[Turkish Journal of Mathematics](https://journals.tubitak.gov.tr/math)

Volume 35 | Number 3 Article 4

1-1-2011

Pseudo PQ-injective modules

ZHANMIN ZHU

Turk J Math
35 (2011) , 391 – 398.
© TÜBİTAK
doi:10.3906/mat-0911-141

Pseudo PQ-injective modules

Zhanmin Zhu

Abstract

A module M_R is called Pseudo PQ-injective (or PPQ-injective for short) if every monomorphism from a principal submodule of M to M extends to an endomorphism of M. Some characterizations and properties of this class of modules are investigated, PPQ-injective modules with some additional conditions are studied, semisimple artinian rings are characterized by PPQ-injective modules.

Key Words: PPQ- injective modules; Endomorphism rings; Strongly Kasch modules; semisimple artinian rings; perfect rings.

1. Introduction

Throughout R is an associative ring with identity and all modules are unitary. Following [6], a right R-module M is called principally quasi-injective (or PQ-injective for short) if every homomorphism from a principal submodule of M to M extends to an endomorphism of M, or equivalently, $\mathbf{l}_M(\mathbf{r}_R(m)) = Sm$ for every $m \in M$, where $S = End(M_R)$. In this paper, we generalized the concept of PQ-injective modules to PPQ-injective modules and give some interesting results on these modules.

As usual, we denote the Jacobson radical of a ring R by $J(R)$ and denote the injective hull of a module M by $E(M)$. Let M be a right R-module, then we denote $S = End(M_R)$. Let $X \subseteq M$, $Y \subseteq M$ and $A \subseteq S$, then we write $\mathbf{r}_R(X) = \{r \in R \mid xr = 0, \text{ for all } x \in X\}$, $\mathbf{l}_S(Y) = \{s \in S \mid sy = 0, \text{ for all } y \in Y\}$, and $\mathbf{r}_M(A) = \{m \in M \mid sm = 0, \text{ for all } s \in A\}$.

2. Pseudo PQ-injective modules

We start with the following definition.

Definition 1 *Let R be a ring. A right R-module M is called Pseudo PQ-injective (or PPQ-injective for short) if every monomorphism from a principal submodule of M to M extends to an endomorphism of M.*

2000 *AMS Mathematics Subject Classification:* 16D50, 16L30, 16P60.

Theorem 2 *The following conditions are equivalent for a module M_R.*

(1) *M is PPQ-injective.*

(2) *$\mathbf{r}_R(m) = \mathbf{r}_R(n), m, n$, in M, implies that $Sm = Sn$.*

(3) *If $m \in M$ and $\alpha, \beta : mR \to M$ are monic, then there exists $s \in S$ such that $\alpha = s\beta$.*

Proof. (1) $\Rightarrow$ (2). If $\mathbf{r}_R(m) = \mathbf{r}_R(n), m, n$ in M, then the mapping $f : mR \to M$; $mr \mapsto nr$ is a monomorphism. Since M is PPQ-injective, there exists $s \in S$ such that s extends f, then $n = f(m) = sm$ and so $Sn \subseteq Sm$. Similarly, $Sm \subseteq Sn$, so $Sm = Sn$.

(2) $\Rightarrow$ (3). Since α, β are monic, we have $\mathbf{r}_R(\alpha(m)) = \mathbf{r}_R(\beta(m))$. By (2), $S\alpha(m) = S\beta(m)$ which shows that $S\alpha = S\beta$, and so there exists $s \in S$ such that $\alpha = s\beta$.

(3) $\Rightarrow$ (1). Take $\beta : mR \to M$ to be the inclusion mapping in (3). $\square$

Example 3 *Let M be one of the following two examples of Pseudo-injective modules which are not quasi-injective: either the Hallet's example or the Teply's example (see [4, p.364]). Since M has five submodules 0, M, N_1, N_2 and $N_1 \oplus N_2$ which are all cyclic, it follows that M is PPQ-injective but not PQ-injective.*

Let M be a right R-module. Following [6], we write $W(S) = \{w \in S \mid ker(w) \subseteq^{ess} M\}$. Note that $W(S)$ is an ideal of S. Recall that a ring R is called semipotent [7] if every right ideal of R not contained in $J(R)$ contains a nonzero idempotent. In order to facilitate, we call a module M_R a principal annihilator module if for every principal submodule K of M_R, there exists a subset A of $End(M_R)$ such that $K = \mathbf{r}_M(A)$. Clearly, M_R is a principal annihilator module if and only if $\mathbf{r}_M(\mathbf{l}_S(K)) = K$ for every principal submodule K of M_R.

