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Weak-projective dimensions

Mohammad Javad Nikmehr, Zahra Poormahmood and Reza Nikandish

Abstract

In this paper, the notions of weak-projective modules and weak-projective dimension over commutative domain R are given. It is shown that over semisimple rings with weak global dimension 1, these modules are equivalent to weak-injective modules. The weak-projective dimension measures how far away a domain is from being a Prüfer domain. Several properties of these modules are also presented.

Key Words: Semi-Dedekind domain; Weak-injective modules; Weak- projective dimension, projective modules; Prüfer domain

1. Introduction

In this note, R will denote a commutative domain with identity and $Q (\neq R)$ will denote its field of quotients. The R -module Q/R will be denoted by K . Lee in [5] studied the structure of weak-injective modules. An R -module M is called *weak-injective* if $\text{Ext}_R^1(N, M) = 0$ for all R -modules N of weak dimension ≤ 1 . In section 2, we introduce a class of R -modules under the name of weak-projective R -modules. We show that weak-projective R -modules are identical to projective R -modules if and only if R is semisimple. Recall that R is called *Prüfer domain* if every finitely generated ideal of R is projective. There are numerous characterizations of Prüfer domains, which can be found in [3]. We show that each weak-projective R -module is *FP*-projective when R is a Noetherian ring. The domain R is called *semi-Dedekind* if every h -divisible R -module is pure-injective. For more details of these domains, we refer the reader to [4].

In section 3, we introduce the concept weak-projective dimension $\text{wpd}(M)$ of an R -module M and give some results. We show that this dimension has the properties that we expect of a “dimension” when the domain is semi-Dedekind.

Throughout this paper, M is an R -module. The notation $(w.)D(R)$ stands for the (weak) global dimension of R . Also, $\text{pd}(M)$ and $\text{id}(M)$ denote the projective and injective dimension of M , respectively. The character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of an R -module M will be denoted by M^b .

2. Weak-projective modules

Recall that an R -module M is called *weak-injective* if $Ext_R^1(N, M) = 0$, for all R -modules N of weak dimension ≤ 1 .

Definition 2.1 *An R -module M is called weak-projective if $Ext_R^1(M, N) = 0$, for every weak-injective R -module N .*

Evidently, direct products and summands of weak-projective R -modules are again weak-projective. All projective R -modules are trivially weak-projective, but the converse is not true. For example, \mathbb{Q}/\mathbb{Z} as a \mathbb{Z} -module is weak-projective, but is not projective. Over a semisimple ring R , weak-projective R -modules are projective. It is obvious that if R is a semisimple ring with $w.D(R) = 1$, then every R -module M is weak-projective if and only if M is weak-injective. Also, if R is semisimple and M is a weak-projective R -module, then $Ext_R^1(M, R) = 0$.

A well-known result states that an R -module F is flat if and only if its character module F^b is injective. The following lemma is an analog of this equivalence.

Lemma 2.2 *(Lee [5, Lemma 3.1]) An R -module A is torsion-free if and only if A^b is weak-injective.*

An R -module M is called *FP-injective* if $Ext_R^1(N, M) = 0$ for all finitely presented R -modules N .

Lemma 2.3 *(Lee[5, Lemma 3.2]) For a domain R , the following are equivalent:*

- (a) R is Prüfer;
- (b) Every weak-injective R -module is FP-injective;
- (c) Every weak-injective R -module is injective.

We may obtain some elementary results on the notion of the weak-projective modules.

Recall that the R -module M is called *FP-projective* [6] if $Ext_R^1(M, N) = 0$, for every FP-injective R -module N .

Lemma 2.4 *If R is a Noetherian ring and M a weak-projective R -module, then M is FP-projective.*

Proof. Let M be a weak-projective R -module. We must prove that $Ext_R^1(M, N) = 0$, for any FP-injective R -module N . Since R is a Noetherian ring, N is an injective R -module, and therefore N is weak-injective. \square

The converse is an easy application of Lemma 2.3.

Lemma 2.5 *Let R be a semi-Dedekind domain and M an R -module. Then the following are equivalent:*

- (a) M is weak-projective;
- (b) $Tor_1^R(M, A) = 0$, for all torsion-free R -modules A ;
- (c) $pd(M) \leq 1$.

