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# Covers and envelopes with respect to a semidualizing module

*Xiaoguang Yan and Xiaosheng Zhu*

## Abstract

Let  $R$  be a commutative ring and  $\mathcal{C}$  be a semidualizing  $R$ -module. For a given class of  $R$ -modules  $\mathcal{Q}$ , we define a class  $\mathcal{Q}_{\mathcal{C}}$  by  $M \in \mathcal{Q}_{\mathcal{C}} \Leftrightarrow \text{Hom}_R(\mathcal{C}, M) \in \mathcal{Q}$ . We prove that if  $\mathcal{Q} \subseteq \mathcal{A}_{\mathcal{C}}(R)$  is a Kaplansky class and closed under direct sums, then  $\mathcal{Q}_{\mathcal{C}}^{\perp}$  is special preenveloping. As corollaries, we can show that  $\mathcal{P}_{\mathcal{C}}^{\perp}$  and  $\mathcal{F}_{\mathcal{C}}^{\perp}$  are both special preenveloping. Finally, we show that  $\mathcal{I}_{\mathcal{C}}^n$  is covering,  $\mathcal{I}_{\mathcal{C}}^{n\perp}$  is enveloping and special preenveloping provided  $R$  is Noetherian.

**Key Words:** Semidualizing module, Kaplansky class, Auslander class, Bass class, (pre)envelope, (pre)cover

## 1. Introduction

(Pre)covers and (pre)envelopes are important notions in relative homological algebra. For example, let  $R$  be a ring, given a precovering class  $\mathcal{F}$ , then for any  $R$ -module we can construct a  $\text{Hom}_R(\mathcal{F}, -)$ -exact complex  $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$  with  $F_i \in \mathcal{F}, i = 1, 2, \dots$ , which we call proper  $\mathcal{F}$ -resolution of  $M$ . Proper resolutions of a given module is unique up to homotopy, hence, one can define relative derived functors through these resolutions. Dually, given a preenveloping class, one can define relative derived functors through the “co-proper” resolutions. So it is useful to show a concrete class is (pre)covering or (pre)enveloping under suitable conditions.

Recently, the notion of semidualizing modules has caught some authors’ attention. Foxby [11], Vasconcelos [26] and Golod [13] initiated the study of semidualizing modules under different names, while Holm and White [18] extended the definition of a semidualizing module to a pair of arbitrary rings. Especially, they defined the so-called  $\mathcal{C}$ -projective,  $\mathcal{C}$ -injective and  $\mathcal{C}$ -flat modules (see Definition 2.3), to characterize the Auslander class  $\mathcal{A}_{\mathcal{C}}(R)$  and the Bass class  $\mathcal{B}_{\mathcal{C}}(R)$  (see Definition 2.4), with respect to a semidualizing module  $\mathcal{C}$ . The notion of  $\mathcal{C}$ -projective ( $\mathcal{C}$ -injective,  $\mathcal{C}$ -flat) modules is fundamental for the study of relative homological algebra with respect to semidualizing modules. Holm and White proved in [18, Proposition 5.10] that the class  $\mathcal{P}_{\mathcal{C}}$  of  $\mathcal{C}$ -projective modules is precovering, the class  $\mathcal{F}_{\mathcal{C}}$  of  $\mathcal{C}$ -flat modules is covering and the class  $\mathcal{I}_{\mathcal{C}}$  of  $\mathcal{C}$ -injective modules is enveloping. We know that  $\mathcal{C}$ -projective ( $\mathcal{C}$ -flat,  $\mathcal{C}$ -injective) modules is build from projective (flat, injective) modules and the semidualizing module  $\mathcal{C}$ . Now for a given class of modules  $\mathcal{Q}$ , define a corresponding class  $\mathcal{Q}_{\mathcal{C}}$  in some way. (For example, define  $\mathcal{Q}_{\mathcal{C}}$  as follows:  $M \in \mathcal{Q}_{\mathcal{C}} \Leftrightarrow \text{Hom}_R(\mathcal{C}, M) \in \mathcal{Q}$ . In

this case, if  $\mathcal{Q}$  is the class of projective modules then  $\mathcal{Q}_{\mathcal{C}}$  is just the class of  $\mathcal{C}$ -projective modules.) In general, the class  $\mathcal{Q}_{\mathcal{C}}$  is difficult to handle, while the given class  $\mathcal{Q}$  may be well-understood. So a natural question is: can one deduce to the existence of (pre)covers or (pre)envelopes of  $\mathcal{Q}_{\mathcal{C}}$  from some properties of  $\mathcal{Q}$ ?

This paper is divided into two sections. In Section 2, we recall some definitions, give some terminology and notations for the use throughout this work. We prove our main results in Section 3. Our main theorem is Theorem 3.6.

