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Rotational embeddings in $E^4$ with pointwise 1-type gauss map

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Rotational embeddings in $\mathbb{E}^4$ with pointwise 1-type gauss map

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Abstract

In the present article we study the rotational embedded surfaces in $\mathbb{E}^4$. The rotational embedded surface was first studied by G. Ganchev and V. Milousheva as a surface in $\mathbb{E}^4$. The Otsuki (non-round) sphere in $\mathbb{E}^4$ is one of the special examples of this surface. Finally, we give necessary and sufficient conditions for the flat Ganchev-Milousheva rotational surface to have pointwise 1-type Gauss map.

Key word and phrases: Rotation surface, gauss map, finite type, Pointwise 1-type.

1. Introduction

Since the late 1970’s, the study of submanifolds of Euclidean space or pseudo-Euclidean space with the notion of finite type immersion has been extensively carried out. An isometric immersion $x: M \rightarrow \mathbb{E}^m$ of a submanifold $M$ in Euclidean $m$-space $\mathbb{E}^m$ is said to be of finite type if $x$ identified with the position vector field of $M$ in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$, that is,

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

where $x_0$ is a constant map $x_1, x_2, ..., x_k$ non-constant maps such that $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are different, then $M$ is said to be of $k$-type. Similarly, a smooth map $\phi$ of an $n$-dimensional Riemannian manifold $M$ of $\mathbb{E}^m$ is said to be of finite type if $\phi$ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of $\Delta$ ([4], [5]). Granted, this notion of finite type immersion is naturally extended in particular to the Gauss map $G$ on $M$ in Euclidean space ([8]). Thus, if a submanifold $M$ of Euclidean space has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$ ([1], [2], [3], [11]). However, the Laplacian of the Gauss map of some typical well-known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space $\mathbb{E}^3$ take a somewhat different form; namely, $\Delta G = f(G + C)$ for some non-constant function $f$ and some constant vector $C$. Therefore, it is worth studying the class of solution surfaces satisfying

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such an equation. A submanifold $M$ of a Euclidean space $E^m$ is said to have pointwise 1-type Gauss map if its Gauss map $G$ satisfies

$$\Delta G = f(G + C)$$

(1)

for some non-zero smooth function $f$ on $M$ and a constant vector $C$. A pointwise 1-type Gauss map is called proper if the function $f$ defined by (1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([6], [9], [12], [13]). In [9], one of the present authors characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Also, together with B.-Y. Chen, they proved that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincides with surfaces of revolution with constant mean curvature. Moreover, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map ([6]).

In [16], D. W. Yoon studied Vranceanu rotation surfaces in Euclidean 4-space $E^4$. He obtained the complete classification theorems for the flat Vranceanu rotation surfaces with 1-type Gauss map and an equation in terms of the mean curvature vector. For more details see also [15].

In this article we will investigate rotational embedded surface with pointwise 1-type Gauss map in Euclidean 4-space $E^4$.

The rotational embedded surface was studied by G. Ganchev and V. Milousheva as a surface in $E^4$ which is defined by the following surface patch with respect to an orthonormal system of coordinates

$$X(s, t) = (f_1(s), f_2(s), f_3(s)\cos t, f_3(s)\sin t),$$

(2)

where $\alpha(s) = (f_1(s), f_2(s), f_3(s))$ is a space curve parametrized by the arc-length, i.e., $(f_1')^2 + (f_2')^2 + (f_3')^2 = 1$ and $f_3(s) > 0$ ([10]).

We prove the following theorem.

**Theorem A.** Let $M$ be a flat rotational embedded surface in Euclidean 4-space $E^4$. Then $M$ has pointwise 1-type Gauss map if and only if

$$f_1(s) = \int \mu \cos(\frac{\lambda}{a\mu} \ln |as+b|)ds,$$

$$f_2(s) = \int \mu \sin(\frac{\lambda}{a\mu} \ln |as+b|)ds,$$

$$f_3(s) = as + b.$$

for some constants $\lambda \neq 0, \mu > 0, a \neq 0$ and $b$.

2. Preliminaries

Let $x : M \to E^n$ be an isometric immersion from an $n$-dimensional connected Riemannian manifold $M$ into an $m$-dimensional Euclidean space $E^m$. Let $\bar{\nabla}$ be the Levi-Civita connection of $E^m$ and $\nabla$ the induced...
connection on $M$. Then the Gaussian and Weingarten formulas are given, respectively, by
\[ \nabla_X Y = \nabla_X Y + h(X, Y), \]
\[ \nabla_X \xi = -A_\xi X + D_X \xi \]
for vector fields $X, Y$ tangent to $M$ and a vector field $\xi$ normal to $M$, where $h$ denotes the second fundamental form, $D$ the normal connection and $A_\xi$ the shape operator in the direction of $\xi$ that is related with $h$ by
\[ \langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle, \]
where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{E}^4$ and that in the submanifold $M$ as well.

