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On the uniqueness of strongly flat covers of cyclic acts

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Abstract

In [1], strongly flat covers of cyclic acts are discussed and it is asked if strongly flat covers are unique. From this point of view, in this paper we give numerous classes of monoids over which strongly flat covers of cyclic acts are unique.

Key Words: Strongly flat, cover, cyclic act

1. Introduction

A right S-act B_S is called a *cover* of a right S-act A_S if there exists an epimorphism $f : B_S \longrightarrow A_S$ such that for any proper subact C_S of B_S the restriction $f|_{C_S} : C_S \longrightarrow A_S$ is not an epimorphism. An epimorphism with this property is called a *coessential epimorphism*. A cover B_S of an act A_S is called a *projective (strongly flat) cover* of A_S if B_S is a projective (strongly flat) act. In [1], Mahmoudi and Renshaw investigate strongly flat (condition (P)) covers of cyclic acts. Recently in [2], monoids over which all right S-acts have strongly flat (condition (P)) covers are characterized, answering a question of Mahmoudi and Renshaw. Another question posed in [1] is whether strongly flat covers of acts are unique. This question remains open up to now. In Section 2 we give some conditions on a monoid under which strongly flat covers of its cyclic acts are unique. The reader is referred to [3] for preliminaries and basic results related to monoids and strongly flat acts. First we present some results that we need in the sequel.

Proposition 1.1 *([2]) Any strongly flat right S-act which has a projective cover is projective.*

Theorem 1.2 *([2]) Let S be a monoid. The following conditions are equivalent:*

(i) all right S-acts have strongly flat covers;

(ii) S satisfies condition (A), and every cyclic right S-act has a strongly flat cover.

For a right congruence σ on a monoid S we define a relation σ_u by $s(\sigma_u)t$ if and only if $(us)\sigma(ut)$. It is easily checked that σ_u is also a right congruence on S.

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Lemma 1.3 *([1]) Let S be a monoid and ρ a right congruence on S. If S/σ is a cover of S/ρ, then there exists $u \in S$ such that $S/\sigma \cong S/\sigma_u$, $\sigma_u \subseteq \rho$ and for all $u \in [1]_\rho$, $uS \cap [1]_{\sigma_u} \neq \emptyset$.*

Theorem 1.4 *([1]) Let S be a monoid. Then the cyclic right S-act S/ρ has a strongly flat cover if and only if $[1]_\rho$ contains a left collapsible submonoid R such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$.*

Proposition 1.5 *([1]) Let S be a monoid. Then every cyclic right S-act has a strongly flat cover if and only if every left unitary submonoid T of S contains a left collapsible submonoid R such that for all $u \in T$, $uS \cap R \neq \emptyset$.*

2. When strongly flat covers of cyclic acts are unique

In this section we discuss the uniqueness of projective, strongly flat and condition (P) covers of acts.

In [4], it is proved that projective covers of acts are unique up to isomorphism. In [1], it is shown that condition (P) covers are not unique , and it is asked if strongly flat covers are unique. This question remains open up to now. We present some classes of monoids over which strongly flat covers of cyclic right S-acts are unique.

Proposition 2.1 *Let S/ρ be a cyclic right S-act such that $[1]_\rho$ is left cancellative. If S/ρ has a strongly flat cover, then S is the only strongly flat cover of S/ρ. In this case $[1]_\rho$ is a subgroup of S.*

Proof. Suppose that S/σ is a strongly flat cover of S/ρ. Using Lemma 1.3, suppose that $\sigma \subseteq \rho$ and for each $u \in [1]_\rho$, $uS \cap [1]_\sigma \neq \emptyset$. Let $R = [1]_\sigma$. So R is left collapsible and left cancellative, and clearly $R = \{1\}$. Now, we show that $\sigma = \Delta_S$. Suppose that $x\sigma y$ for $x, y \in S$. Because S/σ is strongly flat, there exists $v \in R$ such that $vx = vy$. Then $v = 1$ and so $x = y$. Therefore, $\sigma = \Delta_S$, and $S/\sigma = S$. Moreover, If $u \in [1]_\rho$, since $uS \cap [1]_\sigma \neq \emptyset$, there exists $s \in S$ such that $us = 1$. Now, since $[1]_\rho$ is left unitary, $s \in [1]_\rho$. Hence $[1]_\rho$ is a subgroup of S. □

The following corollary immediately holds.