**Theorem** *Let  $R$  be a commutative ring and  $\mathcal{C}$  be a semidualizing  $R$ -module, let  $\mathcal{Q}$  be a Kaplansky class with  $\mathcal{Q} \subseteq \mathcal{A}_{\mathcal{C}}(R)$ . Consider the class  $\mathcal{Q}_{\mathcal{C}}$  defined by  $M \in \mathcal{Q}_{\mathcal{C}} \Leftrightarrow \text{Hom}_R(\mathcal{C}, M) \in \mathcal{Q}$ . If  $\mathcal{Q}$  is closed under direct sums, then  $\mathcal{Q}_{\mathcal{C}}^{\perp}$  is special preenveloping.*

Then, as corollaries of the main theorem, we show that  $\mathcal{P}_{\mathcal{C}}^{n\perp}$  is special preenveloping (see Corollary 3.9 below),  $\mathcal{F}_{\mathcal{C}}^{n\perp}$  is enveloping and if  $R$  is coherent then  $\mathcal{F}_{\mathcal{C}}^n$  is preenveloping (see Corollary 3.11). At the end of this paper we show the existence of (pre)envelops and (pre)covers by  $\mathcal{I}_{\mathcal{C}}^n$  and  $\mathcal{I}_{\mathcal{C}}^{n\perp}$  (see Theorem 3.13).

Throughout this work,  $R$  is a commutative ring with identity and all modules are unitary. So when we say a ring is Noetherian we mean that it is a commutative Noetherian ring.  $\mathcal{C}$  is always a semidualizing module for  $R$ . A subcategory or a class of modules always means a full subcategory of the category of  $R$ -modules, which is closed under isomorphisms. For unexplained concepts and notations, we refer the reader to [1], [9] and [23].

## 2. Preliminaries

In this section, we give some terminology, and recall some definitions and some known results that we need in the sequel. Among these are semidualizing modules, Auslander class, Bass class,  $\mathcal{C}$ -projective,  $\mathcal{C}$ -injective,  $\mathcal{C}$ -flat modules with respect to a semidualizing module  $\mathcal{C}$  and Kaplansky class.

**Definition 2.1** [22, 23] For a given  $R$ -module  $M$ , and a class of  $R$ -modules  $\mathcal{X}$ , an augmented  $\mathcal{X}$ -resolution of  $M$  is an exact sequence  $X^+ : \cdots \rightarrow X_n \xrightarrow{\partial_n^X} X_{n-1} \rightarrow \cdots \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} M \rightarrow 0$  with  $X_i \in \mathcal{X}$ , for all  $i$ . The truncated complex  $X : \cdots \rightarrow X_n \xrightarrow{\partial_n^X} X_{n-1} \rightarrow \cdots \xrightarrow{\partial_1^X} X_0 \rightarrow 0$  is called an  $\mathcal{X}$ -resolution of  $M$ . An  $\mathcal{X}$ -resolution of  $M$  is said to be proper, if the corresponding augmented resolution  $X^+$  is  $\text{Hom}_R(\mathcal{X}, -)$ -exact. The  $\mathcal{X}$ -projective dimension of  $M$  is defined as

$$\mathcal{X}\text{-}pd_R M = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

Dually, we can define  $\mathcal{Y}$ -coresolution of an  $R$ -module  $N$  with a given class of  $R$ -modules  $\mathcal{Y}$  and  $\mathcal{Y}$ -injective dimension of  $N$ , denoted by  $\mathcal{Y}\text{-}id_R N$ . We denote an augmented  $\mathcal{Y}$ -coresolution by  ${}^+Y$  and the corresponding  $\mathcal{Y}$ -coresolution  $Y$  is said to be proper if  ${}^+Y$  is  $\text{Hom}_R(-, \mathcal{Y})$ -exact.

As usual, for an  $R$ -module  $M$ , we denote projective, injective and flat dimension of  $M$  by  $pd_R M$ ,  $id_R M$  and  $fd_R M$ , respectively.

The definition of a semidualizing module has already been extended to arbitrary associative rings (see [18, Definition 2.1]). Note here the ground ring is commutative.

**Definition 2.2** [28, 1.8] An  $R$ -module  $\mathcal{C}$  is semidualizing if it satisfies the following conditions:

- (1)  $\mathcal{C}$  admits a (possibly unbounded) resolution by finitely generated projective modules;
- (2) The natural homothety map  $R \rightarrow \text{Hom}_R(\mathcal{C}, \mathcal{C})$  is an isomorphism; and
- (3)  $\text{Ext}_R^{\geq 1}(\mathcal{C}, \mathcal{C}) = 0$ .

A free  $R$ -module of rank one is semidualizing. If  $R$  is Cohen-Macaulay and admits dualizing module  $D$ , then  $D$  is semidualizing. More examples of semidualizing modules can be found in [5, 18].

In [18], Holm and White defined  $\mathcal{C}$ -projective,  $\mathcal{C}$ -injective and  $\mathcal{C}$ -flat modules in order to study the Auslander class  $\mathcal{A}_{\mathcal{C}}(R)$  and the Bass class  $\mathcal{B}_{\mathcal{C}}(R)$  with respect to a semidualizing module  $\mathcal{C}$ .

