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On almost complex structures in the cotangent bundle

Arif Salimov, Aydin Gezer and Seher Aslancı

Abstract

E. M. Patterson and K. Yano studied vertical and complete lifts of tensor fields and connections from a manifold M_n to its cotangent bundle $T^*(M_n)$. Afterwards, K. Yano studied the behavior on the cross-section of the lifts of tensor fields and connections on a manifold M_n to $T^*(M_n)$ and proved that when φ defines an integrable almost complex structure on M_n, its complete lift ${}^C\varphi$ is a complex structure. The main result of the present paper is the following theorem: Let φ be an almost complex structure on a Riemannian manifold M_n. Then the complete lift ${}^C\varphi$ of φ, when restricted to the cross-section determined by an almost analytic 1-form ω on M_n, is an almost complex structure.

Key word and phrases: Almost complex structure, cotangent bundle, cross-section, Nijenhuis tensor, analytic tensor field.

1. Preliminaries

Let M_n be an n-dimensional manifold and $T^*(M_n)$ its cotangent bundle. We denote by $\Im_s^r(M_n)$ the set of all tensor fields of type (r,s) on M_n. Similarly, we denote by $\Im_s^r(T^*(M_n))$ the corresponding set on $T^*(M_n)$.

In this section, we shall summarize all the basic definitions and results on cross-section in $T^*(M_n)$ that are needed later. Let M_n be an n-dimensional manifold of class C^∞ and $T^*(M_n)$ its cotangent bundle over M_n. If x^i are local coordinates in a neighborhood U of a point $x \in M_n$, then a covector P at x which is an element of $T^*(M_n)$ is expressible in the form (x^i, p_i), where p_i are components of P with respect to the natural frame ∂_i. We may consider $(x^i, p_i) = (x^i, x^{\bar{\imath}}) = x^J$, $i = 1, ..., n;$ $\bar{\imath} = n+1, ..., 2n;$ $J = 1, ..., 2n$ as local coordinates in a neighborhood $\pi^{-1}(U)$(π is the natural projection $T^*(M_n)$ onto M_n).

Now, consider $X \in \Im_0^1(M_n)$ and $\theta \in \Im_1^0(M_n)$, then ${}^CX$ (complete lift) and ${}^V\theta$ (vertical lift) have, respectively, components [5, p. 236], [6]

$$ {}^CX = \begin{pmatrix} X^h \\ -p_m \partial_h X^m \end{pmatrix}, \quad {}^V\theta = \begin{pmatrix} 0 \\ \theta_h \end{pmatrix} \tag{1.1} $$

with respect to the coordinates $(x^h, x^{\bar{h}})$ in $T^*(M_n)$, where X^h and θ_h are local components of X and θ.

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For $\varphi \in \Im_1^1(M_n)$, we can define a vector field $\gamma\varphi \in \Im_0^1(T^*(M_n))$ [5, p.232], [6]:

$$\gamma\varphi = \begin{pmatrix} 0 \\ p_j \varphi_i^j \end{pmatrix} \tag{1.2}$$

where φ_i^j are local components of φ in M_n. Clearly, we have $(\gamma\varphi)\ {}^V f = 0$ for any $f \in \Im_0^0(M_n)$, where ${}^V f = f \circ \pi$ is a vertical lift of f. So that $\gamma\varphi$ is a vertical vector field.