If we define a covariant differentiation $\nabla h$ of the second fundamental form $h$ on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of $M$ by
\[ (\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \]
then we have the Codazzi equation
\[ (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z). \]

Let us now define the Gauss map $G$ of a submanifold $M$ into $G(n, m)$ in $\wedge^n \mathbb{E}^m$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented $n$-planes through the origin of $\mathbb{E}^m$ and $\wedge^n \mathbb{E}^m$ is the vector space obtained by the exterior product of $n$ vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N = \binom{m}{n}$. Let $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ be an adapted local orthonormal frame field in $\mathbb{E}^m$ such that $e_1, e_2, \ldots, e_n$ are tangent to $M$ and $e_{n+1}, e_{n+2}, \ldots, e_m$ normal to $M$. The map $G : M \to G(n, m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge \ldots \wedge e_n)(p)$ is called the Gauss map of $M$, that is a smooth map which carries a point $p$ in $M$ into the oriented $n$-plane in $\mathbb{E}^m$ obtained from the parallel translation of the tangent space of $M$ at $p$ in $\mathbb{E}^m$.

For any real valued function $f$ on $M$ the Laplacian of $f$ is defined by the relation
\[ \Delta f = -\sum_i \langle \nabla_{e_i} \nabla f, e_i \rangle. \]

3. Proof of Theorem

Let $M$ be a rotational embedded surface in $\mathbb{E}^4$ defined by the patch (2). We choose a moving frame $e_1, e_2, e_3, e_4$ such that $e_1, e_2$ are tangent to $M$ and $e_3, e_4$ are normal to $M$ in the following ([10]):
\[ e_1 = \frac{\partial X}{\partial s}, \quad e_2 = \frac{\partial X}{\partial t}, \]
\[ e_3 = \frac{1}{\kappa} (f_1''(s), f_2''(s), f_3''(s) \cos t, f_3''(s) \sin t), \]
\[ e_4 = \frac{1}{\kappa} (f_1'(s)f_2''(s) - f_2'(s)f_3''(s), f_1''(s)f_3'(s) - f_2''(s)f_3'(s) - f_1'(s)f_3''(s), \]
\[ (f_1'(s)f_2''(s) - f_2'(s)f_3''(s)) \cos t, (f_1'(s)f_2''(s) - f_1''(s)f_2'(s)) \sin t), \]
where
\[ \kappa = \sqrt{(f_1'')^2 + (f_2'')^2 + (f_3'')^2} \neq 0 \] (7)
is the curvature of the space curve \( \alpha \).

Hence, the coefficients of the first fundamental form of the surface are
\[ E = \langle X_s(s, t), X_s(s, t) \rangle = 1, \]
\[ F = \langle X_s(s, t), X_t(s, t) \rangle = 0, \]
\[ G = \langle X_t(s, t), X_t(s, t) \rangle = f_3^2(s). \]

Since \( EG - F^2 = f_3^2(s) \) does not vanish, the surface patch \( X(s, t) \) is regular.

We denote by \( \tilde{\alpha} \) the projection of \( \alpha \) on the 2-dimensional plane \( Oe_1e_2 \). So the curvature of \( \tilde{\alpha} \) is defined by \( \kappa_1 = f_1'f_2'' - f_2'f_1''' \). Then with respect to the frame field \( \{e_1, e_2, e_3, e_4\} \), the Gaussian and Weingarten formulas (3)–(4) of \( M \) look like [10]:
\[ \tilde{\nabla}_{e_1}e_1 = \kappa e_3, \]
\[ \tilde{\nabla}_{e_1}e_2 = 0, \]
\[ \tilde{\nabla}_{e_2}e_2 = -\frac{f_3''}{f_3} e_1 - \frac{f_3'''}{\kappa f_3} e_3 - \frac{\kappa_1}{\kappa f_3} e_4, \]
\[ \tilde{\nabla}_{e_2}e_1 = \frac{f_3'}{f_3} e_2 \] (8)
and
\[ \tilde{\nabla}_{e_3}e_3 = -\kappa e_1 + \tau e_4, \]
\[ \tilde{\nabla}_{e_3}e_2 = \frac{f_3''}{\kappa f_3} e_2, \]
\[ \tilde{\nabla}_{e_4}e_4 = -\tau e_3, \]
\[ \tilde{\nabla}_{e_4}e_2 = \frac{\kappa_1}{\kappa f_3} e_2. \] (10)

Where, \( \tau \) is the second curvature of space curve \( \alpha \). The Gauss curvature of \( M \) is obtained by equating
\[ K = -\frac{f_3'''}{f_3}. \] (11)