**Definition 2.3** [18, Definition 5.1] Let  $\mathcal{C}$  be a semidualizing module for a ring  $R$ . An  $R$ -module is  $\mathcal{C}$ -projective ( $\mathcal{C}$ -flat) if it is of the form  $\mathcal{C} \otimes_R P$  ( $\mathcal{C} \otimes_R F$ ) for some projective (flat) module  $P$  ( $F$ ). An  $R$ -module is  $\mathcal{C}$ -injective if it is of the form  $\text{Hom}_R(\mathcal{C}, E)$  for some injective module  $E$ . We set

- $\mathcal{I}_{\mathcal{C}}^n$  = the category of modules with  $\mathcal{I}_{\mathcal{C}}$ -injective dimension less than or equal to  $n$ .
- $\mathcal{P}_{\mathcal{C}}^n$  = the category of modules with  $\mathcal{P}_{\mathcal{C}}$ -projective dimension less than or equal to  $n$ .
- $\mathcal{F}_{\mathcal{C}}^n$  = the category of modules with  $\mathcal{F}_{\mathcal{C}}$ -projective dimension less than or equal to  $n$ .

Note that by Definition 2.1 we say that a module  $M$  has  $\mathcal{P}_{\mathcal{C}}$ -projective dimension less than or equal to  $n$  if and only if there exists an exact sequence  $0 \rightarrow P_n \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$  with each  $P_i$   $\mathcal{C}$ -projective. Similarly, a module  $N$  has  $\mathcal{I}_{\mathcal{C}}$ -injective dimension less than or equal to  $n$  if and only if there exists an exact sequence of the form  $0 \rightarrow N \rightarrow E_0 \rightarrow \dots \rightarrow E_{-n} \rightarrow 0$  with each  $E_i$   $\mathcal{I}_{\mathcal{C}}$ -injective. Moreover, we set:

- $\mathcal{I}^n$  = the category of modules with injective dimension less than or equal to  $n$ .
- $\mathcal{P}^n$  = the category of modules with projective dimension less than or equal to  $n$ .
- $\mathcal{F}^n$  = the category of modules with flat dimension less than or equal to  $n$ .

Over a noetherian ring, Avramov and Foxby [3, 11] and Enochs, Jenda and Xu [10] connected the study of (semi)dualizing modules to associated Auslander class  $\mathcal{A}_{\mathcal{C}}(R)$  and Bass class  $\mathcal{B}_{\mathcal{C}}(R)$  for (semi)dualizing modules, which are subcategories of the category of  $R$ -modules.

**Definition 2.4** [28, 1.14] The Auslander class of  $R$  with respect to  $\mathcal{C}$ , denoted  $\mathcal{A}_{\mathcal{C}}(R)$ , consists of modules  $M$  satisfying:

- (a1)  $\text{Tor}_{\geq 1}^R(\mathcal{C}, M) = 0$ ;
- (a2)  $\text{Ext}_R^{\geq 1}(\mathcal{C}, \mathcal{C} \otimes_R M) = 0$ ; and
- (a3) the canonical map  $\mu_M : M \rightarrow \text{Hom}_R(\mathcal{C}, \mathcal{C} \otimes_R M)$  is an isomorphism.

The Bass class of  $R$  with respect to  $\mathcal{C}$ , denoted  $\mathcal{B}_{\mathcal{C}}(R)$ , consists of modules  $N$  satisfying:

- (b1)  $\text{Ext}_R^{\geq 1}(\mathcal{C}, N) = 0$ ;
- (b2)  $\text{Tor}_{\geq 1}^R(\mathcal{C}, \text{Hom}_R(\mathcal{C}, N)) = 0$ ; and
- (b3) the canonical map  $\nu_N : \mathcal{C} \otimes_R \text{Hom}_R(\mathcal{C}, N) \rightarrow N$  is an isomorphism.

**Remark 2.5** Let  $\mathcal{C}$  be a semidualizing  $R$ -module. If two of the three modules in a short exact sequence are in  $\mathcal{A}_{\mathcal{C}}(R)(\mathcal{B}_{\mathcal{C}}(R))$ , so is the third (see [18, Corollary 6.3.]). The category  $\mathcal{A}_{\mathcal{C}}(R)$  contains modules of finite flat dimension and modules of finite  $\mathcal{I}_{\mathcal{C}}$ -injective dimension, and the category  $\mathcal{B}_{\mathcal{C}}(R)$  contains modules of finite injective dimension and modules of finite  $\mathcal{F}_{\mathcal{C}}$ -projective, hence finite  $\mathcal{P}_{\mathcal{C}}$ -projective dimension (see [18, Corollaries 6.1 and 6.2]).

Next lemma is used frequently, and it goes back to [25 Theorem 2.8].

**Lemma 2.6** *Let  $\mathcal{C}$  be a semidualizing module, then for a given  $R$ -module  $M$ :*

1.  $M \in \mathcal{B}_{\mathcal{C}}(R)$  if and only if  $\text{Hom}_R(\mathcal{C}, M) \in \mathcal{A}_{\mathcal{C}}(R)$ ;
2.  $M \in \mathcal{A}_{\mathcal{C}}(R)$  if and only if  $\mathcal{C} \otimes_R M \in \mathcal{B}_{\mathcal{C}}(R)$ . □

We conclude this section by recalling the definition of Kaplansky class.

