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# A Fredholm alternative-like result on power bounded operators

Ali Ülger, Onur Yavuz

## Abstract

Let  $X$  be a complex Banach space and  $T : X \rightarrow X$  be a power bounded operator, i.e.,  $\sup_{n \geq 0} \|T^n\| < \infty$ . We write  $\mathcal{B}(X)$  for the Banach algebra of all bounded linear operators on  $X$ . We prove that the space  $\text{Ran}(I - T)$  is closed if and only if there exist a projection  $\theta \in \mathcal{B}(X)$  and an invertible operator  $R \in \mathcal{B}(X)$  such that  $I - T = \theta R = R\theta$ . This paper also contains some consequences of this result.

## 1. Introduction

Let  $X$  be a complex Banach space. It is well known that for every compact operator  $K : X \rightarrow X$ , the range of the operator  $I - K$  is closed. However, we cannot expect this to hold for an arbitrary bounded linear operator  $T : X \rightarrow X$ . So it is natural to ask when the range of the operator  $I - T$  is closed. In this paper, we answer this problem for power bounded operators by proving that, for a power bounded operator  $T$ , the range of the operator  $I - T$  is closed if and only if  $I - T$  can be written as a product of two commuting operators  $\theta$  and  $R$  where  $\theta$  is an idempotent and  $R$  is invertible. We also present some consequences of this result and it is essentially self-contained.

## 2. Main results

Let  $T : X \rightarrow X$  be a power bounded operator on  $X$ . If we renorm  $X$  with the norm  $\|x\| := \sup_{n \geq 0} \|T^n x\|$ , then  $T$  becomes a contraction on  $X$  with this new norm, that is,  $\|T\| \leq 1$ . For that reason we will work with a fixed contraction operator  $T$ . Clearly all of the results presented below are valid for power bounded operators. We will denote by  $\mathcal{B}(X)$  the Banach algebra of all bounded linear operators on  $X$ , and by  $\mathcal{B}(X^*)$  the Banach algebra of all bounded linear operators on the dual space  $X^*$ . Note that one can identify  $\mathcal{B}(X^*)$  with the dual space of the projective tensor space  $X^* \hat{\otimes} X$  [1, p. 230, Corollary 2]. So it carries a weak\* topology. The natural duality between the spaces  $\mathcal{B}(X^*)$  and  $X^* \hat{\otimes} X$  is given by  $\langle B, f \otimes x \rangle = \langle B(f), x \rangle$  for every operator  $B \in \mathcal{B}(X^*)$ , every functional  $f \in X^*$ , and every vector  $x \in X$ .

We start with the following observation which will be used in the proof of our main theorem.

**Lemma 2.1** *Let  $T \in \mathcal{B}(X)$  and assume that  $\text{Ran}(T)$  is closed. Then the following are equivalent:*

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1.  $\text{Ker}(T^*) = \text{Ker}(T^{*2})$ .

2.  $\overline{\text{Ran}(T^2)} = \text{Ran}(T)$ .

**Proof.** (1)  $\Rightarrow$  (2): Since

$$\text{Ran}(T)^\perp = \text{Ker}(T^*) = \text{Ker}(T^{*2}) = \text{Ran}(T^2)^\perp$$

and  $\text{Ran}(T)$  is closed, we have  $\overline{\text{Ran}(T^2)} = \text{Ran}(T)$  by Hahn-Banach Theorem.

(2)  $\Rightarrow$  (1): We have

$$\text{Ker}(T^{*2}) = \overline{\text{Ran}(T^2)}^\perp \quad \text{and} \quad \text{Ker}(T^*) = \text{Ran}(T)^\perp.$$

As  $\text{Ran}(T)^\perp = \overline{\text{Ran}(T^2)}^\perp$ , it follows that  $\text{Ker}(T^*) = \text{Ker}(T^{*2})$ . □

The following lemma is proved in ([3], p. 69) for nonexpansive (not necessarily linear) mappings. One can also find a proof of this result in the monograph [2, Lemma 9.4].

**Lemma 2.2** *Let  $T \in \mathcal{B}(X)$  be a contraction and  $S_\lambda = \lambda I + (1 - \lambda)T$  for  $0 < \lambda < 1$ . Then  $\lim_{n \rightarrow \infty} \|S_\lambda^{n+1}(x) - S_\lambda^n(x)\| = 0$  for every  $x \in X$ .*

The following results, which will be needed for the proof of Theorem 2.9, follow without much difficulty from the preceding lemma.