Suppose that there is given a 1-form $\omega \in \Im_1^0(M_n)$ whose local expression is $\omega = \omega_i(x)dx^i$. Then the correspondence $x \to \omega_x$, ω_x being the value of ω at $x \in M_n$, determines a mapping $\beta_\omega : M_n \to T^*(M_n)$, such that $\pi \circ \beta_\omega = id_{M_n}$ and n-dimensional submanifold $\beta_\omega(M_n)$ of $T^*(M_n)$ is called the cross-section determined by ω and its parametric representations are as follows:

$$\begin{cases} x^k = x^k, \\ p_k = \omega_k(x^1, ..., x^n), \end{cases} \tag{1.3}$$

with respect to the coordinates (x^k, p_k) in $T^*(M_n)$. Differentiating (1.3) by x^j, we see that n tangent vector fields B_j to $\beta_\omega(M_n)$ have component

$$B_j^K = (\frac{\partial x^K}{\partial x^j}) = \begin{pmatrix} \delta_j^k \\ \partial_j \omega_k \end{pmatrix} \tag{1.4}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T^*(M_n)$.

On the other hand, the fibre being represented by

$$\begin{cases} x^k = \text{const.}, \\ p_k = p_k. \end{cases} \tag{1.5}$$

On differentiating (1.5) by p_j, we see that n tangent vector fields $C_{\bar{j}}$ to the fibre have components

$$C_{\bar{j}}^K = (\frac{\partial x^K}{\partial p_j}) = \begin{pmatrix} 0 \\ \delta_k^j \end{pmatrix} \tag{1.6}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T^*(M_n)$. $2n$ local vector fields B_j and $C_{\bar{j}}$, being linearly independent, form a frame along the cross-section. We call this the adapted (B, C)-frame along the cross-section [4]. Taking account of (1.1) and (1.2) on the cross-section, we can see that ${}^C X$, ${}^V \theta$ and $\gamma\varphi$ have along $\beta_\omega(M_n)$ components of the form [4], (see also [5])

$${}^C X = \begin{pmatrix} X^j \\ -L_X \omega_j \end{pmatrix}, \ {}^V \theta = \begin{pmatrix} 0 \\ \theta_j \end{pmatrix}, \ \gamma\varphi = \begin{pmatrix} 0 \\ \omega_h \varphi_j^h \end{pmatrix} \tag{1.7}$$

with respect to the adapted (B, C)-frame. Similarly, if $N \in \Im_2^1(M_n)$, then $\gamma N \in \Im_1^1(T^*(M_n))$ is an affinor field along $\beta_\omega(M_n)$ with components [5, p. 232]

$$\gamma N = \begin{pmatrix} 0 & 0 \\ N_{ij}^h \omega_h & 0 \end{pmatrix} \tag{1.8}$$

with respect to the adapted (B, C)-frame, where S_{ij}^h are local components of S in M_n (For applications of γN, see the formula (2.8)).

2. Main results

Let $\varphi \in \Im_1^1(M_n)$ and $\omega \in \Im_1^0(M_n)$. We define an operator

$$\Phi_\varphi : \Im_1^0(M_n) \to \Im_2^0(M_n)$$

associated with φ and applied to the 1-form ω by

$$\begin{aligned}(\Phi_\varphi \omega)(X;Y) &= (L_{\varphi X}\omega - L_X\tilde{\omega})(Y) = \\ &= (\varphi X)(\omega(Y)) - X(\omega(\varphi Y)) + \omega((L_Y\varphi)X),\end{aligned}$$

where $\tilde{\omega}(Y) = (\omega \circ \varphi)(X) = \omega(\varphi Y)$ for any $X, Y \in \Im_0^1(M_n)$.

When φ is an almost complex structure, a 1-form satisfying $\Phi_\varphi\omega = 0$ is said to be almost analytic [5, p. 309].

In a Riemannian connection $\triangledown$, the equation of almost analytic 1-form ω:

$$(\varphi X)(\omega(Y)) - X(\omega(\varphi Y)) + \omega((L_Y\varphi)X) = 0$$

may be written as

$$(\triangledown_{\varphi X}\omega)(Y) - (\triangledown_X\omega)(\varphi Y) - \omega((\triangledown_X\varphi)Y) + \omega((\triangledown_Y\varphi)X) = 0, \tag{2.1}$$

which is equivalent to the condition for the almost analyticity. Thus, the equation (2.1) is an expression of the condition for the 1-form ω to be almost analytic in terms a Riemannian connection ∇.