**Definition 2.7** [8, Definition 2.1] Let  $\mathcal{F}$  be a class of modules. Then  $\mathcal{F}$  is said to be a Kaplansky class if there exists a cardinal number  $\aleph$  such that for any  $M \in \mathcal{F}$  and any  $x \in M$ , there exists a submodule  $F$  of  $M$  such that  $x \in F, F \in \mathcal{F}, M/F \in \mathcal{F}$  and  $|F| \leq \aleph$ , where  $|F|$  denotes the cardinality of  $F$ .

### 3. Main results

We give our main results in this section. First we show the existence of special preenvelops relative to a Kaplansky class. Recall that Enochs and López-Ramos proved that if  $\mathcal{F}$  is a Kaplansky class which is closed under extensions and direct limits, then  $\mathcal{F}^\perp$  is enveloping [8, Theorem 2.8]. Replacing the condition “closed under extensions and direct limits” by “closed under direct sums” we get the following proposition, and the proof is inspired by that of [1, Proposition 2.6]. For the completeness, we give the proof.

**Proposition 3.1** *If  $\mathcal{Q}$  is a Kaplansky class which is closed under direct sums, then  $\mathcal{Q}^\perp$  is special preenveloping.*

**Proof.** Since  $\mathcal{Q}$  is a Kaplansky class, there exists a cardinal number  $\aleph$  such that each  $L \in \mathcal{Q}$  can be written as a direct union of a continuous chain of submodules  $\{L_\alpha, \alpha < \lambda\}$  for some ordinal number  $\lambda$  such that  $L_0 \in \mathcal{Q}, |L_0| \leq \aleph, L_{\alpha+1}/L_\alpha \in \mathcal{Q}$  and  $|L_{\alpha+1}/L_\alpha| \leq \aleph$  when  $\alpha + 1 < \lambda$ . Let  $\mathcal{S}$  be the representative set of modules  $L$  in  $\mathcal{Q}$  with  $|L| \leq \aleph$ . Then by [6, Theorem 1.2], it is not difficult to see that for an  $R$ -module  $M$ ,  $M \in \mathcal{Q}^\perp \Leftrightarrow \text{Ext}_R^1(L, M) = 0$  for all  $L \in \mathcal{S} \Leftrightarrow \text{Ext}_R^1(B, M) = 0$ , where  $B = \bigoplus_{L \in \mathcal{S}} L$ .

Now we only need to employ the method developed in [1, Theorem 2.6].

Take an exact sequence  $0 \rightarrow K \rightarrow P \rightarrow B \rightarrow 0$  with  $P$  projective. For a given  $R$ -module  $M$ , consider the pushout of the map  $\varphi : K^{\text{Hom}(K, M)} \rightarrow M$  defined by  $\varphi(\{x_f\}) = \Sigma f(x)$  and the inclusion map  $K^{\text{Hom}(K, M)} \rightarrow P^{\text{Hom}(K, M)}$ :

$$\begin{array}{ccc}
 K^{\text{Hom}(K, M)} & \longrightarrow & P^{\text{Hom}(K, M)} \\
 \varphi \downarrow & & \downarrow \\
 M & \xrightarrow{\iota} & M_1.
 \end{array}$$

Thus  $\iota$  is the inclusion and  $M_1/M \cong B(\text{Hom}(K, M)) \in \mathcal{Q}$ , since  $\mathcal{Q}$  is closed under direct sums. Moreover, for any morphism  $f : K \rightarrow M$  there is a morphism  $f' : P \rightarrow M_1$  such that the following diagram is commutative:

$$\begin{array}{ccc} K & \longrightarrow & P \\ f \downarrow & & \downarrow f' \\ M & \longrightarrow & M_1. \end{array}$$

Continuing this process, for any ordinal number  $\lambda$  we can construct a continuous chain of modules  $\{M_\alpha, \alpha < \lambda\}$  such that  $M_0 = M, M_{\alpha+1}/M_\alpha \in \mathcal{Q}$  for all  $\alpha + 1 < \lambda$  and for any  $K \rightarrow M_\alpha$  there exists a  $P \rightarrow M_{\alpha+1}$  with the commutative diagram:

$$\begin{array}{ccc} K & \longrightarrow & P \\ \downarrow & & \downarrow \\ M_\alpha & \longrightarrow & M_{\alpha+1}. \end{array}$$

By [1, Proposition 2.1 and Corollary 2.2], for the given module  $M$ , we get an exact sequence  $0 \rightarrow M \rightarrow T \rightarrow L \rightarrow 0$  with  $T = \bigcup_{\alpha < \lambda} M_\alpha \in \mathcal{Q}^\perp$  for some ordinal number  $\lambda$  and  $L = T/M = \bigcup_{\alpha < \lambda} M_\alpha/M = \bigcup_{\alpha < \lambda} L_\alpha$ . Note that  $\{L_\alpha, \alpha < \lambda\}$  is a continuous chain of submodules of  $L$  such that  $L_0 = 0, L_{\alpha+1}/L_\alpha \cong M_{\alpha+1}/M_\alpha \in \mathcal{Q}, \alpha + 1 < \lambda$ . Thus  $\{L_\alpha, \alpha < \lambda\}$  is a  ${}^\perp(\mathcal{Q}^\perp)$ -filtration of  $L$  (see [12, Definition 3.1.1]), since  $\mathcal{Q} \subseteq {}^\perp(\mathcal{Q}^\perp)$ . Therefore,  $L \in {}^\perp(\mathcal{Q}^\perp)$  by *Eklof Lemma* (see [12, Lemma 3.1.2]). So by definition,  $0 \rightarrow M \rightarrow T$  is a special  $\mathcal{Q}^\perp$ -preenvelope of  $M$ . □