**Lemma 2.3** *Let  $T \in \mathcal{B}(X)$  be a contraction. Then*

$$\text{Ker}(I - T) \cap \text{Ran}(I - T) = \{0\}.$$

**Proof.** Let  $S = S_{\frac{1}{2}} = \frac{I+T}{2}$ . Then the range and the kernel of the operator  $I - S$  coincide with those of  $I - T$ . Let  $y \in \text{Ker}(I - S) \cap \text{Ran}(I - S)$ . Since  $(I - S)(y) = 0$ , that is,  $S(y) = y$ , we have  $S^n(y) = y$  for every  $n$ . We have  $(I - S)(x) = y$  for some  $x \in X$ , that is,  $y = x - S(x)$ . By applying the operator  $S^n$  to this equality we get  $y = S^n x - S^{n+1} x$ . By the previous lemma,  $\|S^n x - S^{n+1} x\|$  converges to 0 as  $n \rightarrow \infty$ , which implies that  $y = 0$ . So  $\text{Ker}(I - S) \cap \text{Ran}(I - S) = \{0\}$ . Thus,  $\text{Ker}(I - T) \cap \text{Ran}(I - T) = \{0\}$ . □

**Lemma 2.4** *Let  $T : X \rightarrow X$  be a contraction. Then,*

$$\text{Ker}(I - T^*) = \text{Ker}((I - T^*)^2).$$

**Proof.** By Lemma 2.3, we have  $\text{Ker}(I - T^*) \cap \text{Ran}(I - T^*) = \{0\}$ . Let  $x \in \text{Ker}((I - T^*)^2)$ . Then the element  $y = (I - T^*)(x)$  is in the intersection of the spaces  $\text{Ker}(I - T^*)$  and  $\text{Ran}(I - T^*)$ , which is trivial. Hence  $x \in \text{Ker}(I - T^*)$ , which implies that  $\text{Ker}((I - T^*)^2) \subseteq \text{Ker}(I - T^*)$ . The other inclusion is always true. □

The following lemma will be needed in the proof of Theorem 2.6.

**Lemma 2.5** *Let  $(R_\alpha)$  be a bounded net in  $\mathcal{B}(X^*)$ . Then we have the following:*

1. The net  $(R_\alpha)$  converges to  $R \in \mathcal{B}(X^*)$  in the weak\* topology if and only if  $\langle R_\alpha(f), x \rangle$  converges to  $\langle R(f), x \rangle$  for every  $x \in X$  and  $f \in X^*$ .
2. If  $(R_\alpha)$  converges to  $R$  in the weak\* topology, then  $(R_\alpha \circ Q)$  converges to  $R \circ Q$  in the weak\* topology for every operator  $Q \in \mathcal{B}(X^*)$ .
3. If  $(R_\alpha)$  converges to  $R$  in the weak\* topology, then  $(L^* \circ R_\alpha)$  converges to  $L^* \circ R$  for every operator  $L \in \mathcal{B}(X)$ .

**Proof.** Assertion (1) follows from the fact that the net  $(R_\alpha)$  is bounded and the set of atomic tensors  $f \otimes x$  are total in the space  $X^* \hat{\otimes} X$ . Assertions (2) and (3) follow, respectively, from the identities

$$\langle R_\alpha \circ Q, f \otimes x \rangle = \langle R_\alpha, Q(f) \otimes x \rangle.$$

$$\langle L^* \circ R_\alpha, f \otimes x \rangle = \langle (L^* \circ R_\alpha)(f), x \rangle = \langle R_\alpha(f), L(x) \rangle = \langle R_\alpha, f \otimes L(x) \rangle.$$

□

The next result shows that for a power bounded operator  $T \in \mathcal{B}(X)$ , the kernel of the operator  $I - T^*$  is always complemented in  $X^*$ .

**Theorem 2.6** *Let  $T \in \mathcal{B}(X)$  be a contraction. Then there exists a projection  $P \in \mathcal{B}(X^*)$  whose range is  $\text{Ker}(I - T^*)$  and whose kernel contains  $\text{Ran}(I - T^*)$ .*

**Proof.** Let  $S = \frac{I+T}{2}$ . Since  $\|S\| \leq 1$ , the set  $\{S^{*n} : n \geq 0\}$  is bounded. So by Alaoglu theorem, the sequence  $(S^{*n})$  has a convergent subnet  $(S^{*n_i})$  that converges to an operator  $P$  in  $(\mathcal{B}(X^*), w^*)$ . By Lemma 2.2, we have  $\langle S^{*n+1} f - S^{*n} f, x \rangle \rightarrow 0$  for every  $f \in X^*$  and  $x \in X$ . This, together with the fact that  $P$  is the weak\* limit of the net  $(S^{*n_i})$ , implies that

$$P \circ S^* = S^* \circ P = P.$$

Then  $S^{*n_i} \circ P = P$  for every  $n_i$ . So, passing again to the limit in  $(\mathcal{B}(X^*), w^*)$  and using Lemma 2.5, we get  $P^2 = P$ . This proves that every cluster point of the sequence  $(S^{*n})$  is a projection. As  $S^* = \frac{I+T^*}{2}$ , we also have

