids H and G is a groupoid morphism (f, f_0) such that f and f_0 are smooth [10, 11].*

Example 1 *Let M be a manifold. The product manifold $M \times M$ is a Lie groupoid over M in the following way: α is the second projection and β is the first projection; $1_x = (x, x)$ for all $x \in M$ and $(x, y) \circ (y, z) = (x, z)$ [11].*

Example 2 *If G is a Lie group acting as smooth on manifold M , then we can define action Lie groupoid $G \times M$ with $G \times M$ as manifold of arrows and M as manifold of objects. The source map is the second projection and the target map is given by left action. The composition is defined by*

$$(g_1, m_1) \circ (g_2, m_2) = (g_1 g_2, m_1), \quad \forall g_1 g_2 \in G, \quad \forall m_1, m_2 \in M \quad [11].$$

Example 3 *Let M be a connected manifold and for $x, y \in M$ let*

$$\pi M = \{(x, [a], y) \mid [a] \text{ is the homotopy class of paths } \ni a(0) = x, a(1) = y\}.$$

Then πM is a Lie groupoid over M as follows:

$$\begin{aligned} \alpha(x, [a], y) &= x, \quad \beta(x, [a], y) = y, \quad 1_x = (x, [c]_x, x), \quad (x, [a], y)^{-1} = (y, [a^{-1}], x) \\ m((x, [a], y), (y', [b], z)) &= (x, [a \circ b], z) \Leftrightarrow y = y'. \end{aligned}$$

If πM equipped with the quotient topology of the compact-open topology on the space of paths of M , then $\alpha \times \beta : \pi M \rightarrow M \times M$ is a covering map. It follows that πM is a Lie groupoid over M , and it is called the fundamental groupoid associated to M . Its isotropy groups are the fundamental groups $\pi(M, x)$ for all $x \in M$ [11].

3. Coverings and actions of Lie groupoids

In this section, we will present some results related to coverings and actions of Lie groupoids. It will be appropriate to give them in two parts.

3.1. Coverings of Lie groupoids

We recall the notions of a smooth covering map and covering manifold of a manifold.

Definition 3 *If \widetilde{M} and M are connected manifolds, a smooth covering map $p : \widetilde{M} \rightarrow M$ is a smooth surjective map with the property that every $m \in M$ has a connected neighborhood U such that each component of $p^{-1}(U)$ is mapped smoothly onto U by p . In this context we will also say that U is *canonic*. The manifold M is called the *base of the covering*, and \widetilde{M} is called a *covering manifold of M* .*

Now, let us define the coverings of Lie groupoids.

Definition 4 *Let $p : \widetilde{G} \rightarrow G$ be a morphism of Lie groupoids. For each $\tilde{x} \in \widetilde{G}_0$, if the restriction $\widetilde{G}_{\tilde{x}} \rightarrow G_{p(\tilde{x})}$ of p is a diffeomorphism, p is called the *covering morphism of Lie groupoids* and Lie groupoid \widetilde{G} is called the *covering of Lie groupoid G* .*

Let us give an equivalent criterion to the covering of Lie groupoids.

Let $p : H \rightarrow G$ be a covering morphism of Lie groupoids. Take the pullback

$$G \times_{\alpha \times p_0} H_0 = \{(a, x) \in G \times H_0 \mid \alpha(a) = p_0(x)\}.$$

Since α is a submersion, $G \times_{\alpha \times p_0} H_0$ is a manifold. Then the map $s_p : G \times_{\alpha \times p_0} H_0 \rightarrow H$ is the lifting function assigning to the unique element $h \in H_x$, the pair (a, x) such that $p(h) = a$. It is clear that s_p is inverse of the map $(p, \alpha) : H \rightarrow G \times_{\alpha \times p_0} H_0$.

Thus the morphism $p : H \rightarrow G$ is covering morphism of Lie groupoids iff the morphism (p, α) is a diffeomorphism.

Definition 5 *The Lie subgroup $p(\widetilde{G}\{\tilde{x}\})$ of the Lie group $G\{p(\tilde{x})\}$ is called the *characteristic group of p at \tilde{x}* for any Lie groupoid morphism $p : \widetilde{G} \rightarrow G$ and $\tilde{x} \in \widetilde{G}_0$.*

Now, let us state an important lemma giving the passing from the smooth covering maps to the covering morphisms of Lie groupoids.