Theorem 4 *Let M_R be PPQ-injective. Then*

(1) *$J(S) \subseteq W(S)$.*

(2) *If S is also semipotent, then $J(S) = W(S)$.*

(3) *If $mR \subseteq M$ is simple, then Sm is simple.*

(4) *$Soc(M_R) \subseteq Soc(_SM)$.*

(5) *If M_R is also a principal annihilator module, then $Soc(M_R) = Soc(_SM)$.*

Proof. (1). Let $a \in J(S)$. If $a \notin W(S)$, then $ker(a) \cap K = 0$ for some $0 \neq K \leq M_R$. Take $k \in K$ such that $ak \neq 0$, then $\mathbf{r}_R(k) = \mathbf{r}_R(ak)$. Since M_R is PPQ-injective, $Sk = Sak$. Write $k = bak$, where $b \in S$, then $(1 - ba)k = 0$, and so $k = 0$, a contradiction. Therefore, $J(S) \subseteq W(S)$.

(2). By (1), we need only to prove that $W(S) \subseteq J(S)$. If not, then $W(S)$ contains a nonzero idempotent e because S is semipotent. But $Ker(e) = (1 - e)M$ is not essential in M_R, a contradiction.

(3). Let $mR \subseteq M$ be simple. Then $\mathbf{r}_R(am) = \mathbf{r}_R(m)$ for each $a \in S$ such that $am \neq 0$, so the PPQ-injectivity of M_R implies that $S(am) = Sm$. Which shows that Sm is simple.

(4). Follows from (3).

(5). Suppose that M_R is a principal annihilator module. If Sm is simple, then $\mathbf{l}_S(mb) = \mathbf{l}_S(m)$ for each $b \in R$ such that $mb \neq 0$, and hence $mbR = mR$. It shows that mR is also simple, so $Soc(_SM) \subseteq Soc(M_R)$, and whence $Soc(_SM) = Soc(M_R)$ by (4). $\square$

Recall that a ring R is called left Kasch [7] if every simple left R-module embeds in ${}_RR$, equivalently, $\mathbf{r}_R(T) \neq 0$ for every maximal left ideal T of R. The concept of left Kasch rings were generalized to modules in paper [1]. Following [1], a module ${}_RM$ is said to be Kasch provided that every simple module in $\sigma[M]$ embeds in M, where $\sigma[M]$ is the category consisting of all M-subgenerated left R-modules. Kasch modules have studied by series of authors, see [1, 6, 5]. In paper [12], a module ${}_RM$ is called strongly Kasch if every simple left R-module embeds in M. It is easy to see that ${}_RM$ is strongly Kasch if and only if $\mathbf{r}_M(T) \neq 0$ for every maximal left ideal T of R. And we also recall that a module M is called C_2 [7, p.9] if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. C_2 modules are also called direct injective modules [8, p.368]. Following [11], a module M is called GC_2 if every submodule of M that is isomorphic to M is itself a direct summand of M. Clearly, C_2 modules are GC_2.

Proposition 5 *Let M be a right R-module. Consider the following conditions:*

(1) *S is left Kasch.*

(2) *${}_SM$ is strongly Kasch.*

(3) *M_R is C_2.*

(4) *M_R is GC_2.*

(5) *$W(S) \subseteq J(S)$.*

Then we always have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$.

Proof. $(1) \Rightarrow (2)$. Let K be any maximal left ideal of S. Since S is left Kasch, $\mathbf{r}_S(K) \neq 0$. Choose $0 \neq s \in \mathbf{r}_S(K)$, then $0 \neq sM \subseteq \mathbf{r}_M(K)$ for ${}_SM$ is faithful. So $\mathbf{r}_M(K) \neq 0$, and then ${}_SM$ is strongly Kasch.