**Corollary 3.2** *If  $\mathcal{Q}$  is a Kaplansky class which is closed under direct sums, then  $({}^\perp(\mathcal{Q}^\perp), \mathcal{Q}^\perp)$  is a complete cotorsion theory.*

**Proof.** It is trivial that the pair is a cotorsion theory. From Proposition 3.1, we know that this cotorsion theory has enough injectives, so by [9, Proposition 7.1.7] it is complete. □

**Example 3.3** From [8, Proposition 2.10] we know that the class of Gorenstein flat modules  $\mathcal{GF}$  is a Kaplansky class. Since  $\mathcal{GF}$  closed under direct sums, we can say that every module admits a special  $\mathcal{GF}^\perp$ -preenvelope, and  $({}^\perp(\mathcal{GF}^\perp), \mathcal{GF}^\perp)$  is a complete cotorsion theory.

We know that an  $R$ -module  $M$  is  $\mathcal{C}$ -projective, that is,  $M = \mathcal{C} \otimes_R P$  for some projective module  $P$ , if and only if  $\text{Hom}_R(\mathcal{C}, M)$  is projective. Following this manner, we give the following definition.

**Definition 3.4** For a given class  $\mathcal{Q} \subseteq \mathcal{A}_C(R)$ , we define a class  $\mathcal{Q}_C$  as follows:  $M \in \mathcal{Q}_C \Leftrightarrow \text{Hom}_R(\mathcal{C}, M) \in \mathcal{Q}$ .

By definition and Lemma 2.6, it is easy to see that  $\mathcal{Q}_C \subseteq \mathcal{B}_C(R)$ , for any module  $N, N \in \mathcal{Q} \Leftrightarrow \mathcal{C} \otimes_R N \in \mathcal{Q}_C$ , and every element in  $\mathcal{Q}_C$  has the form  $\mathcal{C} \otimes_R N$  with  $N \in \mathcal{Q}$ . Note that if we take  $\mathcal{Q}$  to be  $\mathcal{P}^n(\mathcal{F}^n)$ , then by [25, Theorem 2.11(c)]  $\mathcal{Q}_C$  is just  $\mathcal{P}_C^n(\mathcal{F}_C^n)$ .

In order to prove our main theorem, we need the following lemma.

**Lemma 3.5** *Let  $\mathcal{Q}$  and  $\mathcal{Q}_C$  be as in Definition 3.4, then  $\mathcal{Q}$  is closed under extensions (direct limits, direct summands, direct products) if and only if  $\mathcal{Q}_C$  is closed under extensions (direct limits, direct summands, direct products).*

**Proof.** We only give the proof for “extensions,” and for the proof of “direct limits” and “direct products” one only need to note that  $\mathcal{C}$  is finitely presented.

First, suppose that  $\mathcal{Q}$  is closed under extensions. Any exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  with  $M', M'' \in \mathcal{Q}_C$  is  $\text{Hom}_R(\mathcal{C}, -)$ -exact, since  $M' \in \mathcal{Q}_C \subseteq \mathcal{B}_C(R)$ . Thus, applying the functor  $\text{Hom}_R(\mathcal{C}, -)$  to the exact sequence, we get another exact sequence  $0 \rightarrow \text{Hom}_R(\mathcal{C}, M') \rightarrow \text{Hom}_R(\mathcal{C}, M) \rightarrow \text{Hom}_R(\mathcal{C}, M'') \rightarrow 0$  with  $\text{Hom}_R(\mathcal{C}, M'), \text{Hom}_R(\mathcal{C}, M'') \in \mathcal{Q}$ . So  $\text{Hom}_R(\mathcal{C}, M) \in \mathcal{Q}$ , then by definition  $M \in \mathcal{Q}_C$ .