Lemma 1 *If $p : \widetilde{M} \rightarrow M$ is a smooth covering map of connected manifolds then the morphism $\pi_1 p : \pi_1 \widetilde{M} \rightarrow \pi_1 M$ of Lie groupoids is a covering morphism.*

Proof. Since the M and \widetilde{M} are manifolds, the fundamental groupoids $\pi_1 M$ and $\pi_1 \widetilde{M}$ are Lie groupoids by Example 3. Also from [1], the $\pi_1 p : \pi_1 \widetilde{M} \rightarrow \pi_1 M$ is a covering morphism of groupoids. For this reason, it is enough to prove that $\pi_1 p$ is smooth and the lifting function $s_{\pi_1 p} : \pi_1 M \times_{\alpha \times \pi_1 p} \pi_1 \widetilde{M} \rightarrow \pi_1 \widetilde{M}$ that is

inverse of the map $(\pi_1 p, \alpha) : \pi_1 \widetilde{M} \rightarrow \pi_1 M \times_{\pi_1 p} \widetilde{M}$, is a diffeomorphism. Firstly let us show that $\pi_1 p$ is smooth. Since $p : \widetilde{M} \rightarrow M$ is smooth, we have $p(U) \subset V$ for any charts (U, φ) on \widetilde{M} and (V, ψ) on M , and $\psi \circ p \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$ is a smooth map. Since the fundamental groupoids $\pi_1 \widetilde{M}$ and $\pi_1 M$ are Lie groupoids, we can define the groupoid morphism $\pi_1 p$ as $\widehat{\psi}^{-1} \circ id \circ \widehat{\varphi}$ by lifted charts $(\widehat{U}, \widehat{\varphi})$ and $(\widehat{V}, \widehat{\psi})$ that are liftings of the charts (U, φ) and (V, ψ) , respectively. $\widehat{\psi}$ and $\widehat{\varphi}$ are smooth, because they are the chart maps of Lie groupoids $\pi_1(M)$ and $\pi_1(\widetilde{M})$, respectively. Clearly, the identity map id is smooth. Thus $\pi_1 p$ is smooth.

Now let us show that $s_{\pi_1 p}$ is a diffeomorphism. By the fact that $\pi_1 p$ is the morphism of Lie groupoids and α is the source map of the Lie groupoid, the map $(\pi_1 p, \alpha) : \pi_1 \widetilde{M} \rightarrow \pi_1 M \times_{\pi_1 p} \widetilde{M}$ is clearly smooth. Furthermore, $(\pi_1 p, \alpha)$ is a bijection, because $\pi_1 p$ is the covering morphism of groupoids. So there exists an inverse $s_{\pi_1 p} : \pi_1 M \times_{\pi_1 p} \widetilde{M} \rightarrow \pi_1 \widetilde{M}$ of $(\pi_1 p, \alpha)$. The map $s_{\pi_1 p}$ is the lifting function the each pair (a, x) assigning to the unique homotopy class $[b]_{\widetilde{x}}$ of smooth paths b such that $\pi_1 p([b]) = [a]$. The homotopy lifting property and unique lifting property imply that $s_{\pi_1 p}$ is well-defined. $s_{\pi_1 p}$ can write as the composition of smooth maps in the following diagram:

$$\begin{array}{ccccccc} \pi_1 M \times_p \widetilde{M} & \xrightarrow{I \times \epsilon} & \pi_1 M \times \pi_1 \widetilde{M} & \xrightarrow{I \times L_{\widetilde{a}}} & \pi_1 M \times \pi_1 \widetilde{M} & \xrightarrow{pr_2} & \pi_1 \widetilde{M} \\ & & & & & & \\ ([a], \widetilde{x}) & \longrightarrow & ([a], [1_{\widetilde{x}}]) & \longrightarrow & ([a], [\widetilde{a}]) & \longrightarrow & [\widetilde{a}] \end{array}$$

From here, $s_{\pi_1 p}$ is a diffeomorphism. So $\pi_1 p$ is a covering morphism of Lie groupoids. □

Lemma 2 *Let M be a connected manifold and let $q : \widetilde{G} \rightarrow \pi_1 M$ be covering morphism of groupoids. Let $\widetilde{M} = \widetilde{G}_0$ and $p = q_0 : \widetilde{M} \rightarrow M$. Let \mathcal{A} denotes an atlas consisting of the liftable charts such that M is smooth. Then the smooth structure over \widetilde{M} is the unique structure such that the followings are hold:*

1. $p : \widetilde{M} \rightarrow M$ is the covering map.
2. There exists an isomorphism $r : \widetilde{G} \rightarrow \pi_1 \widetilde{M}$ which is identical on the objects such that the following diagram is commutative:

$$\begin{array}{ccc} & \pi_1 \widetilde{M} & \\ & \nearrow r & \downarrow \pi_1 p \\ \widetilde{G} & \xrightarrow{q} & \pi_1 M \end{array}$$

Proof. Firstly, let us show that p is a covering map, where \widetilde{M} has the lifted manifold. Let $\widetilde{\mathcal{A}}$ be a collection of the liftings of the elements of \mathcal{A} . So $\widetilde{\mathcal{A}}$ forms an atlas for the charts of the manifold on \widetilde{M} . For any $U \in \mathcal{A}$, $p^{-1}(U)$ is the union of the elements of $\widetilde{\mathcal{A}}$. Hence p is smooth. At the same time, if $\widetilde{U} \in \widetilde{\mathcal{A}}$ then $p(\widetilde{U}) \in \mathcal{A}$.