Corollary 2.2 *Let S be a monoid. If S/ρ is a cyclic right S-act such that $[1]_\rho$ is a subgroup of S, then S/ρ has the unique strongly flat cover S.*

Proposition 2.3 *Let S/ρ be a cyclic right S-act such that $[1]_\rho$ is commutative. If S/ρ has a strongly flat cover, then it is unique up to isomorphism.*

Proof. Let S/ρ have strongly flat covers S/σ_1 and S/σ_2. By Lemma 1.3, suppose that $\sigma_1 \subseteq \rho$, $\sigma_2 \subseteq \rho$ and for each $u \in [1]_\rho$, $uS \cap [1]_{\sigma_1} \neq \emptyset$ and $uS \cap [1]_{\sigma_2} \neq \emptyset$. We show that $\sigma_1 = \sigma_2$. Suppose that $x\sigma_1 y$. Since S/σ_1 is strongly flat, there exists $u \in [1]_{\sigma_1}$ such that $ux = uy$. Then since $[1]_{\sigma_1} \subseteq [1]_\rho$; $u \in [1]_\rho$ also, there exists $s \in S$ such that $us \in [1]_{\sigma_2}$. Moreover, since $[1]_\rho$ is left unitary, $s \in [1]_\rho$, and we have $su = us \in [1]_{\sigma_2}$. Since $sux = suy$, $x\sigma_2\, y$. Thus $\sigma_1 \subseteq \sigma_2$. Similarly, $\sigma_2 \subseteq \sigma_1$, and hence $\sigma_1 = \sigma_2$. □

Proposition 2.4 *Let ρ be a right congruence on a monoid S such that $[1]_\rho$ is a left simple subsemigroup of S with a 1 adjoined. Then a strongly flat cover of S/ρ is unique if it exists.*

Proof. Let S/ρ have strongly flat covers S/σ_1 and S/σ_2. By Lemma 1.3, suppose that $\sigma_1 \subseteq \rho$, $\sigma_2 \subseteq \rho$ and for each $u \in [1]_\rho$, $uS \cap [1]_{\sigma_1} \neq \emptyset$ and $uS \cap [1]_{\sigma_2} \neq \emptyset$. If $[1]_{\sigma_1} = \{1\}$ or $[1]_{\sigma_2} = \{1\}$, clearly $[1]_\rho$ is a subgroup of S, and we get a contradiction by the structure of $[1]_\rho$.

Now, suppose that $[1]_{\sigma_1} \neq \{1\}$ and $[1]_{\sigma_2} \neq \{1\}$. We show that $\sigma_1 = \sigma_2$. Suppose that $x\sigma_1 y$. Since S/σ_1 is strongly flat, there exists $u \in [1]_{\sigma_1}$ such that $ux = uy$. If $u = 1$, then $x = y$ and $x\sigma_2 y$. Suppose that $u \neq 1$. Take $1 \neq t \in [1]_{\sigma_2}$. So there exists $s \in [1]_\rho$ such that $su = t$. Then $tx = sux = suy = ty$, and so $x\sigma_2 y$. Thus $\sigma_1 \subseteq \sigma_2$. Similarly, $\sigma_2 \subseteq \sigma_1$. □

In view of the previous propositions we deduce the following theorem.

Theorem 2.5 *Over the following monoids strongly flat covers of cyclic right S-acts are unique if they exist.*

(i) *S is a left cancellative monoid.*

(ii) *S is commutative.*

(iii) *S is a left simple semigroup with a 1 adjoined.*

Lemma 2.6 *Let S/ρ be a cyclic strongly flat right S-act such that $[1]_\rho$ contains a left zero. Then S/ρ is projective.*

Proof. Let z be a left zero element of $[1]_\rho$. We show that $\rho = \ker \lambda_z$. Suppose that $x\rho y$. Since S/ρ is strongly flat, there exists $u \in [1]_\rho$ such that $ux = uy$. So $zx = zux = zuy = zy$, and so $x(\ker \lambda_z)y$. Clearly, if $zx = zy$ then $x\rho y$. Hence S/ρ is projective, being isomorphic to zS. □

Lemma 2.7 *If a left collapsible submonoid R of a monoid S contains a right zero z, then z is a zero element of R.*

Proof. Let z be a right zero element of a left collapsible submonoid R. Suppose $x \in R$. Then there exists $u \in R$ such that $ux = uz = z$. For $u, z \in R$ there exists $v \in R$ such that $vu = vz = z$. So $zx = vux = vz = z$, and z is a zero element of R. □

Proposition 2.8 *Let S/ρ be a cyclic right S-act . Then S/ρ has a unique strongly flat cover if:*

(i) *$[1]_\rho$ contains a left zero; or*

(ii) *$[1]_\rho$ contains a right zero.*

Proof. (i) Let $R = [1]_\rho$, since R contains a left zero, clearly by Theorem 1.4 S/ρ has a strongly flat cover.

