Turkish Journal of Mathematics

Volume 35 | Number 3 Article 11

1-1-2011

Weingarten quadric surfaces in a Euclidean 3-space

MIN HEE KIM

DAE WON YOON

Turk J Math
35 (2011) , 479 – 485.

doi:10.3906/mat-0910-89

Weingarten quadric surfaces in a Euclidean 3-space

Min Hee Kim and Dae Won Yoon

Abstract

In this paper, we study quadric surfaces in a Euclidean 3-space. Furthermore, we classify quadric surfaces in a Euclidean 3-space in terms of the Gaussian curvature and the mean curvature.

Key Words: Quadric surface, Weingarten surface, Gaussian curvature, mean curvature.

1. Introduction

A Weingarten surface is a surface on which there exists the Jacobi equation $\Phi(k_1, k_2) = \det\begin{pmatrix} (k_1)_s & (k_1)_t \\ (k_2)_s & (k_2)_t \end{pmatrix} = 0$ between the principal curvatures k_1, k_2 on a surface, or equivalently, the Jacobi equation $\Psi(H, K) = 0$ between the Gaussian curvature K and the mean curvature H on a surface, where $(k_1)_s = \frac{\partial k_1}{\partial s}$ and $(k_2)_t = \frac{\partial k_2}{\partial t}$.

On the other hand, if a surface satisfies a linear equation $ak_1 + bk_2 = c$ or $aK + bH = c$ for some real numbers a, b, c with $(a, b) \neq (0, 0)$, then it is said to be a linear Weingarten surface.

For the study of these surfaces, W. Kühnel ([5]) investigated ruled Weingarten surface in a Euclidean 3-space $\mathbb{E}^3$. F. Dillen and W. Kühnel ([2]) and Y. H. Kim and D. W. Yoon ([4]) gave a classification of ruled Weingarten surfaces and ruled linear Weingarten surfaces in a Minkowski 3-space $\mathbb{E}^3_1$, respectively. D. W. Yoon ([10]) classified ruled linear Weingarten surface in $\mathbb{E}^3$. Recently, M. I. Munteanu and I. Nistor ([9]) and R. Lopéz ([6, 7]) studied polynomial translation (linear) Weingarten surfaces and a cyclic linear Weingarten surface in a Euclidean 3-space, respectively. In [8] R. Lopéz classified all parabolic linear Weingarten surfaces in hyperbolic 3-space.

In this paper, we study quadric surfaces in a Euclidean 3-space and prove the following classification theorem.

Theorem A. *Let M be a quadric surface in a Euclidean 3-space with non-zero Gaussian curvature everywhere. If M satisfies the Jacobi equation with respect to the Gaussian curvature K and the mean curvature H, that is,*

$$\Psi(K, H) = 0, \tag{1.1}$$

2000 *AMS Mathematics Subject Classification:* 53B25, 53C40.

then M is an open part of one of a hyperboloid of two sheets, a hyperboloid of one sheet, an ellipsoid or an elliptic paraboloid.

Throughout this paper, we assume that all objects are smooth and all surfaces are Riemannian, unless otherwise mentioned.

2. Weingarten quadric surfaces in $\mathbb{E}^3$

A subset M of a Euclidean 3-space $\mathbb{E}^3$ is called a quadric surface if it is the set of points (x_1, x_2, x_3) satisfying the following equation of the second degree:

$$\sum_{i=1}^{3} a_{ij} x_i x_j + \sum_{i=1}^{3} b_i x_i + c = 0, \tag{2.1}$$

where a_{ij}, b_i, c are all real numbers. Suppose that M is not a plane. Then A is not a zero matrix and we may assume without loss of generality that the matrix $A = (a_{ij})$ is symmetric. By applying a coordinate transformation in $\mathbb{E}^3$ if necessary, M is either ruled surface, or one of the following two kinds ([1]):

$$x_3^2 - a x_1^2 - b x_2^2 = c, \quad abc \neq 0 \tag{2.2}$$

or

$$x_3 = \frac{a}{2} x_1^2 + \frac{b}{2} x_2^2, \quad a > 0,\ b > 0. \tag{2.3}$$

If a surface satisfies the equation (2.2), it is said to be a *quadric surface of the first kind* and we call a surface satisfying (2.3) a *quadric surface of the second kind.*