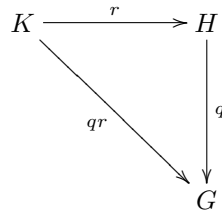
Let \tilde{U} be a lifting of the set $U \in \mathcal{A}$. Then the restriction $p|_{\tilde{U}}$ is a bijection. Thus p is a diffeomorphism. Also since \tilde{U} is open in \tilde{M} , \tilde{U} is the canonic in \tilde{M} and U is the canonic in M . Consequently p is a covering morphism.

Now let us define the morphism $r : \tilde{G} \rightarrow \pi_1 \tilde{M}$ which is identical on the objects. Take $\tilde{a} \in \tilde{G}(\tilde{x}, \tilde{y})$ and $q(\tilde{a}) \in \pi_1 M(x, y)$. Let $a : I \rightarrow M$ be a representation of $q(\tilde{a})$. Then the morphism a induces a morphism $\pi_1 a : \pi_1 I \rightarrow \pi_1 M$ satisfying the condition $(\pi_1 a)(\iota) = q(\tilde{a})$, where ι is the unique element of $\pi_1 I(0, 1)$. Since I is the 1-transitive, $\pi_1 a$ can be lifted to the morphism $a' : (\pi_1 I, 0) \rightarrow (\tilde{G}, \tilde{x})$, where $a'(\iota)$ is the lifting of $q(\tilde{a})$ and $a'(\iota) = \tilde{a}$. By the Example 3, $a'_0 : I \rightarrow \tilde{M}$ is smooth and $r(\tilde{a})$ can be defined as $r(\tilde{a}) = [a'_0]$. Clearly, $r(\tilde{a}) \in \pi_1 \tilde{M}(\tilde{x}, \tilde{y})$. Also, $r(\tilde{a})$ is independent of the choice of representation a . When the different representations a_1, a_2 are equivalent, the liftings \tilde{a}_1, \tilde{a}_2 are also equivalent. Let us suppose $b \in \tilde{G}(\tilde{y}, \tilde{z})$. Then $r(b\tilde{a})$ and $r(b)r(\tilde{a})$ are both liftings of $q(b\tilde{a})$. Thus, $r(b\tilde{a}) = r(b)r(\tilde{a})$. This shows that r is a morphism of groupoids. Further, it follows from the definition of r , $(\pi_1 p)r = q$. By [1], we know that r is a covering morphism. This implies that $r : \tilde{G}(\tilde{x}, \tilde{y}) \rightarrow \pi_1 \tilde{M}(\tilde{x}, \tilde{y})$ is one-to-one for each $\tilde{x}, \tilde{y} \in \tilde{M}$. Since each $c \in \pi_1 \tilde{M}(\tilde{x}, \tilde{y})$ is covered by the element $\tilde{c} \in St_{\tilde{G}} \tilde{x}$, it is also onto. Consequently r is an isomorphism. The uniqueness of the smooth structure can easily be checked. That is, the smooth structure of M is lifted to the same smooth structure on \tilde{M} by q and $\pi_1 p$, because r is identical on the objects. \square

Proposition 1 *Let $r : K \rightarrow H$ and $q : H \rightarrow G$ be Lie groupoid morphisms. Then there exist the following.*

1. *If q and r are the covering morphisms of Lie groupoids, then so is qr .*
2. *If q and qr are the covering morphisms of Lie groupoids, then so is r .*
3. *If r and qr are the covering morphisms of Lie groupoids and r_0 is surjective, then q is the covering morphism of Lie groupoids.*

Proof. We consider the following diagram.



For brevity, we say $qr = p$.

1. If q and r are the covering morphism of Lie groupoids, then the maps $q' : St_H y \rightarrow St_G z$ and $r' : St_K x \rightarrow St_H y$ are diffeomorphisms. Since the composition of diffeomorphisms will be also a diffeomorphism, $q'r' : St_K x \rightarrow St_G z$ is a diffeomorphism. This implies that $qr : K \rightarrow G$ is the covering morphism of Lie groupoids.