$$T^* \circ P = P \circ T^* = P.$$

So  $\text{Ran}(P) \subseteq \text{Ker}(I - T^*)$ . On the other hand, for  $f \in \text{Ker}(I - T^*)$ , we have  $T^*(f) = f$ , so  $S^*(f) = f$ . Hence  $S^{*n_i} f = f$ , which implies that  $P(f) = f$ . Hence  $\text{Ran}(P) = \text{Ker}(I - T^*)$ . To prove the inclusion  $\text{Ran}(I - T^*) \subseteq \text{Ker}(P)$ , let  $f \in X^*$  be an arbitrary element and  $g = f - T^*(f)$ . Then, we have  $P(g) = (P \circ T^*)(f) = 0$ . Hence  $\text{Ran}(I - T^*) \subseteq \text{Ker}(P)$ . □

As an important corollary of this theorem we present the following result.

**Corollary 2.7** *Let  $T$  be a power bounded operator and  $S = \frac{I+T}{2}$ . Then*

1.  $\overline{\text{Ran}(I - T)} = X$  if and only if  $S^n(x) \rightarrow 0$  weakly for every  $x \in X$ .
2.  $\text{Ran}(I - T) = X$  if and only if  $\|S^n\| \rightarrow 0$ .

**Proof.** (1): First assume that  $\langle S^n x, f \rangle \rightarrow 0$  for every  $x \in X$  and  $f \in X^*$ . Then  $\langle x, S^{*n} f \rangle \rightarrow 0$  for every  $x \in X$  and  $f \in X^*$ , that is, the sequence  $(S^{*n})$  converges to 0 in the weak\* topology of  $\mathcal{B}(X^*)$ . Then the projection  $P$  obtained in Theorem 2.6 is trivial, which in turn implies that  $\text{Ker}(I - T^*) = \{0\}$ . Thus,  $\overline{\text{Ran}(I - T)} = X$ .

Conversely, if  $\overline{\text{Ran}(I - T)} = X$ , then, since every weak\* cluster point of the sequence  $(S^{*n})$  is a projection on  $\text{Ker}(I - T^*)$ , the only weak\* cluster point of the sequence  $(S^{*n})$  is 0. This implies that the sequence  $(S^{*n})$  itself converges to 0 in the weak\* topology since the sequence  $(S^{*n})$  is bounded.

(2): We first note that by a result by Katznelson-Tzafriri [5, Theorem 1], we have  $\sigma(S) \cap \{z \in \mathbb{C} : |z| = 1\} \subseteq \{1\}$ . Note also that by Lemma 2.3, the operator  $(I - T)$  is invertible if and only if it is onto. Thus,

$$\begin{aligned} \text{Ran}(I - T) = X &\Leftrightarrow 1 \notin \sigma(T) \Leftrightarrow 1 \notin \sigma(S) \\ &\Leftrightarrow \sigma(S) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset \\ &\Leftrightarrow \|S^n\| \rightarrow 0. \end{aligned}$$

□

The following corollary, which is of independent interest, will be needed for the proof of our main theorem.

**Corollary 2.8** *Let  $T \in \mathcal{B}(X)$  be a contraction and assume that  $\text{Ran}(I - T^*)$  is closed. Then  $\text{Ran}((I - T^*)^2)$  is also closed.*

**Proof.** By Lemma 331 of [4, p.274], it is enough to prove that the space  $\text{Ran}(I - T^*) + \text{Ker}(I - T^*)$  is closed. Let  $((I - T^*)(f_n) + g_n)$  be a sequence in  $\text{Ran}(I - T^*) + \text{Ker}(I - T^*)$  that converges to  $f \in X^*$ . Let  $P$  denote the projection obtained in Theorem 2.6. Since  $\text{Ran}(I - T^*)$  is a subset of  $\text{Ker}(P)$ , we have  $P((I - T^*)(f_n) + g_n) = P g_n \rightarrow P f$ . As  $\text{Ker}(I - T^*) = \text{Ran}(P)$ , the sequence  $(g_n)$  converges to  $P(f)$ . Thus, the sequence  $(I - T^*)(f_n)$  converges to  $f - P(f)$  which must be in  $\text{Ran}(I - T^*)$ , as the space  $\text{Ran}(I - T^*)$  is closed. □

We can now prove the main result of this paper.