Conversely, suppose that  $\mathcal{Q}_C$  is closed under extensions. Any exact sequence  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  with  $N', N'' \in \mathcal{Q}$  is  $\mathcal{C} \otimes_R -$ -exact, since  $N'' \in \mathcal{Q} \subseteq \mathcal{A}_C(R)$ . Thus, applying the functor  $\mathcal{C} \otimes_R -$  to the exact sequence, we get another exact sequence  $0 \rightarrow \mathcal{C} \otimes_R N' \rightarrow \mathcal{C} \otimes_R N \rightarrow \mathcal{C} \otimes_R N'' \rightarrow 0$  with  $\mathcal{C} \otimes_R N', \mathcal{C} \otimes_R N'' \in \mathcal{Q}_C$ , so  $\mathcal{C} \otimes_R N \in \mathcal{Q}_C$ . Therefore  $N \in \mathcal{Q}$ . □

Now we are in position to give our main theorem:

**Theorem 3.6** *Let  $\mathcal{Q}$  be a Kaplansky class with  $\mathcal{Q} \subseteq \mathcal{A}_C(R)$ . If  $\mathcal{Q}$  is closed under direct sums (hence  $\mathcal{Q}^\perp$  is special preenveloping by Proposition 3.1), then  $\mathcal{Q}_C^\perp$  is special preenveloping.* □

We see that if we could prove that  $\mathcal{Q}_C$  is a Kaplansky class, then by Proposition 3.1 and Lemma 3.5,  $\mathcal{Q}_C^\perp$  is special preenveloping. But, in general, we do not know whether  $\mathcal{Q}_C$  is a Kaplansky class. (we give an affirmative answer for a special case, see Proposition 3.10). But note that, in Proposition 3.1, the crucial step to show the existence of special  $\mathcal{Q}^\perp$ -preenvelops is to understand that each  $M \in \mathcal{Q}$  can be written as a direct union of a continuous chain of submodules  $\{M_\alpha, \alpha < \lambda\}$  for some ordinal number  $\lambda$  such that  $M_0 \in \mathcal{Q}, |M| \leq \aleph, M_{\alpha+1}/M_\alpha \in \mathcal{Q}$  and  $|M_{\alpha+1}/M_\alpha| \leq \aleph$  for a fixed cardinal number  $\aleph$ . In fact, for any class  $\mathcal{T}$  which has the above property, we can prove, as in Proposition 3.1, that  $\mathcal{T}^\perp$  is special preenveloping. So, in order to prove the theorem we only need to prove the following lemma.

**Lemma 3.7** *Let  $\mathcal{Q}_C$  be the class in Theorem 3.6, then there exists a cardinal number  $\bar{\aleph}$  such that each  $N \in \mathcal{Q}_C$  can be written as a direct union of a continuous chain of submodules  $\{N_\alpha, \alpha < \lambda\}$  for some ordinal number  $\lambda$  such that  $N_0 \in \mathcal{Q}_C, |N_0| \leq \bar{\aleph}, N_{\alpha+1}/N_\alpha \in \mathcal{Q}_C$  and  $|N_{\alpha+1}/N_\alpha| \leq \bar{\aleph}$  for all  $\alpha + 1 < \lambda$ .*

**Proof.** Let  $\aleph$  be a cardinal number which implements the Kaplansky property for  $\mathcal{Q}$ . Set  $\bar{\aleph} = |R| \cdot \aleph_0 \cdot \aleph$ . For each  $M \in \mathcal{Q}_C$ ,  $\text{Hom}_R(\mathcal{C}, M) \in \mathcal{Q}$  by definition. Then  $\text{Hom}_R(\mathcal{C}, M) = \bigcup_{\alpha < \lambda} M_\alpha$  for some ordinal number  $\lambda$  such that  $M_0 \in \mathcal{Q}, |M| \leq \aleph, M_{\alpha+1}/M_\alpha \in \mathcal{Q}$  and  $|M_{\alpha+1}/M_\alpha| \leq \aleph$  for  $\alpha + 1 < \lambda$ . The first isomorphism of the following sequence is from the fact that  $M \in \mathcal{Q}_C \subseteq \mathcal{B}_C(R)$ :

$$M \cong \mathcal{C} \otimes_R \text{Hom}_R(\mathcal{C}, M) \cong \mathcal{C} \otimes_R \bigcup_{\alpha < \lambda} M_\alpha \cong \mathcal{C} \otimes_R \varinjlim M_\alpha \cong \varinjlim \mathcal{C} \otimes_R M_\alpha.$$

We now show that  $\{N_\alpha = \mathcal{C} \otimes_R M_\alpha, \alpha < \lambda\}$  is the desired continuous chain. For each  $\alpha < \lambda$  we know that  $M_{\alpha+1}/M_\alpha \in \mathcal{Q} \subseteq \mathcal{A}_C(R)$ . Thus applying the functor  $\mathcal{C} \otimes_R -$  to the exact sequence  $0 \rightarrow M_\alpha \rightarrow M_{\alpha+1} \rightarrow$

$M_{\alpha+1}/M_\alpha \rightarrow 0$  leaves us an exact sequence  $0 \rightarrow \mathcal{C} \otimes_R M_\alpha \rightarrow \mathcal{C} \otimes_R M_{\alpha+1} \rightarrow \mathcal{C} \otimes_R (M_{\alpha+1}/M_\alpha) \rightarrow 0$ . This exact sequence shows that  $\{\mathcal{C} \otimes_R M_\alpha, \alpha < \lambda\}$  is indeed a continuous chain and that  $\mathcal{C} \otimes_R M_{\alpha+1}/\mathcal{C} \otimes_R M_\alpha \cong \mathcal{C} \otimes_R (M_{\alpha+1}/M_\alpha) \in \mathcal{Q}_\mathcal{C}$ . Finally, it is not difficult to see that  $\mathcal{C} \otimes_R M_0 \in \mathcal{Q}_\mathcal{C}, |\mathcal{C} \otimes_R M_0| \leq \bar{\aleph}$  and  $|\mathcal{C} \otimes_R M_{\alpha+1}/\mathcal{C} \otimes_R M_\alpha| = |\mathcal{C} \otimes_R (M_{\alpha+1}/M_\alpha)| \leq \bar{\aleph}$ .  $\square$