2. If q and $p = qr$ are the covering morphism of Lie groupoids, then the maps $q' : St_{Hy} \rightarrow St_{Gz}$ and $p' : St_{Kx} \rightarrow St_{Gz}$ are diffeomorphisms. Let us consider a map $r' : St_{Kx} \rightarrow St_{Hy}$. Since q' is a diffeomorphism, from the equality $p' = q'r'$ we have $(q')^{-1}p' = r'$. Hence r' is a diffeomorphism, because the left side of the equality is a diffeomorphism. Consequently, r is a covering morphism of Lie groupoids.
3. The proof is similar to that of (2).

□

Let M and \widetilde{M} be connected manifolds. $SCov(M)$ is a category whose objects are smooth covering maps $p : \widetilde{M} \rightarrow M$ and a morphism from $p : \widetilde{M} \rightarrow M$ to $q : \widetilde{N} \rightarrow M$ is a map $r : \widetilde{M} \rightarrow \widetilde{N}$ satisfying the condition $p = q \circ r$.

If M is a connected manifold, we know that the fundamental groupoid $\pi_1 M$ is a Lie groupoid. Then we constitute a category as follows:

$LGdCov(\pi_1 M)$ has smooth covering morphisms $p : \widetilde{G} \rightarrow \pi_1 M$ of Lie groupoids as objects, where $\widetilde{M} = \widetilde{G}_0$ is connected manifold. A morphism from $p : \widetilde{G} \rightarrow \pi_1 M$ to $q : \widetilde{H} \rightarrow \pi_1 M$ is a morphism $r : \widetilde{G} \rightarrow \widetilde{H}$ of Lie groupoids satisfying the condition $p = q \circ r$. Now we can state our first main result.

Theorem 1 *The categories $LGdCov(\pi_1 M)$ and $SCov(M)$ are equivalent.*

Proof. Firstly, let us define a functor $\Gamma : SCov(M) \rightarrow LGdCov(\pi_1 M)$ in the following way: let M and \widetilde{M} be connected manifolds. Let $p : \widetilde{M} \rightarrow M$ be a covering map of the manifolds. Then from Lemma 1, $\pi_1 p : \pi_1 \widetilde{M} \rightarrow \pi_1 M$ is a covering morphism of Lie groupoids. That is, $\Gamma(p) = \pi_1 p$ is a covering morphism of Lie groupoids.

Secondly, let us define a functor $\Phi : LGdCov(\pi_1 M) \rightarrow SCov(M)$ in the following way: let $q : \widetilde{G} \rightarrow \pi_1 M$ be covering morphism of Lie groupoids, where M and $\widetilde{G}_0 = \widetilde{M}$ are connected manifolds. By the Lemma 2, there exists a lifted manifold structure on \widetilde{M} so that p is the covering map of the manifolds for any $p = q_0 : \widetilde{M} \rightarrow M$. Thus $\Phi(q) = q_0 = p$ is a covering map of the manifolds.

The natural equivalences $\Gamma\Phi \simeq 1_{LGdCov(\pi_1 M)}$ and $\Phi\Gamma \simeq 1_{SCov(M)}$ are showed as similar to the algebraic equivalence which is given in [1].

□

Let G be a Lie groupoid. Then we obtain a category of the coverings of G denoted by $LGdCov(G)$ whose objects are the covering morphisms $p : H \rightarrow G$ of Lie groupoids and a morphism from $p : H \rightarrow G$ to $q : K \rightarrow G$ is a morphism $p : H \rightarrow K$ of Lie groupoids such that $p = qr$.

From Proposition 1, it seems that each Lie groupoid morphism r in the category $LGdCov(G)$ is a covering morphism.

3.2. Actions of Lie groupoids

In this part, we will present some basic concepts and results related to actions of Lie groupoids on the manifolds and relations between coverings and actions of Lie groupoids. First, let us give a definition of action

of Lie groupoids on the manifolds.

Definition 6 Let G be a Lie groupoid and let M be a manifold. Let $w : M \rightarrow G_0$ be a submersion. A left action of G on M via w is a smooth map $\phi : G_\alpha \times_w M \rightarrow M, (a, x) \mapsto a \cdot x$ satisfying the conditions

$$i) w(a \cdot x) = \beta(a) \quad ii) b \cdot (a \cdot x) = (b \circ a) \cdot x \quad iii) (1_{w(x)}) \cdot x = x,$$

for any $a, b \in G, x \in M$. In this case, we also call the manifold M a left G -manifold. Similarly, we can define a right action of G on M [12].