Putting
\[
\begin{align*}
A(s) &= -\left( \kappa^2 + \frac{(f_3'')^2 + \kappa_1^2}{\kappa^2 f_3^2} \right), \\
B(s) &= -\left( \kappa' + \frac{f_3''' f_3'}{\kappa f_3^2} + \frac{\kappa f_3'}{f_3^3} \right), \\
D(s) &= -\left( \kappa \tau + \frac{\kappa_1 f_3'}{\kappa f_3^2} \right),
\end{align*}
\] (12)
we get, by using (6), (8) and (10),

\[-\Delta G = A(s)e_1 \wedge e_2 + B(s)e_2 \wedge e_3 + D(s)e_2 \wedge e_4.\]  

(13)

We now suppose that the rotational embedded surface \(M\) is of pointwise 1-type Gauss map in \(E^4\). From (1) and (13),

\[f + f \langle C, e_1 \wedge e_2 \rangle = -A(s),\]
\[f \langle C, e_2 \wedge e_3 \rangle = -B(s),\]
\[f \langle C, e_2 \wedge e_4 \rangle = -D(s).\]  

(14)

Since \(\Delta G\) is a linear combination of \(e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4\) and \(e_3 \wedge e_4\), we also have

\[f \langle C, e_1 \wedge e_3 \rangle = 0,\]
\[f \langle C, e_1 \wedge e_4 \rangle = 0,\]
\[f \langle C, e_3 \wedge e_4 \rangle = 0.\]  

(15)

By differentiating (15) covariantly with respect to \(s\), we get

\[
\frac{f_3'}{f_3} < C, e_2 \wedge e_3 > + \frac{f_3''}{\kappa_f} < C, e_1 \wedge e_2 > = 0, \\
\frac{f_3'}{f_3} < C, e_2 \wedge e_4 > + \frac{\kappa_1}{\kappa_f} < C, e_1 \wedge e_2 > = 0, \\
\frac{f_3''}{\kappa_f} < C, e_2 \wedge e_4 > - \frac{\kappa_1}{\kappa_f} < C, e_2 \wedge e_3 > = 0. 
\]

(16)

Since \(M\) is flat, (11) implies \(f_3'' = 0\). Thus \(f_3(s) = as + b\) for some constants \(a \neq 0\) and \(b\). Hence, substituting \(f_3'' = 0\) into (16) and using (14) we obtain,

\[f_3'B(s) = 0,\]
\[f_3'D + \frac{\kappa_1}{\kappa}(A(s) + f) = 0,\]
\[\kappa_1B(s) = 0.\]  

(17)

Suppose \(Q = \{p \in M : B(s) \neq 0\}\) is a non-empty set. Then, from the third formula of (16) we have \(\kappa_1 = f_1'f_2'' - f_2'f_1'' = 0\). Consequently, using this equality with \((f_1')^2 + (f_2')^2 + (f_3')^2 = 1\), we get \((f_1')^2 + (f_2')^2 = 1 - a^2\). Therefore, \(f_1', f_2', f_3'\) are constant functions and \(\kappa = \sqrt{(f_1'')^2 + (f_2'')^2 + (f_3'')^2} = 0\), which is a contradiction. So, \(B(s) = 0\). Furthermore, if we make use of the second equation of (12) with \(f_3'' = 0\), then we obtain \(\kappa = \frac{\lambda}{\alpha \kappa + \beta}\), where \(\lambda\) is a nonzero constant. We may put

\[f_1' = \mu \cos \theta(s), f_2' = \mu \sin \theta(s)\]  

(18)

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for some function $\theta(s)$, where $1 - a^2 = \mu^2$. Furthermore, substituting (18), $\kappa = \frac{\lambda}{as+b}$ and $f_3 = as + b$ into (7) with some computation implies $\frac{da}{ds} = \frac{\lambda}{a\mu} \ln |as + b| > 0$. Solving this equation, we get $\theta(s) = \frac{\lambda}{a\mu} \ln |as + b|$. So, we obtain

\[
\begin{align*}
    f_1(s) &= \int \mu \cos \left(\frac{\lambda}{a\mu} \ln |as + b|\right) ds, \\
    f_2(s) &= \int \mu \sin \left(\frac{\lambda}{a\mu} \ln |as + b|\right) ds, \\
    f_3(s) &= as + b.
\end{align*}
\]

The converse is easily verified. Thus, our theorem is proved.

**Corollary 3.1** Let $M$ be a rotational embedded surface in Euclidean 4-space given by the surface patch (2). Then the Gauss map of $M$ cannot be harmonic.

**Proof.** Suppose the Gauss map of the rotational embedded surface is harmonic. Then by (13), $A(s) = B(s) = D(s) = 0$. Thus, from the first equation of (12) we get $\kappa = 0$, which is a contradiction. \qed

**References**


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