**“Proof of Theorem 3.6”:** By Lemma 3.7 we can prove the theorem as we did for Proposition 3.1. Note that  $\mathcal{Q}_\mathcal{C}$  is closed under direct sums by Lemma 3.5.  $\square$

**Remark 3.8** Our first application of Theorem 3.6 goes to  $\mathcal{A}_\mathcal{C}(R)$ , the *Auslander class* of  $R$  with respect to  $\mathcal{C}$ . By [7, Proposition 3.10], we know that over a Noetherian ring  $\mathcal{A}_\mathcal{C}(R)$  is a Kaplansky class which is also closed under direct sums, thus  $\mathcal{A}_\mathcal{C}(R)^\perp$  is special preenveloping by Theorem 3.6 (This also can be obtained by [7, Theorem 3.11] and Wakamatsu’s lemma). In particular, when  $\mathcal{C}$  is a dualizing module over a Noetherian and local Cohen-Macaulay ring  $R$ , then  $\mathcal{A}_\mathcal{C}(R)$  is just  $\mathcal{GP}_{<\infty}$ , the class of modules with finite Gorenstein projective dimension (see [10, Corollary 2.4]). Therefore, in this case, we can say that  $\mathcal{GP}_{<\infty}^\perp$  is special preenveloping.

Now we take  $\mathcal{Q}$  to be  $\mathcal{P}^n$ , then  $\mathcal{Q}_\mathcal{C}$  is  $\mathcal{P}_\mathcal{C}^n$  by [25, Theorem 2.11(c)], and we get the following corollary.

**Corollary 3.9**  $\mathcal{P}_\mathcal{C}^{n\perp}$  is a special preenveloping class.

**Proof.**  $\mathcal{P}^n$  is closed under direct sums and from [1, Proposition 4.1] we know that  $\mathcal{P}^n$  is a Kaplansky class. Thus  $\mathcal{P}_\mathcal{C}^{n\perp}$  is special preenveloping by Theorem 3.6.  $\square$

When  $\mathcal{Q}$  is a special kind of Kaplansky class, we can show that  $\mathcal{Q}_\mathcal{C}$  is also a Kaplansky class.

**Proposition 3.10** If  $\mathcal{Q} \subseteq \mathcal{A}_\mathcal{C}(R)$  is closed under pure submodules and pure quotients (in this case  $\mathcal{Q}$  is a Kaplansky class by [16, Proposition 3.2]), then  $\mathcal{Q}_\mathcal{C}$  is a Kaplansky class.

**Proof.** Suppose that  $M \in \mathcal{Q}_\mathcal{C}$ , then  $M = \mathcal{C} \otimes_R N$  for some  $N \in \mathcal{Q}$ . Choose an element  $x \in M$ , assume that  $x = \sum_{i=1}^k c_i \otimes n_i$  with  $c_i \in \mathcal{C}$  and  $n_i \in N$ ,  $i = 1, 2, \dots, k$ . Let  $N'$  be the submodule of  $N$  generated by  $n_1, \dots, n_k$ . Then by [20, Lemma 2.7], we get a pure submodule  $\bar{N}$  of  $N$  which contains  $N'$  and  $|\bar{N}| \leq \sup\{\aleph_0, |N'|, |R|\}$ . Note that  $N'$  is finitely generated, so  $|N'| \leq \aleph_0 \cdot |R|$ . Set  $\aleph' = \aleph_0 \cdot |R|$ , then  $|\bar{N}| \leq \aleph'$ . Since  $\bar{N}$  is pure in  $N$ , both  $\bar{N}$  and  $N/\bar{N}$  are in  $\mathcal{Q}$ . Therefore,  $\mathcal{C} \otimes_R \bar{N} \in \mathcal{Q}_\mathcal{C}$ . Furthermore, the exact sequence  $0 \rightarrow \mathcal{C} \otimes_R \bar{N} \rightarrow \mathcal{C} \otimes_R N \rightarrow \mathcal{C} \otimes_R (N/\bar{N}) \rightarrow 0$  implies that  $\mathcal{C} \otimes_R \bar{N}$  is a submodule of  $M (= \mathcal{C} \otimes_R N)$  which contains  $x$  and  $M/\mathcal{C} \otimes_R \bar{N} \cong \mathcal{C} \otimes_R N/\mathcal{C} \otimes_R \bar{N} \cong \mathcal{C} \otimes_R (N/\bar{N}) \in \mathcal{Q}_\mathcal{C}$ . Finally,  $|\mathcal{C} \otimes_R \bar{N}| \leq \aleph_0 \cdot \aleph_0 \cdot \aleph' = \bar{\aleph}$  which is independent of the choice of  $\mathcal{C} \otimes_R \bar{N}$ . So there is a cardinal  $\bar{\aleph}$  such that for each module  $M \in \mathcal{Q}_\mathcal{C}$  and an element  $x \in M$ , we have a submodule  $M' (= \mathcal{C} \otimes_R \bar{N}) \subseteq M$  such that  $|M'| \leq \bar{\aleph}$  and  $M', M/M' \in \mathcal{Q}_\mathcal{C}$ . Thus  $\mathcal{Q}_\mathcal{C}$  is a Kaplansky class.  $\square$