Remark: *A tensor field* $\eta \in \Im_2^0(M_n)$ *which satisfies*

$$\eta(\varphi X, Y) = \eta(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_n)$ *is said to be pure.* Applications of this type tensor fields are studied by many authors (for example see [1–3]).

From (2.1), taking the alternation with respect to X and Y, we find that

$$(\triangledown_{\varphi X}\omega)(Y) - (\triangledown_{\varphi Y}\omega)(X) + (\triangledown_Y\omega)(\varphi X) - (\triangledown_X\omega)(\varphi Y) = 0,$$

i.e. $(\triangledown_X\omega)Y - (\triangledown_Y\omega)X = (\wedge\nabla\omega)(X,Y)$ is the pure 2-form with respect to the structure φ for an almost analytic 1-form ω on a Riemannian manifold.

We calculate

$$\begin{aligned}-\omega((\triangledown_X\varphi)Y) + \omega((\triangledown_Y\varphi)X) = -\omega((\triangledown_X\varphi)Y) \\ +(\triangledown_X\omega)(\varphi Y) - (\triangledown_X\omega)(\varphi Y) + \omega((\triangledown_Y\varphi)X) \\ +(\triangledown_Y\omega)(\varphi X) - (\triangledown_Y\omega)(\varphi X) = -(\triangledown_X\omega \circ \varphi)Y \\ +(\triangledown_X\omega)(\varphi Y) + (\triangledown_Y\omega \circ \varphi) - (\triangledown_Y\omega)(\varphi X).\end{aligned} \tag{2.2}$$

By virtue of (2.2), the equation (2.1) is written as

$$(\nabla_Y\tilde{\omega})X-(\nabla_X\tilde{\omega})Y=(\nabla_Y\omega)(\varphi X)-(\nabla_{\varphi X}\omega)(Y). \tag{2.3}$$

If we substitute φX into X, then the equation (2.3) may also be written as

$$-((\nabla_Y\omega)X-(\nabla_X\omega)Y)=(\nabla_Y\tilde{\omega})\varphi X-(\nabla_{\varphi X}\tilde{\omega})Y$$

or

$$(\nabla_Y\tilde{\tilde{\omega}})X-(\nabla_X\tilde{\tilde{\omega}})Y=(\nabla_Y\tilde{\omega})\varphi X-(\nabla_{\varphi X}\tilde{\omega})Y, \tag{2.4}$$

where $\tilde{\tilde{\omega}}=\tilde{\omega}\circ\varphi$. The equation (2.4) is condition that $\tilde{\omega}\in\Im_1^0(M_n)$ be almost analytic.

From equations (2.3) and (2.4), we have

Theorem 1 *If a 1-form ω on a Riemannian manifold with an almost complex structure φ is almost analytic, then the 1-form $\tilde{\omega}=\omega\circ\varphi$ is also almost analytic.*

We shall now prove the following proposition.

Proposition *In a Riemannian manifold, the condition*

$$\Phi_\varphi\tilde{\omega}=(\Phi_\varphi\omega)\circ\varphi+\omega\circ N_\varphi$$

holds, where N_φ is the Nijenhuis tensor of φ.

Proof. We shall now apply the operator Φ_φ to the 1-form $\tilde{\omega}=\omega\circ\varphi$

$$(\Phi_\varphi\tilde{\omega})(X;Y)=(L_{\varphi X}\tilde{\omega}-L_X(\tilde{\omega}\circ\varphi))(Y)=(L_{\varphi X}(\omega\circ\varphi)-L_X((\omega\circ\varphi)\circ\varphi))(Y)$$

$$=((L_{\varphi X}\omega)\circ\varphi+\omega\circ(L_{\varphi X}\varphi)-(L_X(\omega\circ\varphi))\circ\varphi-(\omega\circ\varphi)\circ(L_X\varphi))(Y)$$