Proof. (a) \Rightarrow (b) The isomorphism $Ext_R^1(M, A^b) \cong Hom_{\mathbb{Z}}(Tor_1^R(M, A), \mathbb{Q}/\mathbb{Z})$, together with Lemma 2.2, proves the result.

(b) \Rightarrow (a) This follows from [4, Lemma 4.1].

(b) \Leftrightarrow (c) See [4, Lemma 4.9]. □

It is easy to check that the quotient \mathbb{Z} -module $\mathbb{Z}/2\mathbb{Z}$ is weak-projective.

Combining Lemma 2.5, with the simple fact that an R -module D is divisible if and only if D^b is torsion-free gives the next corollary.

Corollary 2.6 *Let R be a semi-Dedekind domain and M an R -module. Then M is weak-projective if and only if $Tor_1^R(M, D^b) = 0$, for all divisible R -modules D .*

The following fact can be easily verified, so we omit its proof.

Lemma 2.7 *If R is a Prüfer domain, then every R -module is weak-projective.*

Lemma 2.8 *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence such that A and C are weak-projective R -modules. Then B is weak-projective.*

Proof. Let N be a weak-injective R -module. From the induced exact sequence

$$Ext_R^1(C, N) \rightarrow Ext_R^1(B, N) \rightarrow Ext_R^1(A, N),$$

we have $Ext_R^1(B, N) = 0$, since $Ext_R^1(C, N) = Ext_R^1(A, N) = 0$. □

Corollary 2.9 *If every submodule and quotient of an R -module M is weak-projective, then M is weak-projective.*

From the previous corollary we have the following example.

Example 2.10 *The \mathbb{Z} -module \mathbb{Q} is weak-projective.*

Recall that R is called a *Matlis domain* if the projective dimension of Q (or, equivalently, K) is 1. The R -module C is called *Matlis cotorsion* if $Ext_R^1(Q, C) = 0$, and M is called *strongly flat* if $Ext_R^1(M, C) = 0$ for every Matlis cotorsion R -module C .

The next result gives a relationship between weak-projective R -modules and strongly flat R -modules.

Lemma 2.11 *If R is a Matlis domain and M a strongly flat R -module, then M is weak-projective.*

Proof. If M is a strongly flat R -module, then $Ext_R^1(M, N) = 0$, for all Matlis cotorsion R -modules N . It is easy to see that if R is a Matlis domain, then every weak-injective R -module is Matlis cotorsion. □

Lemma 2.12 *Let R be a semi-Dedekind domain. If M is a projective R -module and N a weak-projective R -module, then $M \otimes_R N$ is weak-projective.*

Proof. The isomorphism $Tor_n^R(M \otimes N, A) \cong M \otimes Tor_n^R(N, A)$, together with Lemma 2.5, proves the result. □

The converse is true when R is a local semi-Dedekind domain.

In what follows, $\sigma_M : M \rightarrow E(M)$ denotes the injective envelope of an R -module M . Recall that an injective envelope $\sigma_M : M \rightarrow E(M)$ has the unique mapping property (see [1]) if for any homomorphism $f : M \rightarrow N$ with N injective, there exists a unique homomorphism $g : E(M) \rightarrow N$ such that $g\sigma_M = f$.

Corollary 2.13 *The following statements are equivalent:*

- (a) R is a Prüfer domain;
- (b) Every R -module is weak-projective;
- (c) $\text{Ext}_R^1(M, N) = 0$, for all weak-injective R -modules N ;
- (d) Every weak-injective R -module has an injective envelope with the unique mapping property.

Proof. It is enough to show that (d) \Rightarrow (a).

(d) \Rightarrow (a) Let M be any weak-injective R -module. We have the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & M & \xrightarrow{\sigma_M} & E(M) & \xrightarrow{\gamma} & L & \longrightarrow & 0 \\
 & & \downarrow 0 & & \downarrow \sigma_L \gamma & & \downarrow \sigma_L & & \\
 & & E(L) & \cong & E(L) & \cong & E(L) & &
 \end{array}$$

Note that $\sigma_L \gamma \sigma_M = 0 = 0 \sigma_M$, so $\sigma_L \gamma = 0$ by (d). Therefore $L = \text{im}(\gamma) \subseteq \ker(\sigma_L) = 0$, and hence M is injective. Thus (a) follows. □

We end this section with the following characterizations of weak-projective R -modules.