Remark 1 We note that it is easy to find examples appropriate to the above structure given in Definition 6. In fact, such examples with a slightly different structure can be find [10, 11]. More precisely, let G be a Lie groupoid acting on manifold M via $w : M \rightarrow G_0$. By this action, we can define a Lie groupoid denoted by $G \times M$ whose objects is M . It is called the action Lie groupoid. The set of objects M and the set of morphism $G_\alpha \times_w M$ are manifolds, because α and w are submersions. A morphism from an object x to an object y is a pair (a, x) such that $a \cdot x = y$.

Proposition 2 The first projection $p : G \times M \rightarrow G$ with $w : M \rightarrow G_0$ on the objects is a covering morphism of Lie groupoids.

Proof. By the definition of $p, p((b, y) \circ (a, x)) = p((b \circ a, x)) = b \circ a = p((b, y)) \circ p((a, x))$. Also, since p is given by w on objects, it follows $p((1_{w(x)}, x) = 1_{w(x)} = 1_{p((1_{w(x)}, x))}$, whence p is a groupoid morphism. Furthermore, by [1] p is the covering morphism of groupoids. So there exists a lifting function $s_p : G_\alpha \times_{p_0=w} (G \times M)_0 \rightarrow G_\alpha \times_{p_0=w} M$. It follows $s_p : G_\alpha \times_{p_0=w} M \rightarrow G_\alpha \times_{p_0=w} M$, because $(G \times M)_0 = M$. It is obvious that s_p is a diffeomorphism, since it is identity map. Thus p is the covering morphism of Lie groupoids. \square

Remark 2 In the light of Remark 1 and Proposition 2, it is possible to combine concepts of action and covering for Lie groupoids. Indeed, let $p : H \rightarrow G$ be a covering morphism of Lie groupoids. We take $M = H_0$ and $w = p_0 : H_0 \rightarrow G_0$. Thus we obtain an action $\phi : G_\alpha \times_{p_0} H_0 \rightarrow H_0, (a, \tilde{x}) \mapsto a \cdot \tilde{x} = \tilde{\beta}(\tilde{a})$ of G on $M = H_0$ by $w = p_0$. In here, since p is the covering morphism of Lie groupoids, there exists only one lifting \tilde{a} of a , its source is \tilde{x} , such that $p(\tilde{a}) = a$ and $p_0(\tilde{x}) = x$ for $\tilde{x} \in M = H_0$ and $a \in G_{p_0(\tilde{x})}$. Now we can show that ϕ is an action. For any $a, b \in G, x \in G_0, \tilde{x} \in H_0, \tilde{a}, \tilde{b} \in H, \tilde{c} \in H_{\tilde{x}},$

- i) $w(a \cdot \tilde{x}) = p_0(a \cdot \tilde{x}) = p_0(\tilde{\beta}(\tilde{a})) = \beta(a),$
- ii) $b \cdot (a \cdot \tilde{x}) = b \cdot \tilde{\beta}(\tilde{a}) = \tilde{\beta}(\tilde{b}) = \tilde{\beta}(\tilde{b} \circ \tilde{a}) = (b \circ a) \cdot \tilde{x},$
- iii) $1_{p_0(\tilde{x})} \cdot \tilde{x} = \tilde{\beta}(\tilde{c}) = \tilde{x},$ where c is an arrow starting at \tilde{x} .

Now let us prove smoothness of the action. Since $p : H \rightarrow G$ is the covering morphism of Lie groupoids, p and p_0 are smooth, and also $s_p : G_\alpha \times_{p_0} H_0 \rightarrow H$ is a diffeomorphism. Thus w is smooth and hence the action $\phi : G_\alpha \times_{p_0} H_0 \rightarrow H_0, (a, \tilde{x}) \mapsto a \cdot \tilde{x} = \tilde{\beta}(\tilde{a})$, defined as the composition of the target map $\tilde{\beta}$ of Lie groupoid and s_p , is smooth. Finally G acts smoothly on H_0 .

So we obtain the category $LGdOp(G)$ whose objects are smooth actions (M, w) and a morphism from (M, w) to (M', w') is a smooth map $f : M \rightarrow M'$ such that $w' \circ f = w$ and $f(a \cdot x) = a \cdot f(x)$.

After these preliminaries, we can now give the second main result of this paper which concerns other equivalence of categories .