$$=(L_{\varphi X}\omega-L_X(\omega\circ\varphi))(\varphi Y)+(\omega\circ(L_{\varphi X}\varphi)-(\omega\circ\varphi)\circ(L_X\varphi))(Y)$$

$$=(L_{\varphi X}\omega-L_X(\omega\circ\varphi))(\varphi Y)+\omega((L_{\varphi X}\varphi)Y)-\omega(\varphi(L_X\varphi)Y)$$

$$=(\Phi_{\varphi X}\omega)(\varphi Y)+\omega([\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]+\varphi^2[X,Y])$$

$$=(\Phi_\varphi\omega)(X;\varphi Y)+\omega(N_\varphi(X,Y)). \tag{2.5}$$

Thus, the proof is complete. □

We note that the 1-form ω in Proposition *is not necessary to be almost analytic*, in general. In particular, if the 1-form ω is almost analytic, then from Theorem 1 and Proposition, we have

Theorem 2 *For an almost analytic 1-form ω on a Riemannian manifold with an almost complex structure φ, we have the following equation.*

$$\omega\circ N_\varphi=0.$$

Let $\varphi \in \Im_1^1(M_n)$. Then, the complete lift ${}^C\varphi$ of φ along the cross-section ω to $T^*(M_n)$ has local components of the form

$$ {}^C\varphi = \begin{pmatrix} \varphi_i^h & 0 \\ (\partial_i \varphi_h^a - \partial_h \varphi_i^a)\omega_a - \varphi_i^t \partial_t \omega_h + \varphi_h^t \partial_i \omega_t & \varphi_i^h \end{pmatrix} $$

with respect to the adapted (B,C)-frame [4]. We consider that the local vector fields

$$ {}^C X_{(i)} = {}^C(\frac{\partial}{\partial x^i}) = {}^C(\delta_i^h \frac{\partial}{\partial x^h}) = \begin{pmatrix} X^i \\ 0 \end{pmatrix} $$

and

$$ {}^V X^{(\bar{\imath})} = {}^V(dx^i) = {}^V(\delta_h^i dx^h) = \begin{pmatrix} 0 \\ \delta_h^i \end{pmatrix} $$

$i = 1, ..., n;\ \bar{\imath} = n+1, ..., 2n$ span the module of vector fields in $\pi^{-1}(U)$. Hence, any tensor fields is determined in $\pi^{-1}(U)$ by their actions on ${}^C X$ and ${}^V\theta$ for any $X \in \Im_0^1(M_n)$ and $\theta \in \Im_1^0(M_n)$. The complete lift ${}^C\varphi$ has the properties

$$ \begin{cases} {}^C\varphi({}^C X) = {}^C(\varphi(X)) + \gamma(L_X \varphi), \\ {}^C\varphi({}^V\theta) = {}^V(\varphi(\theta)), \end{cases} \tag{2.6} $$

which characterize ${}^C\varphi$, where $\varphi(\theta) \in \Im_1^0(M_n)$.

Theorem 3 *Let M_n be a Riemannian manifold with an almost complex structure φ. Then the complete lift ${}^C\varphi \in \Im_1^1(T^*(M_n))$ of φ, when restricted to the cross-section determined by an almost analytic 1-form ω on M_n, is an almost complex structure.*

Proof. Let $\varphi, \psi \in \Im_1^1(M_n)$ and $N \in \Im_2^1(M_n)$, using (1.7), (1.8) and (2.6), we have

$$ \gamma(\varphi \mp \psi) = \gamma(\varphi) \mp \gamma(\psi), \quad {}^C\varphi(\gamma\psi) = \gamma(\psi \circ \varphi), \quad (\gamma N)({}^C X) = \gamma N_X \tag{2.7} $$

where N_X is the tensor field of type (1,1) on M_n defined by $N_X(Y) = N(X,Y)$ for any $Y \in \Im_0^1(M_n)$.