$(2) \Rightarrow (3)$. Let K be a submodule of M_R and $\sigma : eM \to K$ be an isomorphism, where $e^2 = e \in S$. Then there exists $s \in S$ such that $se = \sigma e$ and $K = seM$. Let $a = se$, then $ae = a$ and $K = aM$. We claim that $Sa = Se$. If not let $Sa \subseteq L \subseteq^{max} Se$. By the strongly Kasch hypothesis of ${}_SM$, there exists a monomorphism $\alpha : Se/L \to {}_SM$. Write $m = \alpha(e + L)$, then $em = e\alpha(e + L) = \alpha(e + L) = m$ and $am = a\alpha(e + L) = \alpha(ae + L) = \alpha(a + L) = \alpha(0) = 0$. Noting that $Ker(a) = Ker(e)$, we have $m = em = 0$, and hence $e \in L$. This contradiction shows that $Sa = Se$. Write $e = ba$, then $a = aba$, and hence K is a direct summand of M_R.

$(3) \Rightarrow (4)$. Obvious.

$(4) \Rightarrow (5)$. See [14, Corollary 6]. □

Theorem 6 *Let M_R be a finitely cogenerated PPQ-injective module. Then the following statements are equivalent:*

(1) *${}_SM$ is strongly Kasch.*

(2) *M_R is C_2.*

(3) *M_R is GC_2.*

(4) *$W(S) = J(S)$.*

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ by Proposition 5. $(3) \Rightarrow (4)$ by Theorem 4(1) and Proposition 5.

(4) $\Rightarrow$ (1). Since M_R is finitely cogenerated, $Soc(M_R)$ is finitely generated and essential in M_R. Assume that (4) holds. Observe first that $J(S) \subseteq \mathbf{l}_S(Soc(M_R))$ because $Soc(M_R) \subseteq Soc({}_SM)$ by Theorem 4(4); and $\mathbf{l}_S(Soc(M_R)) \subseteq W(S)$ because $Soc(M_R) \subseteq^{ess} M_R$. Using (4), it follows that $J(S) = \mathbf{l}_S(Soc(M_R))$. Let $\mathrm{Soc}(M_R) = x_1R \oplus \cdots \oplus x_nR$, where each x_iR is simple, then

$$J(S) = \mathbf{l}_S(\mathrm{Soc}(M_R)) = \cap_{i=1}^n \mathbf{l}_S(x_i).$$

Since Sx_i is simple by Theorem 4(3), each $\mathbf{l}_S(x_i)$ is a maximal left ideal of S. Therefore S is semilocal. Noting that the map $S \to M^n$ given by $s \mapsto (sx_1, sx_2, \cdots, sx_n)$ is a left S-homomorphism with kernel $J(S)$, $S/J(S)$ embeds in ${}_SM^n$. Note that the ring $S/J(S)$ is semisimple and hence left Kasch, and every simple left S-module K, regarded as a left $S/J(S)$-module, is simple, so as a left $S/J(S)$-module, K embeds in the left $S/J(S)$-module $S/J(S)$, which follows that K embeds in $S/J(S)$ as left S-modules. Therefore, ${}_SK$ embeds in the left S-module ${}_SM^n$ and hence embeds in ${}_SM$. □

Let M and N be two right R-modules, then we call M pseudo principally N-injective (or PP-N-injective for short) if every monomorphism from a principal submodule of N to M extends to an homomorphism of N to M. Clearly, M is PPQ-injective if and only if M is PP-M-injective.

Proposition 7 *Let M, N be two right R-modules and N' be a submodule of N. If M is PP-N-injective, then*

(1) *Every direct summand of M is PP-N-injective.*

(2) *M is PP-N'-injective.*

Proof. (1). Let $M = M_1 \oplus M_2$. Then for every principal submodule K of N and every monomorphism f of K to M_1, since M is PP-N-injective, f extends to a homomorphism of N to M. Which follows that f extends to a homomorphism of N to M_1 because M_1 is a direct summand of M.

(2) It is obvious. □

By Proposition 7, we have immediately the following corollary.

Corollary 8 *Every direct summand of a PPQ-injective module is PPQ-injective.*

Following [12], we call a right R-module M minimal quasi-injective if every homomorphism from a simple submodule of M to M can be extended to an endomorphism of M.

Theorem 9 *The following statements are equivalent for a ring R:*

(1) *R is a semisimple artinian ring.*

(2) *R is a right V-ring and every minimal quasi-injective right R-module is PPQ-injective.*

(3) *Every right R-module is PPQ-injective.*

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Since R is a right V-ring, every simple right R-module is injective and hence is a direct summand of each module containing it. So every right R-module is minimal quasi-injective, and then (3) follows from (2).