Let ℓ be a class of R -modules and M an R -module. A homomorphism $\phi \in \text{Hom}_R(N, M)$ with $N \in \ell$ is called an ℓ -precover of M if the induced map

$$\text{Hom}_R(1_{N'}, \phi) : \text{Hom}_R(N', N) \rightarrow \text{Hom}_R(N', M)$$

is surjective for all $N' \in \ell$. An ℓ -precover $\phi \in \text{Hom}_R(N, M)$ is called an ℓ -cover if each $\gamma \in \text{Hom}_R(N, N)$ satisfying $\phi = \phi\gamma$ is an automorphism of N . The class ℓ is called a *precover(cover) class* if every R -module has an ℓ -precover(ℓ -cover).

The ℓ -preenvelope, ℓ -envelope, preenvelope and envelope classes are defined dually (see [9]). In particular, if ℓ is the class of weak-injective R -modules, an ℓ -envelope is called a *weak-injective envelope*.

Proposition 2.14 *If M is an R -module, then the following are equivalent:*

- (a) M is weak-projective;
- (b) M is projective with respect to every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, where A is weak-injective;
- (c) For every exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$, where F is weak-injective, $K \rightarrow F$ is a weak-injective preenvelope of K ;
- (d) M is cokernel of a weak-injective preenvelope $K \rightarrow F$ with F projective.

Proof. (a) \Rightarrow (b) Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence, where A is weak-injective. Then $Ext_R^1(M, A) = 0$ by (a). Thus $Hom_R(M, B) \rightarrow Hom_R(M, C) \rightarrow 0$ is exact, and (b) holds.

(b) \Rightarrow (a) For every weak-injective R -module N , there is a short exact sequence $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$ with E injective, which induces an exact sequence $Hom_R(M, E) \rightarrow Hom_R(M, L) \rightarrow Ext_R^1(M, N) \rightarrow 0$. Since $Hom_R(M, E) \rightarrow Hom_R(M, L) \rightarrow 0$ is exact by (b), we have $Ext_R^1(M, N) = 0$, and (a) follows.

(a) \Rightarrow (c) is easy to verify.

(c) \Rightarrow (d) Let $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ be an exact sequence with P projective. Note that P is weak-injective by hypothesis, thus $K \rightarrow P$ is a weak-injective preenvelope.

(d) \Rightarrow (a) By (d), there is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$, where $K \rightarrow P$ is a weak-injective preenvelope with P projective. It gives rise to the exactness of $Hom_R(P, N) \rightarrow Hom_R(K, N) \rightarrow Ext_R^1(M, N) \rightarrow 0$, for each weak-injective R -module N . Note that $Hom_R(P, N) \rightarrow Hom_R(K, N) \rightarrow 0$ is exact by (d). Hence $Ext_R^1(M, N) = 0$, as desired. \square

3. The weak-projective dimension over semi-Dedekind domains

We begin this section with the definition of weak-injective dimension.

Definition 3.1 (a) For any R -module M , let weak-injective dimension $wid(M)$ of M , denote the smallest integer $n \geq 0$ such that $Ext_R^{n+1}(N, M) = 0$ for every R -module N of weak dimension ≤ 1 . (If no such n exists, set $wid(M) = \infty$).

(b) $wid(R) = \sup\{wid(M) : M \text{ is an } R\text{-module}\}$.

Lemma 3.2 Let R be a semi-Dedekind domain. For an R -module M , the following statements are equivalent:

(a) $wid(M) \leq n$;

(b) $Ext_R^{n+1}(N, M) = 0$ for all R -modules N of weak dimension ≤ 1 ;

(c) If the sequence $0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n \rightarrow 0$ is exact with E_0, E_1, \dots, E_{n-1} weak-injective, then also E_n is weak-injective.

Proof. (a) \Rightarrow (b) Use induction on n . Clear if $wid(M) = n$. If $wid(M) \leq n - 1$ resolve N by $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$ with K and P flat. K have weak dimension ≤ 1 by [4, Corollary 4.4], and $Ext_R^{n+1}(N, M) \cong Ext_R^n(K, M) = 0$ by induction hypothesis.

(b) \Leftrightarrow (c) follows from the isomorphism $Ext_R^{n+1}(N, M) \cong Ext_R^1(N, E_n)$.