Theorem 2 *Let G be a Lie groupoid. Then the category $LGdCov(G)$ of coverings of G and the category $LGdOp(G)$ of actions of G on the manifolds are equivalent.*

Proof. Firstly, let us define a functor $\Gamma : LGdOp(G) \rightarrow LGdCov(G)$ in the following way. Let us denote the smooth action of Lie groupoid G on a manifold M via the smooth map $w : M \rightarrow G_0$ by

$$\begin{aligned} \phi : G_{\alpha \times_w} M &\rightarrow M \\ (a, x) &\mapsto \phi(a, x) = a \cdot x. \end{aligned}$$

From Remark 1, there exists an action Lie groupoid $G \times M$ where the manifold of objects is M . A covering morphism $p : G \times M \rightarrow G$ of Lie groupoids is defined by w on manifold of objects and first projection on manifold of morphisms. Thus $\Gamma(M, w)$ is a covering morphism of Lie groupoids.

Secondly, let us define a functor $\Phi : LGdCov(G) \rightarrow LGdOp(G)$ in the following way. For any covering morphism $p : \tilde{G} \rightarrow G$ of Lie groupoids, we say that $M = \tilde{G}_0$ and $w = p_0 : \tilde{G}_0 \rightarrow G_0$. From Remark 2, we obtain the smooth action $\phi : G_{\alpha \times_{p_0}} \tilde{G}_0 \rightarrow \tilde{G}_0$, $(a, \tilde{x}) \mapsto a \cdot \tilde{x} = \tilde{\beta}(\tilde{a})$ of Lie groupoid G onto $M = \tilde{G}_0$ via the smooth map $w = p_0$. Thus $\Phi(p)$ is a smooth action of Lie groupoid G on a manifold.

It is obvious that $\Phi\Gamma = 1_{LGdOp(G)}$ and $\Gamma\Phi = 1_{LGdCov(G)}$. □

Now we will give some side results related to actions of Lie groupoids on a manifold and coverings of Lie groupoids. For this, firstly we have to define another concept induced by action of Lie groupoids on a manifold.

Definition 7 *Let G, H be Lie groupoids and M be a manifold. Let M be a left G -manifold via w and be a right H -manifold via w' . Let $a \cdot x$ and $x \cdot b$ be defined for any $x \in M$, $a \in G$ and $b \in H$. If the conditions*

$$i) w'(a \cdot x) = w'(x) \quad ii) w(x \cdot b) = w(x) \quad iii) a \cdot (x \cdot b) = (a \cdot x) \cdot b,$$

are satisfied, the manifold M is called a G - H -manifold and is denoted by (w', M, w) . So, it is said that G and H act on manifold M by w - w' .

Example 4 *Let G be a Lie groupoid. Then G acts on itself via β - α . The action is given by the composition of G . Indeed, the smooth left action of G on $M = G$ via $w = \beta : G \rightarrow G_0$ is defined by $\phi : G_{\alpha \times_{w=\beta}} G \rightarrow G$, $(a, b) \mapsto a \cdot b = a \circ b$. The smooth right action of G on $M = G$ via $w' = \alpha : G \rightarrow G_0$ is defined by $\phi' : G_{\beta \times_{w'=\alpha}} G \rightarrow G$, $(a, b) \mapsto b \cdot a = b \circ a$. Finally let us show that the above conditions are satisfied via the following:*

- i) $w'(a \cdot b) = \alpha(a \cdot b) = \alpha(a \circ b) = \alpha(b) = w'(b)$,
- ii) $w(b \cdot a) = \beta(b \cdot a) = \beta(b \circ a) = \beta(b) = w(b)$,
- iii) $a \cdot (b \cdot c) = a \cdot (b \circ c) = a \circ (b \circ c) = (a \circ b) \circ c = (a \circ b) \cdot c = (a \cdot b) \cdot c$.

Thus, the Lie groupoid G is a G - G -manifold.

The proof of the following proposition is omitted, but can be found in [4].

Proposition 3 *Let M be a G -manifold where the action of G is proper and free. Then M/G has a unique smooth structure and the quotient map $r : M \rightarrow M/G$ is a submersion.*

From now on, we will assume that the actions of Lie groups on manifolds are proper and free.

Theorem 3 *Let H be a Lie group and G be a Lie groupoid. If M is a G - H -manifold, then the action of G on M determines a left G -manifold on the orbit space M/H .*