(3) $\Rightarrow$ (1). Let K be any principal right R-module. Since $K \oplus E(K)$ is PPQ-injective, by proposition 7(1), K is PP-$K \oplus E(K)$-injective, and hence K is PP-$E(K)$-injective by proposition 7(2). Therefore, $K = E(K)$ is injective. This proves the theorem. □

A module M is called C_3 [7] if, whenever N and K are direct summands of M with $N \cap K = 0$ then $N \oplus K$ is also a direct summand of M. We call a module M PC_2 if every principal submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. And we call a module M PC_3 if, whenever N and K are direct summands of M with $N \cap K = 0$ and K is principal, then $N \oplus K$ is also a direct summand of M.

Theorem 10 *Every PPQ-injective module is PC_2 and PC_3.*

Proof. Let M_R be PPQ-injective with $S = End(M_R)$. If K is a principal submodule of M and $K \cong eM$, where $e^2 = e \in S$, then eM is PP-M-injective by proposition 7 and hence K is also PP-M-injective, which follows that K is a direct summand of M because K is principal. This proves PC_2. Now let N and K be direct summands of M with $N \cap K = 0$ and K principal. Write $N = eM$ and $K = fM$, where e, f are idempotents in S, then $eM \oplus fM = eM \oplus (1-e)fM$. Since $(1-e)fM \cong fM$ is principal, $(1-e)fM = hM$ for some $h^2 = h \in S$ by PC_2. Let $g = e + h - he$, then $g^2 = g$ and $eM \oplus fM = gM$, as required. □

Recall that a right R-module M is said to be weakly injective [3] if for every finitely generated submodule $N_R \subseteq E(M)$, we have $N \subseteq X_R \subseteq E(M)$ for some $X_R \cong M$.

Corollary 11 *Let M_R be a cyclic module. Then M is injective if and only if it is weakly injective and PPQ-injective.*

Proof. We need only to prove the sufficiency. Let $x \in E(M)$, then there exists $X \subseteq E(M)$ such that $M + xR \subseteq X \cong M$. Since M is PPQ-injective, X is PPQ-injective too. By Theorem 10, X is PC_2 and hence M is a direct summand of X because M is a cyclic submodule of X. But $M \subseteq^{ess} E(M)$, so $M \subseteq^{ess} X$. Thus $M = X$, and then $x \in M$. Therefore, $M = E(M)$ is injective. □

Recall that a module M_R is regular [10] if for every $m \in M$, mR is projective and is a direct summand of M. Clearly, a ring R is regular if and only if the module R_R is regular.

Proposition 12 *Let M_R be a projective module whose cyclic submodules are its images. Then M is regular if and only if M is PC_2 and mR is M-projective for every $m \in M$.*

Proof. $\Rightarrow$. If M is regular. Then every cyclic submodule of M is projective and is a direct summand of M, so the necessity is obvious.

$\Leftarrow$. Since mR is M-projective and is an image of M for every $m \in M$, mR is isomorphic to a direct summand of M. But M is PC_2, mR is a direct summand of M. Observing that M is projective, mR is also projective. □

A ring R is a right PP ring if every principal right ideal of R is projective. The next result extends [9, Theorem 3] from a right P-injective ring to a right C_2 ring.

Corollary 13 *A ring R is regular if and only if R is a right C_2 and right PP ring.*

Corollary 14 *Let M_R be a PPQ-injective cyclic module, then*

(1) *M_R is a C_2 module.*

(2) *$J(S) = W(S)$.*

(3) *If M_R has finite Goldie dimension then S is semilocal.*

(4) *If M_R is uniform, then S is local.*

Proof. (1) Since M_R is cyclic, each direct summand of M_R is also cyclic, so (1) follows because M_R is PC_2 by Theorem 10.

(2) By Theorem 4(1), $J(S) \subseteq W(S)$. But M_R is C_2 by (1), so $W(S) \subseteq J(S)$ by [8, 41.22]. Therefore, $J(S) = W(S)$.

(3) Let s be any injective endomorphism of M. Then $s^k M \cong M$ for each positive integer k, and so $s^k M$ is a direct summand of M_R for M_R is a C_2 module by (1). Since M_R has finite Goldie dimension, it contains no infinite direct sum of its submodules, and thus it satisfies the descending conditions on direct summands. Hence $s^n M = s^{n+1} M$ for some positive integer n. This follows that s is bijective. Therefore, S is semilocal by [2, Theorem 3].