(b) \Rightarrow (a) are trivial. \square

Definition 3.3 For an R -module M , let $wpd(M)$ denotes the smallest integer $n \geq 0$ such that $Ext_R^{n+1}(M, N) = 0$ for every weak-injective R -module N and call $wpd(M)$ the weak-projective dimension of M . If no such n exists, set $wpd(M) = \infty$.

Put $\text{rwpD}(R) = \sup\{\text{wpd}(M) : M \text{ is a right } R\text{-module}\}$ and call $\text{rwpD}(R)$ the right weak-projective dimension of R . Similarly, we have $\text{lwpD}(R)$ (we drop the unneeded letters r and l , because R is commutative).

M is called weak-projective if $\text{wpd}(M) = 0$, i.e., $\text{Ext}_R^1(M, N) = 0$ for every weak-injective R -module N .

Remark 3.4 For every ring R and every R -module M , the inequalities $\text{wpD}(R) \leq D(R)$ and $\text{wpd}(M) \leq \text{pd}(M)$ are valid. It is easy to see that $\text{wpd}(M) = \text{pd}(M)$ for any R -module M if and only if every weak-projective R -module is projective.

Proposition 3.5 Let R be a semi-Dedekind domain. For any R -module M and an integer $n \geq 0$, the following are equivalent:

- (a) $\text{wpd}(M) \leq n$;
- (b) $\text{Ext}_R^{n+1}(M, N) = 0$ for any weak-injective R -module N ;
- (c) $\text{Ext}_R^{n+j}(M, N) = 0$ for any weak-injective R -module N and $j \geq 1$;
- (d) There exists an exact sequence $0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$, where each P_i is weak-projective.

Proof. (c) \Rightarrow (a) is obvious.

(b) \Rightarrow (c) For any weak-injective R -module N , there is a short exact sequence $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$, where E is injective. Then the sequence $\text{Ext}_R^{n+1}(M, L) \rightarrow \text{Ext}_R^{n+2}(M, N) \rightarrow \text{Ext}_R^{n+2}(M, E) = 0$ is exact. Note that L is weak-injective by Lemma 3.2, so $\text{Ext}_R^{n+1}(M, L) = 0$ by (b). Hence $\text{Ext}_R^{n+2}(M, N) = 0$, and (c) follows by induction.

The proof of (a) \Rightarrow (b) is similar to that of (b) \Rightarrow (c).

(a) \Leftrightarrow (d) is straightforward. □

Proposition 3.6 For an R -module M , the following are equivalent:

- (a) $\text{wpD}(R) = 0$;
- (b) $\text{Tor}_1^R(M, A) = 0$, for all torsion-free R -modules A ;
- (c) M has weak dimension ≤ 1 ;
- (d) R is Prüfer;
- (e) Every R -module is weak-projective.

Proof. (a) \Rightarrow (b) The isomorphism $\text{Ext}_R^1(M, A^b) \cong \text{Hom}_{\mathbb{Z}}(\text{Tor}_1^R(M, A), \mathbb{Q}/\mathbb{Z})$, together with Lemma 2.2, proves the result.

(b) \Rightarrow (c) see [5, Corollary 2.4].

(c) \Rightarrow (d) is trivial.

(d) \Rightarrow (e) see Lemma 2.7.

(e) \Rightarrow (a) is trivial. □

Remark 3.7 (a) By Proposition 3.6, $wpD(R)$ measures how far away a domain R is from being a Prüfer domain.

(b) It is well known that R is semihereditary domain if and only if R is Prüfer domain.

The proof of the next proposition is standard homological algebra.

Proposition 3.8 Let R be a semi-Dedekind domain, $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ an exact sequence of R -modules. If two of $wpd(A)$, $wpd(B)$, and $wpd(C)$ are finite, so is the third. Moreover,

(a) $wpd(B) \leq \max\{wpd(A), wpd(C)\}$.

(b) $wpd(A) \leq \max\{wpd(B), wpd(C) - 1\}$.

(c) $wpd(C) \leq \max\{wpd(B), wpd(A) + 1\}$.

Corollary 3.9 Let R be a semi-Dedekind domain.

(a) If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of R -modules, where $0 < wpd(A) < \infty$ and B is weak-projective, then $wpd(C) = wpd(A) + 1$.

(b) $wpd(R) = n$ if and only if $\sup\{wpd(I) : I \text{ is any ideal of } R\} = n - 1$ for any integer $n \geq 2$.