Proof. We consider the G - H -manifold (w', M, w) . That is, let (M, w) be a left G -manifold and (w', M) be a right H -manifold. It is enough to show that the action $\phi' : G_{\alpha \times w}M/H \rightarrow M/H$ induced from the action $\phi : G_{\alpha \times w}M \rightarrow M$ is smooth. From Proposition 3, the quotient map $r : M \rightarrow M/H$ is a submersion. Hence r is open. Thus $1 \times r : G \times M \rightarrow G \times M/H$ is an open map, and hence is a submersion. The submanifold $G_{\alpha \times w}M$ is $1 \times r$ -saturated subset of $G \times M$. From $1 \tilde{\times} r = (1 \times r) |_{G_{\alpha \times w}M} : G_{\alpha \times w}M \rightarrow G_{\alpha \times w}M/H$, the map $1 \tilde{\times} r$ is open and surjective. Also the object manifold G_0 is Hausdorff and the subspace $G_{\alpha \times w}M$ is closed in $G \times M$. Since the quotient map $1 \times r$ is a submersion, $G \times M/H$ has a quotient manifold structure and closed sets of $G \times M/H$ are the sets $(1 \times r)(G_{\alpha \times w}M)$ for the saturated closed sets U of $G \times M$. The set $(1 \times r)(G_{\alpha \times w}M) = G_{\alpha \times w}M/H$ is a closed subset of $G \times M/H$, because $G_{\alpha \times w}M$ is a saturated closed subset of $G \times M$. If K is a closed subset of $G_{\alpha \times w}M/H$, then K is also closed subset of $G \times M/H$ since $G_{\alpha \times w}M/H$ is the closed subset of $G \times M/H$. By the fact that $1 \times r$ is smooth, $(1 \times r)^{-1}(K)$ is a closed subset of $G \times M$. Also since $1 \tilde{\times} r$ is surjective and $G_{\alpha \times w}M$ is the closed subset of $G \times M$, by the structural properties of submanifolds $(1 \tilde{\times} r)^{-1}(K)$ is closed in $G_{\alpha \times w}M$ and $1 \tilde{\times} r$ is smooth. Thus $1 \tilde{\times} r : G_{\alpha \times w}M \rightarrow G_{\alpha \times w}M/H$ is quotient map. Since $\phi' \circ (1 \tilde{\times} r) = r \circ \phi$ is smooth, ϕ' is smooth. \square

Let G be a Lie groupoid. By Example 4, G is a G - G -manifold by β - α . Now we take any $x \in G_0$ and let $N\{x\}$ be closed subgroup of Lie group $G\{x\}$. Hence $N\{x\}$ is also a Lie group. Then G_x is a G - $N\{x\}$ -manifold. Thus we define the space $G_x/N\{x\} = G_{N\{x\}}$ of left cosets of $N\{x\}$.

Corollary 1 *Let G be a Lie groupoid and $N\{x\}$ be Lie subgroup of $G\{x\}$. Then the space $G_{N\{x\}}$ admits a left G -manifold structure by left multiplication.*

Theorem 4 *Let G be a transitive Lie groupoid. We take any $x \in G_0$. Let $N\{x\}$ be a Lie subgroup of the object group $G\{x\}$. Then there exist a transitive Lie groupoid H , a smooth covering morphism $p : H \rightarrow G$ and an $\tilde{x} \in H_0$ such that $p(H\{\tilde{x}\}) = N\{x\}$.*

Proof. Let $M = G_x/N\{x\} = G_{N\{x\}} = \{a \circ N\{x\} \mid a \in G_x\}$ and let us define the map $w : M \rightarrow G_0$ by $a \circ N\{x\} \mapsto \beta(a)$. Then the map $\phi : G_{\alpha \times w}M \rightarrow M$ defined by $(b, a \circ N\{x\}) \mapsto b \circ a \circ N\{x\}$ gives the smooth action of Lie groupoid G on the manifold M via the smooth map w . Indeed, since G_x is a closed submanifold of G , G_x is a G - $N\{x\}$ -manifold via $\beta |_{G_x - \alpha} |_{G_x}$. Thus $M = G_x/N\{x\}$ is a manifold. Further, w is smooth, because it is given by the target map β of Lie groupoid. For all $a \in G(x, y)$ and $b \in G(y, z)$, we have $\alpha(b) = w(a \circ N\{x\}) = \beta(a)$. Hence $b \circ a$ is defined and $b \circ a \in G_x$. Also we have

$w(b \circ (a \circ N\{x})) = w(b \circ a \circ N\{x}) = \beta(b \circ a) = \beta(b)$. That is, the first condition of the action is satisfied. For any $c \in G(z, z')$,

$$\begin{aligned} c(b \circ (a \circ N\{x})) &= c(b \circ a \circ N\{x}) = c \circ (b \circ a \circ N\{x}) \\ &= (c \circ b) \circ (a \circ N\{x}) = (c \circ b) \cdot (a \circ N\{x}). \end{aligned}$$