(4) Let $s \in S$ and $S \neq Ss$. Then $Ker(s) \neq 0$ by [14, Theorem 4] since M_R is GC_2. So, since M is uniform, $Ker(s) \subseteq^{ess} M$. Thus $s \in W(S) = J(S)$. This means that S is local. □

Theorem 15 *Let M_1 be a cyclic module, and let $M_1 \oplus M_2$ be a PPQ-injective module and $\sigma : M_1 \to M_2$ be a monomorphism. Then σ splits and M_1 is PQ-injective.*

Proof. Since $\alpha : \sigma(M_1) \to M_1 \oplus M_2$ given by $\alpha(\sigma(x)) = (x, 0), x \in M_1$, is a monomorphism, it can be extended to an endomorphism α^* of $M_1 \oplus M_2$. If $\iota : M_2 \to M_1 \oplus M_2$ and $\pi : M_1 \oplus M_2 \to M_1$ are natural injection and projection, respectively, then $\tau = \pi\alpha^*\iota$ is such that $\tau\sigma = 1_{M_1}$. Hence σ splits. Let $M_2 = \sigma(M_1) \oplus N_1$. Then $M_1 \oplus M_2 = M_1 \oplus \sigma(M_1) \oplus N_1$, and so $N = M_1 \oplus \sigma(M_1)$ is PPQ-injective by Corollary 8. Let K be any principal submodule of M_1 and $f : K \to M_1$ be an R-homomorphism, then the mapping $\beta : K \to M_1 \oplus \sigma(M_1)$ given by $\beta(x) = (x, \sigma f(x)), x \in K$, is a monomorphism. Hence it can be extended to an endomorphism γ of N. Let $q : M_1 \to N$ and $p : N \to \sigma(M_1)$ are natural injective and projection respectively, then $\mu = \tau p \gamma q$ is an endomorphism of M_1 which extend f. Hence M_1 is PQ-injective. □

Corollary 16 *If M is a cyclic right R-module such that $M \oplus M$ is PPQ-injective, then M is PQ-injective.*

The proofs of the following theorems, Theorems 17 and 18 are similar to the proofs of Propositions 1.2 and 1.5 in [6] respectively, here we omit them.

Theorem 17 *Let M_R be PPQ-injective and let $m, n \in M$.*

(1) *If nR embeds in mR, then Sn is an image of Sm.*

(2) *If $nR \cong mR$, then $Sn \cong Sm$.*

Theorem 18 *Let M_R be PPQ-injective with $S = End(M_R)$, and assume that the sum $\sum_{i=1}^{n} Sm_i$ is direct, $m_i \in M$. Then any monomorphism $\alpha : \sum_{i=1}^{n} m_iR \to M$ can be extended to M.*

By using the same way as the proof of [13, Theorem 2.9], we have the following proposition.

Proposition 19 *Let M be a right R-module which has the following two properties:*

(a) $J(S) \subseteq W(S)$.

(b) *If $s \notin W(S)$, then the inclusion $ker(s) \subset ker(s - sts)$ is strict for some $t \in S$.*

Then the following conditions are equivalent:

(1) *S is right perfect.*

(2) *For any sequence $\{s_1, s_2, \ldots\} \subseteq S$, the chain $ker(s_1) \subseteq ker(s_2s_1) \subseteq \cdots$ terminates.*

Lemma 20 *Let M_R be PPQ-injective. If $s \notin W(S)$, then the inclusion $ker(s) \subset ker(s - sts)$ is strict for some $t \in S$.*

Proof. If $s \notin W(S)$, then $ker(s) \cap mR = 0$ for some $0 \neq m \in M$. Thus $r_R(m) = r_R(sm)$, and so $Sm = S(sm)$ as left S-modules because M_R is PPQ-injective. Write $m = t(sm)$, where $t \in S$, then $(s - sts)m = 0$. Therefore, the inclusion $ker(s) \subset ker(s - sts)$ is strict. □

By Theorem 4, Proposition 19 and Lemma 20, we have immediately the following theorem.

Theorem 21 *Let M_R be a PPQ-injective module, then the following conditions are equivalent:*

(1) *S is right perfect.*

(2) *For any sequence $\{s_1, s_2, \ldots\} \subseteq S$, the chain $ker(s_1) \subseteq ker(s_2s_1) \subseteq \cdots$ terminates.*

Following [13], for a module M_R, we call a submodule K of M a kernel submodule if $K = ker(f)$ for some $f \in End(M_R)$, and we call a submodule K of M an annihilator submodule if $K = \mathbf{r}_M(A)$ for some subset A of $End(M_R)$.