Proof. (a) is true by Proposition 3.8.

(b) For an ideal of R , consider the exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$. Then (b) follows from (a). \square

Theorem 3.10 Let R be a semi-Dedekind domain. Then The following values are identical:

(a) $wpd(R)$;

(b) $\sup\{wpd(M) : M \text{ is a cyclic } R\text{-module}\}$;

(c) $\sup\{wpd(M) : M \text{ is any } R\text{-module}\}$;

(d) $\sup\{id(F) : F \text{ is a weak-injective } R\text{-module}\}$.

Proof. (b) \leq (a) \leq (c) are obvious.

(c) \leq (d) We may assume $\sup\{id(F) : F \text{ is a weak-injective } R\text{-module}\} = m < \infty$. Let M be any R -module and N any weak-injective R -module. Since $id(N) \leq m$, it follows that $Ext_R^{m+1}(M, N) = 0$. Hence $wpd(M) \leq m$.

(d) \leq (b) We may assume $\sup\{wpd(M) : M \text{ is a cyclic } R\text{-module}\} = n < \infty$. Let N be a weak-injective R -module and I any ideal, then $wpd(R/I) \leq n$. By Proposition 3.5, $Ext_R^{n+1}(R/I, N) = 0$, and so $id(N) \leq n$. \square

Proposition 3.11 Let R be a semi-Dedekind domain. Then the following are equivalent:

(a) $wpd(R) \leq 1$;

(b) Every submodule of a (weak-)projective R -module is weak-projective;

(c) Every ideal of R is weak-projective.

Proof. (a) \Rightarrow (b) Let N be a submodule of a weak-projective R -module M . Then, for any weak-injective R -module L , we get an exact sequence

$$0 = Ext_R^1(M, L) \rightarrow Ext_R^1(N, L) \rightarrow Ext_R^2(M/N, L).$$

Note that the last term is zero by (a), hence $Ext_R^1(N, L) = 0$, and (b) follows.

(b) \Rightarrow (c) is trivial.

(c) \Rightarrow (a) Let I be an ideal of R . The exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ implies $wpd(R/I) \leq 1$ by Proposition 3.5. So (a) follows from Theorem 3.10 (b). \square

It is well known that if M is finitely generated projective R -module, then $Hom_R(M, R)$ is finitely generated projective R -module. Here we have the following corollary.

Corollary 3.12 *If R is a semi-Dedekind domain with $wpD(R) \leq 1$, then the dual module $Hom_R(M, R)$ of any finitely generated R -module M is weak-projective.*

In addition, if $w.D(R) = 1$, then the following are equivalent:

- (a) *Every torsion-free R -module is weak-projective;*
- (b) *M^b is weak-projective for every injective R -module M ;*
- (c) *N^{bb} is weak-projective for every torsion-free R -module N .*

Proof. Let M be a finitely generated R -module. Then there exists an exact sequence $P \rightarrow M \rightarrow 0$ with P finitely generated projective. So we have an R -module exact sequence $0 \rightarrow Hom_R(M, R) \rightarrow Hom_R(P, R)$. Note that $Hom_R(P, R)$ is projective, therefore $Hom_R(M, R)$ is weak-projective by Proposition 3.11.

Also, if $w.D(R) = 1$, then (a) \Rightarrow (b) \Rightarrow (c) are clear.

(c) \Rightarrow (a) Let N be any torsion-free R -module. There exists an exact sequence $0 \rightarrow N \rightarrow N^{bb}$. Since $wpD(R) \leq 1$ and N^{bb} is weak-projective by (c), we have that N is weak-projective by Proposition 3.11. \square

A ring R is called semi-Artinian if every nonzero cyclic R -module has a nonzero socle. The following proposition shows that we may compute the weak-projective dimension of semi-Artinian ring using just the weak-projective dimension of simple modules.