Now, let S/σ be a strongly flat cover of S/ρ, and let z be a left zero element of $[1]_\rho$. Since $zS \cap [1]_\sigma \neq \emptyset$, $z \in [1]_\sigma$. By Lemma 2.6, S/σ is projective. Now, since every strongly flat cover of S/ρ is also a projective cover and projective covers are unique, then a strongly flat cover of S/ρ is unique.

(ii) Let ρ be a right congruence on S, and let z be a right zero element of $[1]_\rho$. Let $R = \{1, z\}$. So R is left collapsible and for each $u \in [1]_\rho$, $uz = z \in R$. Then, by Theorem 1.4, S/ρ has a strongly flat cover.

Now, we show that a strongly flat cover of S/ρ is unique. Let S/σ be a strongly flat cover of S/ρ. Since $zS \cap [1]_\sigma \neq \emptyset$, there exists $s \in S$ such that $zs \in [1]_\sigma$. Put $w = zs$. Then w is also a right zero element of $[1]_\rho$. Now, by Lemma 2.7, since $[1]_\sigma$ is left collapsible, w is a zero element of $[1]_\sigma$. Thus by Lemma 2.6, S/σ is projective. Hence S/σ is the unique strongly flat cover of S/ρ up to isomorphism. □

Recall that a semigroup S is called right (left) nil if for each $s \in S$ there exists $n \in \mathbb{N}$ such that s^n is a right (left) zero element of S. Furthermore, a monoid S is called right (left) elementary if $S = G\dot{\cup}N$ such that G is a group and N is either empty or a right (left) nil semigroup.

Theorem 2.9 *Over a right (left) elementary monoid every cyclic right S-act has a unique strongly flat cover.*

Proof. Let S/ρ be a cyclic right S-act and $T = [1]_\rho$. If $T \subseteq G$, clearly T is a subgroup of S, and so S/ρ has a unique strongly flat cover by Corollary 2.2. Now, suppose that $s \in T \cap N$. So there exists $n \in \mathbb{N}$ such that $z = s^n$ is a right (left) zero. Hence, since T contains a right (left) zero, by Proposition 2.8 S/ρ has a unique strongly flat cover. □

Corollary 2.10 *Over a right (left) nil semigroup with a 1 adjoined every cyclic right S-act has a unique strongly flat cover.*

It is well-known that a regular semigroup S is left inverse if for any idempotent elements $e, f \in E(S)$ ($E(S)$ is the set of idempotent elements of S), we have $efe = ef$.

Theorem 2.11 *Over a periodic left inverse monoid every cyclic right S-act has a unique strongly flat cover.*

Proof. Suppose that T is a left unitary submonoid of S. So $R = E(T)$ is a submonoid of T. R is left collapsible since $(ef)e = (ef)f$ for each $e, f \in R$. Moreover, since S is periodic, $uS \cap R \neq \emptyset$ for each $u \in T$. Thus every cyclic right S-act has a strongly flat cover.

Now, let S/ρ have strongly flat covers S/σ_1 and S/σ_2. By Lemma 1.3, suppose that $\sigma_1 \subseteq \rho$, $\sigma_2 \subseteq \rho$ and for each $u \in [1]_\rho$, $uS \cap [1]_{\sigma_1} \neq \emptyset$ and $uS \cap [1]_{\sigma_2} \neq \emptyset$. Suppose that $x\sigma_1 y$. There exists $u \in [1]_{\sigma_1}$ such that $ux = uy$. Since S is periodic, there exists $m \in \mathbb{N}$ such that $e = u^m$ is an idempotent. So $e \in [1]_{\sigma_1}$ and $ex = ey$. Then $eS \cap [1]_{\sigma_2} \neq \emptyset$. So $es \in [1]_{\sigma_2}$ for some $s \in S$. There exists $n \in \mathbb{N}$ such that $f = (es)^n$ is an idempotent. We have $ef = f$ and $f \in [1]_{\sigma_2}$. Now, $ex = ey$ implies that $fx = efx = efex = efey = efy = fy$. Thus $x\sigma_2 y$, and so $\sigma_1 \subseteq \sigma_2$. Similarly, $\sigma_2 \subseteq \sigma_1$. □

Theorem 2.12 *Over a Clifford monoid every cyclic right S-act has a unique strongly flat cover.*

Proof. Suppose that T is a left unitary submonoid of S. Since $ef = fe$ for each $e, f \in E(S)$, $R = E(T)$ is a left collapsible submonoid of T. Let $u \in T$. There exist $e \in E(S)$ and $u^{-1} \in S$ such that $u^{-1}u = uu^{-1} = e$. Since $ue = u$ and T is left unitary, $e \in R$ and so $uS \cap R \neq \emptyset$. Thus every cyclic right S-act has a strongly flat cover.