Lemma 22 *Let M be a right R-module. If M has ACC on annihilator submodules, then $W(S)$ is nilpotent.*

Proof. As $W(S) \supseteq W^2(S) \supseteq \cdots$, we get $\mathbf{r}_M(W(S)) \subseteq \mathbf{r}_M(W^2(S)) \subseteq \cdots$, so let $\mathbf{r}_M(W^n(S)) = \mathbf{r}_M(W^{n+1}(S))$, we show that $W^n(S) = 0$. Suppose that $W^n(S) \neq 0$, then $W^{n+1}(S) \neq 0$. Let $W^n(S)a \neq 0$ for some $a \in S$, and choose $Ker(b)$ maximal in $\{Ker(b) \mid W^n(S)b \neq 0\}$. If $z \in W(S)$ then $Ker(z) \subseteq^{ess} M_R$, so $Ker(z) \cap bM \neq 0$, say $0 \neq bm$ with $zbm = 0$. Thus $Ker(b) \subsetneq Ker(zb)$, so, by the choice of b, $W^n(S)zb = 0$. As $z \in W(S)$ is arbitrary, this shows that $W^{n+1}(S)b = 0$, whence $bM \subseteq \mathbf{r}_M(W^{n+1}(S)) = \mathbf{r}_M(W^n(S))$. It follows that $W^n(S)b = 0$, a contradiction. □

Corollary 23 *Let M_R be a PPQ-injective module. Then*

(1) *If M_R satisfies ACC on kernel submodules, then S is right perfect.*

(2) *If M_R satisfies ACC on annihilator submodules, then S is semiprimary.*

Proof. (1) follows from Theorem 21. (2) follows from (1), Theorem 4(1) and Lemma 22. □

Acknowledgment

The author are very grateful to the referees for their helpful comments.

References

[1] Albu, T., and Wisbauer, R.: Kasch modules. In: Advances in Ring Theory (Eds.: S.K. Jain and S.T. Rizvi) 1-16. Birkhäuser(1997).

[2] Herbera, D., and Shamsuddin, A.: Modules with semi-local endomorphism rings. Proc. Amer. Math. Soc. 123, 3593-3600 (1995).

[3] Jain, S.K., and López-permouth, S.R.: Rings whose cyclics are essentially embeddable in projectives. J. Algebra 128, 257-269 (1990).

[4] Jain, S.K., and Singh, S.: Quasi-injective and Pseudo-injective modules. Canad. Math. Bull. 18, 359-366 (1975)

[5] Kosan, M.T.: Quasi-Dual Modules. Turkish J. Math. 30, 177-185 (2006).

[6] Nicholson, W.K., Park, J.K., and Yousif, M.F.: Principally quasi-injective modules. Comm. Algebra 27, 1683-1693 (1999).

[7] Nicholson, W.K., and Yousif, M.F.: Quasi-Frobenius Rings, Cambridge Tracts in Math., Cambridge University Press, 2003.

[8] Wisbauer, R.: Foundations of Module and Ring Theory. Pennsylvania. Gordon and Breach Science 1991.

[9] Xue, W. M.: On PP rings. Kobe J. Math. 7, 77-80 (1990).

[10] Zelmanowitz, J.: Regular modules. Trans. Amer. Math. Soc. 163, 341-355 (1972).

[11] Zhou, Y. Q.: Rings in which certain ideals are direct summands of annihilators. J. Aust. Math. Soc. 73, 335-346 (2002).

[12] Zhu, Z. M. and Tan, Z. S.: Minimal quasi-injective modules. Sci. Math. Jpn. 62, 465-469 (2005).

[13] Zhu, Z. M., Xia, Z. S., and Tan, Z. S.: Generalizations of principally quasi-injective modules and quasiprincipally injective modules. Int. J. Math. Math. Sci., 1853-1860 (2005).

[14] Zhu, Z. M., Yu, J. X.: On GC_2 modules and their endomorphism rings. Linear and Multilinear Algebra 56, 511-515 (2008).

Zhanmin ZHU
Department of Mathematics,
Jiaxing University, Jiaxing,
Zhejiang Province, 314001, P.R. CHINA
e-mail: zhanmin_zhu@hotmail.com

Received: 12.11.2009