Proposition 3.13 *If R is a semi-Artinian semi-Dedekind domain, then $wpD(R) = \sup\{wpd(M) : M \text{ is a simple } R\text{-module}\}$.*

Proof. It suffices to show that $wpD(R) \leq \sup\{wpd(M) : M \text{ is a simple } R\text{-module}\}$. We may assume that $\sup\{wpd(M) : M \text{ is a simple } R\text{-module}\} = n < \infty$. Let N be a weak-injective R -module and I a maximal ideal of R . Consider the injective resolution of N

$$0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots \rightarrow E^{n-1} \rightarrow E^n \rightarrow \dots$$

Write $L = \text{coker}(E^{n-2} \rightarrow E^{n-1})$. Then $Ext_R^1(R/I, L) = Ext_R^{n+1}(R/I, N) = 0$ by Proposition 3.5. Therefore L is injective by [8, Lemma 4], since R is semi-Artinian. So $id(N) \leq n$, and hence $wpD(R) \leq n$ by Theorem 3.10. \square

Proposition 3.14 *Let R be a semi-Dedekind domain. Then $\sup\{pd(M) : M \text{ is a weak-projective } R\text{-module}\} \leq wiD(R)$.*

Proof. Let M be a weak-projective R -module. It is enough to show that $pd(M) \leq wiD(R)$. We may assume that $wiD(R) = n < \infty$. M admits a projective resolution

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow \cdots P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Let N be any R -module. We have $wid(N) \leq n$, thus by Lemma 3.2, there is an exact sequence

$$0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{n-1} \rightarrow E^n \rightarrow 0,$$

where E^0, E^1, \dots, E^n are weak-injective. Therefore we form a double complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & Hom_R(M, E^n) & \rightarrow & Hom_R(P_0, E^n) & \rightarrow & \cdots \rightarrow Hom_R(P_n, E^n) \rightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & Hom_R(M, E^1) & \rightarrow & Hom_R(P_0, E^1) & \rightarrow & \cdots \rightarrow Hom_R(P_n, E^1) \rightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & Hom_R(M, E^0) & \rightarrow & Hom_R(P_0, E^0) & \rightarrow & \cdots \rightarrow Hom_R(P_n, E^0) \rightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & \rightarrow & Hom_R(P_0, N) & \rightarrow & \cdots \rightarrow Hom_R(P_n, N) \rightarrow \cdots \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0. \end{array}$$

Note that all rows are exact except for the bottom row, since M is weak-projective and all E^i are weak-injective; also note that all columns are exact except for the left column since all P_i are projective.

Using a spectral sequence argument, we know that the two complexes

$$0 \rightarrow Hom_R(P_0, N) \rightarrow Hom_R(P_1, N) \rightarrow \cdots \rightarrow Hom_R(P_n, N) \rightarrow \cdots$$

and

$$0 \rightarrow Hom_R(M, E^0) \rightarrow Hom_R(M, E^1) \rightarrow \cdots \rightarrow Hom_R(M, E^n) \rightarrow 0$$

have isomorphic homology groups. Thus $Ext_R^{n+j}(M, N) = 0$ for all $j \geq 1$. Hence $pd(M) \leq n$. □

It is known that $D(R) = \sup\{pd(M) : M \text{ is a weak-projective } R\text{-module}\}$ if R is a Prüfer domain, and it is easy to see that $D(R) = wpD(R)$ if R is a semisimple ring. In general, we have

Proposition 3.15 *Let R be a semi-Dedekind domain and M be an R -module. Then $D(R) \leq \sup\{pd(M) : M \text{ is a weak-projective } R\text{-module}\} + wpD(R)$.*

Proof. We may assume without loss of generality that $wpD(R)$ is finite. Let $wpD(R) = m < \infty$ and $\sup\{pd(M) : M \text{ is a weak-projective } R\text{-module}\} = n < \infty$. If M is an R -module, then $wpd(M) \leq m$ by Theorem 3.10. So M admits a weak-projective resolution

$$0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where each P_i is weak-projective, $i = 0, 1, 2, \dots, m$. Let $K_i = Ker(P_i \rightarrow P_{i-1})$, $i = 0, 1, 2, \dots, m - 1$, $P_{-1} = M$, $K_{m-1} = P_m$. Then we have the following short exact sequence

$$\begin{aligned} 0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow K_{m-2} \rightarrow 0, \\ 0 \rightarrow K_{m-2} \rightarrow P_{m-2} \rightarrow K_{m-3} \rightarrow 0, \\ \vdots \\ 0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0. \end{aligned}$$

It follows that $pd(K_{m-2}) \leq 1 + n$, $pd(K_{m-3}) \leq 2 + n, \dots, pd(M) \leq m + n$, and hence $D(R) \leq m + n$. This completes the proof. \square

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