So, the second condition of the action is satisfied. Finally, $1_y(a \circ N\{x}) = 1_y \circ a \circ N\{x} = a \circ N\{x}$, that is, the third condition is satisfied. Now let us show that the action is smooth. For this, we have to show that the action $\phi : G_\alpha \times_\beta G_x/N\{x} \rightarrow G_x/N\{x}$ induced from the left action $\phi' : G_\alpha \times_\beta G \rightarrow G$ is smooth. Let $r : G_x \rightarrow G_x/N\{x}$ be quotient map. r is a submersion, because $G_x/N\{x}$ is a manifold. Then r is an open map. Hence $1 \times r : G \times G_x \rightarrow G \times G_x/N\{x}$ is also an open map and a submersion. $G_\alpha \times_w G_x$ is a $(1 \times r)$ -saturated subset of $G \times G_x$. Considering the map $1 \tilde{\times} r = (1 \times r) |_{G_\alpha \times_w G_x} : G_\alpha \times_w G_x \rightarrow G_\alpha \times_w G_x/N\{x}$, $1 \tilde{\times} r$ is open and surjective. Furthermore, we have that $G_\alpha \times_w G_x$ is closed in $G \times G_x$ by the fact that the object manifold G_0 is Hausdorff. Since $(1 \times r)$ is a submersion, $G \times G_x/N\{x}$ has a quotient manifold, and the closed subsets of $G \times G_x/N\{x}$ are the sets $(1 \times r)(U)$ for the closed saturated sets U of $G \times G_x$. The set $(1 \times r)(G_\alpha \times_w G_x) = G_\alpha \times_w G_x/N\{x}$ is closed subset of $G \times G_x/N\{x}$, because $G_\alpha \times_w G_x$ is the closed saturated subset of $G \times G_x$. Furthermore, since $G_\alpha \times_w G_x/N\{x}$ is a closed subset of $G \times G_x/N\{x}$, if K is a closed subset of $G_\alpha \times_w G_x/N\{x}$ then K is also closed subset of $G \times G_x/N\{x}$. It follows from the fact that $1 \times r$ is smooth, $(1 \times r)^{-1}(K)$ is a closed subset of $G \times G_x$. Since $1 \tilde{\times} r$ is surjective and $G_\alpha \times_w G_x$ is a closed subset of $G \times G_x$, $(1 \tilde{\times} r)^{-1}(K)$ is closed in $G_\alpha \times_w G_x$ and $1 \tilde{\times} r$ is smooth. Thus $1 \tilde{\times} r : G_\alpha \times_w G_x \rightarrow G_\alpha \times_w G_x/N\{x}$ is quotient map. Since $\phi \circ (1 \tilde{\times} r) = r \circ \phi'$ is smooth, ϕ is smooth. From Example 1, we obtain an action Lie groupoid whose object manifold is $(G \times M)_0 = M$. The morphisms of this groupoid are the pairs $(b, a \circ N\{x})$ satisfying the condition $b(a \circ N\{x}) = a' \circ N\{x}$. Also the source map is defined by $\alpha(b, a \circ N\{x}) = a \circ N\{x}$, the target map is defined by $\beta(b, a \circ N\{x}) = b \circ a \circ N\{x}$, the object map is defined by $a \circ N\{x} \mapsto (1_{\beta(a)}, a \circ N\{x})$, the inverse map is defined by $(b, a \circ N\{x}) \mapsto (b^{-1}, b(a \circ N\{x}))$ and finally the composition is given by $(b, a' \circ N\{x}) \circ (c, a \circ N\{x}) = (b \circ c, a \circ N\{x})$. It is obvious that the source map and the target map are smooth. The object map is given by $(1_{(\)} \circ w) \times 1 : M \rightarrow G \times M$ is smooth, where 1 is identity map and $1_{(\)}$ is the object map of Lie groupoid. The inverse map is also smooth, because it is defined as multiplication of the inverse map of Lie groupoid and the action. Finally the composition is defined by the diagram

$$\begin{array}{ccc} (G \times M) \times (G \times M) & \xrightarrow{pr_2} & G \times M & \xrightarrow{L_b \times 1} & G \times M \\ ((b, a \circ N\{x}), (b', a' \circ N\{x})) & \xrightarrow{m} & & & (b', a' \circ N\{x}) \\ & \searrow pr_2 & & & \nearrow L_b \times 1 \\ & & (b \circ b', a' \circ N\{x}) & & \end{array}$$

Since the second projection pr_2 , the left translation L_b and the identity map 1 are smooth, the composition of the maps is also smooth. The projection $p : G \times M \rightarrow G$ is defined by w on objects and by $(a, x) \mapsto a$ on morphisms. So by Proposition 2, it follows that p is a covering morphism of Lie groupoids. Taking $H = G \times M$

and $\tilde{x} = N\{x\}$, we obtain $p(H\{\tilde{x}\} = N\{x\})$. □

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