Let $x : M \longrightarrow \mathbb{E}^3$ be a quadric surface of the first kind in $\mathbb{E}^3$. Then M is parametrized by

$$x(u, v) = (u, v, (au^2 + bv^2 + c)^{\frac{1}{2}}). \tag{2.4}$$

Let's denote the function $au^2 + bv^2 + c$ by W. Then, the components E, F and G of the first fundamental form are given by

$$E = 1 + \frac{a^2u^2}{W}, \quad F = \frac{abuv}{W}, \quad G = 1 + \frac{b^2v^2}{W}.$$

For later use, we define smooth function q

$$q = ||x_u \times x_v||^2 = 1 + \frac{a^2u^2}{W} + \frac{b^2v^2}{W}. \tag{2.5}$$

Then, the unit normal vector filed U of the surface M is given by

$$U = \frac{1}{q^{\frac{1}{2}}} \left(-\frac{au}{W^{\frac{1}{2}}}, -\frac{bv}{W^{\frac{1}{2}}}, 1 \right),$$

leading to the components of the second fundamental form on M

$$e = \frac{1}{q^{\frac{1}{2}} W^{\frac{3}{2}}} (aW - a^2u^2), \quad f = -\frac{abuv}{q^{\frac{1}{2}} W^{\frac{3}{2}}}, \quad g = \frac{1}{q^{\frac{1}{2}} W^{\frac{3}{2}}} (bW - b^2v^2).$$

Hence, the Gaussian curvature K and the mean curvature H are given respectively, by

$$K = \frac{1}{q^2W^2}abc, \tag{2.6}$$

$$H = \frac{1}{2q^{\frac{3}{2}}W^{\frac{3}{2}}}H_1, \tag{2.7}$$

where $H_1 = (a+b)c + (ab + a^2b)u^2 + (ab + ab^2)v^2$. From (2.6) a quadric surface of the first kind given by (2.4) has a non-zero Gaussian curvature everywhere.

Differentiating K and H with respect to u and v respectively, we get

$$\begin{cases} K_u &= -\frac{4a^2(a+1)bc}{q^3W^3}u, \\ K_v &= -\frac{4ab^2(b+1)c}{q^3W^3}v, \end{cases} \tag{2.8}$$

$$\begin{cases} H_u &= -\frac{1}{4q^{\frac{5}{2}}W^{\frac{7}{2}}}\{6a(a+1)uH_1W - 4uW(ab+a^2b)(W+a^2u^2+b^2v^2)\}, \\ H_v &= -\frac{1}{4q^{\frac{5}{2}}W^{\frac{7}{2}}}\{6b(b+1)vH_1W - 4vW(ab+ab^2)(W+a^2u^2+b^2v^2)\}. \end{cases} \tag{2.9}$$

Suppose that M is a quadric surface of the first kind satisfying the condition (1.1) . Then, we have

$$K_uH_v - K_vH_u = 0. \tag{2.10}$$

Equation (2.10) together with (2.8) and (2.9) becomes

$$\begin{aligned} &a^2(a+1)bcu\{6b(b+1)vH_1W - 4vW(ab+ab^2)(W+a^2u^2+b^2v^2)\} \\ &- ab^2(b+1)cv\{6a(a+1)uH_1W - 4uW(ab+a^2b)(W+a^2u^2+b^2v^2)\} = 0. \end{aligned} \tag{2.11}$$

The direct computation of the left hand side of (2.11) gives a polynomial in u and v with constants as the coefficients by adjusting the power of the functions W and H_1. Therefore, the coefficients of u^5v and uv^5 in (2.11) give, respectively

$$-4a^4b^2c(a+1)^2(b+1)(a-b) = 0,$$
$$-4a^2b^4c(a+1)(b+1)^2(a-b) = 0.$$

Thus, we have $a=-1$, $b=-1$ or $a=b$ because of $abc \neq 0$. In this case, equation (2.11) holds identically.

1. Case $a=b$. Then a parametrization of M is given by

$$x(u,v) = (u, v, (au^2 + av^2 + c)^{\frac{1}{2}}).$$

(1-i) If $a, c > 0$, then M is an open part of a hyperboloid of two sheets defined by

$$-\frac{x^2}{p^2} - \frac{y^2}{p^2} + z^2 = r^2 \tag{2.12}$$

for some non-zero constants p and r.