Now, let S/ρ have strongly flat covers S/σ_1 and S/σ_2. Suppose that $\sigma_1 \subseteq \rho$, $\sigma_2 \subseteq \rho$ and for each $u \in [1]_\rho$, $uS \cap [1]_{\sigma_1} \neq \emptyset$ and $uS \cap [1]_{\sigma_2} \neq \emptyset$. Suppose that $x\sigma_1 y$. There exists $u \in [1]_{\sigma_1}$ such that $ux = uy$.

There exist $e \in E(S)$ and $u^{-1} \in S$ such that $u^{-1}u = uu^{-1} = e$. Since $ue = u$ and $[1]_{\sigma_1}$ is left unitary, $e \in [1]_{\sigma_1} \subseteq [1]_\rho$ and so $eS \cap [1]_{\sigma_2} \neq \emptyset$. Thus $es \in [1]_{\sigma_2}$ for some $s \in S$. Since $es = se$, $(es)x = (es)y$. Therefore, $x\sigma_2 y$, and so $\sigma_1 \subseteq \sigma_2$. Similarly, $\sigma_2 \subseteq \sigma_1$. □

In Lemma 1.3 of [2] it is shown that condition (A) is equivalent to every right S-act containing a minimal generating set. Therefore, the following theorem holds.

Theorem 2.13 *Let S be a monoid. Then all right S-acts have unique strongly flat covers if and only if*

(i) S satisfies condition (A), and

(ii) every cyclic right S-act has a unique strongly flat cover.

Proof. Necessity is clear by Theorem 1.2.

Sufficiency. Let A_S be a right S-act and $\{a_i \mid i \in I\}$ be a minimal generating set of A_S by condition (A). By part (ii) suppose that B_i is a strongly flat cover of a_iS with a coessential epimorphism $f_i : B_i \longrightarrow a_iS$ for each $i \in I$. Put $B = \coprod_{i\in I} B_i$ and define $f : B_S \longrightarrow A_S$ by $f(b) = f_i(b)$ for $b \in B_i$. We show that f is a coessential epimorphism. Observe that $f^{-1}[\{a_i\}] \subseteq B_i$ for each i. For suppose $f(c) = a_i$ where $c \in B_j$ $(j \neq i)$. Then $a_i = f_j(c) \in a_jS$, contradicting the fact that $\{a_i | \ i \in I\}$ is a minimal generating set of A. Now, suppose $a_i = f_i(b_i)$ for some $b_i \in B_i$, so $B_i = b_iS$. If C were a proper subact of B, then for some $i \in I$ we would have $b_i \in B_i \setminus (C \cap B_i)$. If $f|_C$ were onto, then $f_i|_{C\cap B_i}$ would also be onto, and therefore the contradiction $B_i = C \cap B_i$ would result.

Now, suppose that C_S is another strongly flat cover of A_S with a coessential epimorphism $g : C_S \longrightarrow A_S$. There exists $c_i \in C_S$ such that $g(c_i) = a_i$ for each $i \in I$. First we show that $C_S = \coprod_{i\in I} c_iS$. Suppose that $c_is = c_jt$ for some $i \neq j$ and $s,t \in S$. Since C is strongly flat, there exist $d \in C$ and $u, v \in S$ such that $c_i = du$, $c_j = dv$, and $us = vt$. So $a_i = g(d)u$ and $a_j = g(d)v$. Suppose $g(d) \in a_kS$, then either $k \neq i$ or $k \neq j$, and we get a smaller generating set, which is a contradiction. So $C_S = \coprod_{i\in I} c_iS$ and $g|_{c_iS} : c_iS \longrightarrow a_iS$ is coessential. Thus by part (ii) $c_iS \cong B_i$ for each $i \in I$, and so $B_S \cong C_S$. Hence A_S has a unique strongly flat cover. □

Suppose S is a monoid such that all cyclic right S-acts have projective covers. Then clearly all cyclic right S-acts have strongly flat covers. On the other hand, by Proposition 1.1, all cyclic strongly flat right S-acts are projective. Therefore, since projective covers are unique, strongly flat covers of cyclic right S-acts are unique. For example, by Theorem 3.9 of [1], over a left group with a 1 adjoined each cyclic right S-acts has a unique strongly flat cover.

By the above argument and Theorem 2.13, over a right perfect monoid strongly flat covers of right S-acts are unique up to isomorphism. Then, in light of Example 2.9 of [2], we deduce the following corollary.

Corollary 2.14 *Let S be a monoid. Then strongly flat covers are unique over S if:*

(i) S is a group; or

(ii) S is finite; or

(iii) S is a rectangular band with a 1 adjoined; or

(iv) S is a left (right) zero semigroup with a 1 adjoined; or

(v) *S is a right group with a 1 adjoined.*

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