If $X \in \Im_0^1(M_n)$, then from (2.6) and (2.7), we have

$$ ({}^C\varphi)^2({}^C X) = ({}^C\varphi \circ {}^C\varphi)({}^C X) = {}^C\varphi({}^C\varphi({}^C X)) = {}^C\varphi({}^C(\varphi(X)) $$

$$ +\gamma(L_X\varphi)) = {}^C\varphi({}^C(\varphi(X))) + {}^C\varphi(\gamma(L_X\varphi)) = {}^C(\varphi(\varphi(X))) $$

$$ +\gamma(L_{\varphi X}\varphi) + \gamma((L_X\varphi)\circ\varphi) = {}^C((\varphi\circ\varphi)(X)) + \gamma(L_{\varphi X}\varphi + (L_X\varphi)\circ\varphi) $$

$$ = {}^C(\varphi^2)({}^C X) - \gamma(L_X(\varphi\circ\varphi)) + \gamma(L_{\varphi X}\varphi + (L_X\varphi)\circ\varphi) $$

$$ = {}^C(\varphi^2)({}^C X) + \gamma(L_{\varphi X}\varphi - \varphi(L_X\varphi)) = {}^C(\varphi^2)({}^C X) + \gamma(N_{\varphi, X}) $$

$$= {}^{C}(\varphi^2)({}^{C}X) + (\gamma N_\varphi)({}^{C}X), \tag{2.8}$$

where $N_{\varphi,X}(Y) = (L_{\varphi X}\varphi - \varphi(L_X\varphi))(Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + \varphi^2[X, Y] = N_\varphi(X, Y)$ is nothing but the Nijenhuis tensor constructed by φ and γN_φ has local coordinates of the form $\gamma N_\varphi = \begin{pmatrix} 0 & 0 \\ N_{ij}^h \omega_h & 0 \end{pmatrix}$ (see (1.8)).

Similarly, if $\theta \in \Im_1^0(M_n)$, then by (2.6), we have

$$({}^{C}\varphi)^2({}^{V}\theta) = ({}^{C}\varphi \circ {}^{C}\varphi)({}^{V}\theta) = {}^{C}\varphi({}^{C}\varphi({}^{V}\theta)) = {}^{C}\varphi({}^{V}(\varphi(\theta))$$

$$= {}^{V}(\varphi(\varphi(\theta)) = {}^{V}((\varphi \circ \varphi)(\theta)) = {}^{C}(\varphi^2)({}^{V}\theta) \tag{2.9}$$

By virtue of Theorem 2, we can easily say that $\gamma N_\varphi = 0$. From (2.8), (2.9) and linearity of the complete lift, we have

$$({}^{C}\varphi)^2 = {}^{C}(\varphi^2) = {}^{C}(-I_{M_n}) = -I_{T^*(M_n)}.$$

This completes the proof. □

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References

[1] Magden A., Salimov A. A.: Complete lifts of tensor fields on a pure cross-section in the tensor bundle, J. Geom., 93(1-2), 128-138 (2009).

[2] Salimov A.A.: Almost ψ-holomorphic tensors and their properties, Dokl. Akad. Nauk., 324(3), 533-536 (1992).

[3] Salimov A. A., Iscan M., Etayo F.: Paraholomorphic B-manifold and its properties, Topology Appl., 154(4), 925-933 (2007).

[4] Yano K.: Tensor fields and connections on cross-sections in the cotangent bundle, Tohoku Math. J., 19(1), 32-48 (1967).

[5] Yano K. and Ishihara S.: Tangent and Cotangent Bundles, Marcel Dekker, Inc., New York 1973.

[6] Yano K. and Patterson E. M.: Vertical and complete lifts from a manifold to its cotangent bundle, J. Math. Soc. Japan, 19, 91-113 (1967).

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