(1-ii) If $a > 0$ and $c < 0$, then then M is an open part of a hyperboloid of one sheet given by

$$\frac{x^2}{p^2} + \frac{y^2}{p^2} - z^2 = r^2. \tag{2.13}$$

(1-iii) If $a < 0$ and $c > 0$, then M is given by

$$\frac{x^2}{p^2} + \frac{y^2}{p^2} + z^2 = r^2, \tag{2.14}$$

which is the equation of an ellipsoid.

The case of $a, c < 0$ can never occur.

2. Case $a = -1$. Then a parametrization of M is given by

$$x(u,v) = (u, v, (-u^2 + bv^2 + c)^{\frac{1}{2}}).$$

(2-i) If $b, c > 0$, then M is an open part of a hyperboloid of one sheet defined by

$$x^2 - \frac{y^2}{p^2} + z^2 = r^2 \tag{2.15}$$

for some non-zero constants p and r.

(2-ii) If $b > 0$ and $c < 0$, then then M is an open part of a hyperboloid of two sheets given by

$$-x^2 + \frac{y^2}{p^2} - z^2 = r^2. \tag{2.16}$$

(2-iii) If $b < 0$ and $c > 0$, then M is given by

$$x^2 + \frac{y^2}{p^2} + z^2 = r^2 \tag{2.17}$$

which is the equation of an ellipsoid.

The case of $b, c < 0$ can never occur.

3. Case $b = -1$. Then a parametrization of M is given by

$$x(u,v) = (u, v, (au^2 - v^2 + c)^{\frac{1}{2}}).$$

(3-i) If $a, c > 0$, then M is an open part of a hyperboloid of one sheet defined by

$$-\frac{x^2}{p^2} + y^2 + z^2 = r^2 \tag{2.18}$$

for some non-zero constants p and r.

(3-ii) If $a > 0$ and $c < 0$, then then M is an open part of a hyperboloid of two sheets given by

$$\frac{x^2}{p^2} - y^2 - z^2 = r^2. \tag{2.19}$$

(3-iii) If $a < 0$ and $c > 0$, then M is given by

$$\frac{x^2}{p^2} + y^2 + z^2 = r^2 \tag{2.20}$$

which is the equation of an ellipsoid.

The case of $b, c < 0$ can never occur.

Thus, we have the following theorems.

Theorem 2.1. *If M is a Weingarten quadric surface of the first kind in a Euclidean 3-space, then M is an open part of one of the following surfaces:*

1. *a hyperboloid of two sheets of the form (2.12), (2.16) or (2.19).*
2. *a hyperboloid of one sheet of the form (2.13), (2.15) or (2.18).*
3. *an ellipsoid of the form (2.14), (2.17) or (2.20).*

Theorem 2.2. *Let M be a quadric surface of the first kind in a Euclidean 3-space. Then the Gaussian curvature K and the mean curvature H of M are related by the relation*

$$[(a+b)c + (ab + a^2b)u^2 + (ab + ab^2)v^2]^2 K = 4abc[c + (a + a^2)u^2 + (b + b^2)v^2]H^2$$

for some non-zero constants a, b, c.

Proof. It is obvious by (2.6) and (2.7). □

Let $x : M \longrightarrow \mathbb{E}^3$ be a quadric surface of the second kind in $\mathbb{E}^3$. Then M is parametrized by

$$x(u, v) = (u, v, \frac{a}{2}u^2 + \frac{b}{2}v^2). \tag{2.21}$$

On the other hand, the components E, F and G of the first fundamental form are obtained by

$$E = 1 + a^2u^2, \quad F = abuv, \quad G = 1 + b^2v^2.$$

We define the smooth function q as follows:

$$q = ||x_u \times x_v||^2 = 1 + a^2u^2 + b^2v^2, \tag{2.22}$$

which implies that the unit normal vector field U of the surface M is given by

$$U = \frac{1}{q^{\frac{1}{2}}}(-au, -bv, 1).$$

From this, the components of the second fundamental form on M are obtained by

$$e = \frac{a}{q^{\frac{1}{2}}}, \quad f = 0, \quad g = \frac{b}{q^{\frac{1}{2}}}.$$

Making use of the data described above, the Gaussian curvature K and the mean curvature H write as respectively, as

$$K = \frac{ab}{q^2}, \tag{2.23}$$

$$H = \frac{1}{2q^{\frac{3}{2}}} H_1, \tag{2.24}$$

where $H_1 = a^2bu^2 + ab^2v^2 + a + b$. Since $a, b > 0$, a quadric surface of the second kind given by (2.21) has a positive Gaussian curvature everywhere.