**Theorem 2.9** *Let  $T \in \mathcal{B}(X)$  be a contraction. Then  $\text{Ran}(I - T)$  is closed if and only if there exist a projection  $\theta \in \mathcal{B}(X)$  and an invertible operator  $R \in \mathcal{B}(X)$  such that  $I - T = \theta \circ R = R \circ \theta$ .*

**Proof.** Assume that the space  $\text{Ran}(I - T)$  is closed. Then the space  $\text{Ran}(I - T^*)$  is closed as well. Therefore, by Corollary 2.8, the space  $\text{Ran}((I - T^*)^2)$  is also closed, which in turn implies that the space  $\text{Ran}((I - T)^2)$  is closed. Note that  $\text{Ker}(I - T^*) = \text{Ker}((I - T^*)^2)$  by Lemma 2.4. Hence, it follows from Lemma 2.1 that the range of the operator  $(I - T)^2$  coincides with the range of the operator  $I - T$ . So for every  $x \in X$ , there exists  $y \in X$  such that  $(I - T)(x) = (I - T)^2(y)$ . Thus  $x - (I - T)y$  is in  $\text{Ker}(I - T)$ , which, together with Lemma 2.3, proves that  $\text{Ran}(I - T) \oplus \text{Ker}(I - T) = X$ . Now, define  $R : X \rightarrow X$  as follows:

$$R(z + y) = (I - T)(z) + y \quad \text{where } z \in \text{Ran}(I - T) \quad \text{and} \quad y \in \text{Ker}(I - T).$$

The mapping  $R$  is well-defined, linear, and bounded. We claim that it is invertible. To see that it is onto, let  $x = z + y \in X$ , where  $z \in \text{Ran}(I - T)$  and  $y \in \text{Ker}(I - T)$ . Then  $z = (I - T)w$  for some  $w \in X$ .

So  $R(w + y) = x$ . Now, to see that it is one-to-one, let  $x = z + y \in \text{Ker}(R)$ , where  $z \in \text{Ran}(I - T)$  and  $y \in \text{Ker}(I - T)$ . Then  $(I - T)(z) = -y$ . Thus  $(I - T)(z) = y = 0$  since the only point in the intersection of the spaces  $\text{Ker}(I - T)$  and  $\text{Ran}(I - T)$  is 0. This also implies that  $z$  is in the intersection of the spaces  $\text{Ker}(I - T)$  and  $\text{Ran}(I - T)$ , and so it is 0 as well. Hence  $x = z + y = 0$ . Let  $\theta$  be the projection with range  $\text{Ran}(I - T)$  and kernel  $\text{Ker}(I - T)$ . Then  $I - T = R \circ \theta = \theta \circ R$ . The reverse implication is clear.  $\square$

The following corollary, which is reminiscent of the Fredholm Alternative, is an immediate consequence of the preceding theorem.

**Corollary 2.10** *Let  $T \in \mathcal{B}(X)$  be a contraction and assume that  $\text{Ran}(I - T)$  is closed. Then  $\dim(\text{Ker}(I - T)) = \text{codim}(\text{Ran}(I - T))$ .*

We will prove below an analogue of Theorem 2.9 for the operator algebra  $\mathcal{B}(X)$ . In what follows we will denote by  $R_{(I-T)}$  the operator defined on  $\mathcal{B}(X)$  by  $R_{(I-T)}(A) = A \circ (I - T)$ , and we will denote by  $\mathcal{B}(X) \circ (I - T)$  its image.

**Corollary 2.11** *Let  $T \in \mathcal{B}(X)$  be a contraction and assume that  $\text{Ran}(I - T)$  is closed. Then  $\mathcal{B}(X) = \text{Ker}(R_{(I-T)}) \oplus \mathcal{B}(X) \circ (I - T)$ .*

**Proof.** By Theorem 2.9, we have  $I - T = \theta \circ R$ , where  $\theta$  is a bounded projection and  $R$  is an invertible operator in  $\mathcal{B}(X)$ . Consider the operator

$$\begin{aligned} \Theta : \mathcal{B}(X) &\rightarrow \mathcal{B}(X) \\ A &\mapsto A \circ \theta, \end{aligned}$$

which is a bounded projection on  $\mathcal{B}(X)$ . Using the decomposition  $I - T = R \circ \theta$ , one can easily see that  $\text{Ran}(\Theta) = \mathcal{B}(X) \circ (I - T)$  and  $\text{Ker}(\Theta) = \text{Ker}(R_{(I-T)})$ .  $\square$

Let  $A(T)$  be the norm closed subalgebra of  $\mathcal{B}(X)$  generated by an operator  $T \in \mathcal{B}(X)$  and the identity operator  $I$ , which is clearly a commutative Banach algebra. The proof of the preceding corollary also shows that the following holds.

**Corollary 2.12** *Let  $T \in \mathcal{B}(X)$  be a contraction. Then the ideal  $A(T) \circ (I - T)$  is closed in  $A(T)$  if and only if the range of the operator  $I - T$  is closed in  $X$ .*

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