Differentiating K and H with respect to u and v respectively, we get

$$\begin{cases} K_u &= -\frac{4a^3bu}{q^3}, \\ K_v &= -\frac{4ab^3v}{q^3}, \end{cases} \tag{2.25}$$

$$\begin{cases} H_u &= -\frac{1}{q^{\frac{5}{2}}}(-\frac{3}{2}a^2uH_1 + a^2buq), \\ H_v &= -\frac{1}{q^{\frac{5}{2}}}(-\frac{3}{2}b^2vH_1 + ab^2vq). \end{cases} \tag{2.26}$$

Suppose that M is a Weingarten quadric surface of the second kind. Then, it satisfies

$$K_uH_v - K_vH_u = 0. \tag{2.27}$$

From (2.25) and (2.26) equation (2.27) writes as

$$a^3b^3((2a^2b - 2a^3)u^3v - (2ab^2 - 2b^3)uv^3 - (2a - 2b)uv) = 0. \tag{2.28}$$

This yields immediately $a = b$. Thus, M is given by

$$z = \frac{a}{2}x^2 + \frac{a}{2}y^2, \tag{2.29}$$

and this means that it is an elliptic paraboloid.

Thus, we have this theorem:

Theorem 2.3. *Let M be a Weingarten quadric surface of the second kind in a Euclidean 3-space. Then, M is an open part of an elliptic paraboloid given by (2.29).*

Theorem 2.4. *Let M be a quadric surface of the second kind in a Euclidean 3-space. Then the Gaussian curvature K and the mean curvature H of M are related by the relation*

$$(a^2bu^2 + ab^2v^2 + a + b)K = 4ab(a^2u^2 + b^2v^2 + 1)H^2 \tag{2.30}$$

for some non-zero positive constants a, b.

Proof. It is obvious by (2.23) and (2.24). □

Combining Theorem 2.1, Theorem 2.3 and main theorem in [5], we obtain the following, theorem.

Theorem 2.5 (Classification). *Let M be a Weingarten quadric surface in a Euclidean 3-space with non-zero Gaussian curvature everywhere. Then, M is an open part of one of the following surfaces:*

1. *a hyperboloid of two sheets of the form (2.12), (2.16) or (2.19).*
2. *a hyperboloid of one sheet of the form (2.13), (2.15) or (2.18).*
3. *an ellipsoid of the form (2.14), (2.17) or (2.20).*
4. *an elliptic paraboloid of the form (2.29).*

References

[1] Chen, B.-Y. and Dillen, F.: Quadrics of finite type. J. Geom. 38, 16-22 (1990).

[2] Dillen, F. and Kühnel, W.: Ruled Weingarten surfaces in Minkowski 3-space. Manuscripta Math. 98, 307-320 (1999).

[3] GaÍvez, J. A., Martińez, A. and Milań, F.: Linear Weingarten surfaces in $\mathbb{R}^3$. Monatsh. Math. 138, 133-144 (2003).

[4] Kim, Y. H. and Yoon, D. W.: Classification of ruled surfaces in Minkowski 3-spaces. J. Geom. Phys. 49, 89-100 (2004).

[5] Kühnel, W.: Ruled W-surfaces, Arch. Math. 62, 475-480 (1994).

[6] López, R.: On linear Weingarten surfaces. International J. Math. 19, 439-448 (2008).

[7] López, R.: Special Weingarten surfaces foliated by circles. Monatsh. Math. 154, 289-302 (2008).

[8] López, R.: Parabolic Weingarten surfaces in hyperbolic space. Publ. Math. Debrecen 74, 59-80 (2009).

[9] Munteanu, M. I. and Nistor, A. I.: Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space. arXiv:0809.4745v1 [math.DG] 27 Sep 2008.

[10] Yoon, D. W.: Some properties of the helicoid as ruled surfaces. JP Jour. Geom. Topology 2, 141-147 (2002).

Min Hee KIM, Dae Won YOON
Department of Mathematics Education and RINS,
Gyeongsang National University,
Jinju 660-701, South-KOREA
e-mail: 1205mh@hanmail.net, dwyoon@gnu.ac.kr

Received: 23.10.2009