Recall that  $\mathcal{F}^n$  denotes the class of modules  $M$  with  $fd_R M \leq n$  and  $\mathcal{F}_\mathcal{C}^n$  denotes the class of modules  $N$  with  $\mathcal{F}_\mathcal{C}\text{-}pd_R M \leq n$ . As an application of the Proposition 3.10 we have the following result.

**Corollary 3.11**  $\mathcal{F}_\mathcal{C}^{n\perp}$  is enveloping. Moreover, if the ring  $R$  is coherent, then  $\mathcal{F}_\mathcal{C}^n$  is preenveloping.



**Proof.** Note that if we take  $\mathcal{Q}$  to be  $\mathcal{F}^n$ , then  $\mathcal{Q}_{\mathcal{C}}$  is just  $\mathcal{F}_{\mathcal{C}}^n$ . Since  $\mathcal{F}^n$  is the class which is closed under extensions, direct limits, pure submodules and pure quotients, then  $\mathcal{F}_{\mathcal{C}}^n$  is a Kaplansky class which is closed under extensions and direct limits by Lemma 3.5 and Proposition 3.10. Thus  $\mathcal{F}_{\mathcal{C}}^{n\perp}$  is enveloping by [8, Theorem 2.8].

If  $R$  is coherent, then  $\mathcal{F}^n$  is closed under direct products. So  $\mathcal{F}_{\mathcal{C}}^n$  is closed under direct products by Lemma 3.5, thus  $\mathcal{F}_{\mathcal{C}}^n$  is preenveloping by [8, Theorem 2.5].  $\square$

**Remark 3.12** In fact, we know that  $M \in \mathcal{F}_{\mathcal{C}}^n \Leftrightarrow \mathcal{F}_{\mathcal{C}}\text{-}pd_R M \leq n \Leftrightarrow fd_R \text{Hom}_R(\mathcal{C}, M) \leq n \Leftrightarrow id_R(\text{Hom}_{\mathbb{Z}}(\text{Hom}_R(\mathcal{C}, M), \mathbb{Q}/\mathbb{Z})) \leq n \Leftrightarrow id_R(\mathcal{C} \otimes_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \leq n \Leftrightarrow \mathcal{I}_{\mathcal{C}}\text{-}id_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq n \Leftrightarrow \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{I}_{\mathcal{C}}^n$ . Therefore,  $(\mathcal{F}_{\mathcal{C}}^n, \mathcal{I}_{\mathcal{C}}^n)$  is a duality pair (see [17, 2.1]) and  $\mathcal{F}_{\mathcal{C}}^n$  is closed under pure submodules and pure quotients by [17, Theorem 3.1]. Furthermore, we say that  $(\mathcal{F}_{\mathcal{C}}^n, \mathcal{I}_{\mathcal{C}}^n)$  is a coproduct closed duality pair, since  $\mathcal{F}_{\mathcal{C}}^n$  is closed under direct sums. Thus  $\mathcal{F}_{\mathcal{C}}^n$  is a covering class by [17, Theorem 3.1(b)].

Similarly, for a given class  $\mathcal{T} \subseteq \mathcal{B}_{\mathcal{C}}(R)$  we can define a class  $\mathcal{T}_{\mathcal{C}}$  as follows:  $M \in \mathcal{T}_{\mathcal{C}} \Leftrightarrow \mathcal{C} \otimes_R M \in \mathcal{T}$ . Then we can prove that if  $\mathcal{T}$  is a Kaplansky class which is closed under direct sums, then both  $\mathcal{T}^{\perp}$  and  $\mathcal{T}_{\mathcal{C}}^{\perp}$  are special preenveloping. For example, we can take  $\mathcal{T}$  to be  $\mathcal{B}_{\mathcal{C}}(R)$ , the Bass class of  $R$  with respect to  $\mathcal{C}$ , where  $\mathcal{C}$  is a semidualizing module over a Noetherian ring, and to be  $\mathcal{GI}_{<\infty}$ , the class of modules with finite Gorenstein injective dimension over a Noetherian local Cohen-Macaulay ring. In particular, if we take  $\mathcal{T}$  to be  $\mathcal{I}^n$ , then by [25, Theorem 2.11(b)]  $\mathcal{T}_{\mathcal{C}}$  is  $\mathcal{I}_{\mathcal{C}}^n$ . Then we can show that every module admits a special  $\mathcal{I}_{\mathcal{C}}^{n\perp}$ -preenvelope. In fact, we claim that the existence of (pre)envelops ((pre)covers) by  $\mathcal{I}_{\mathcal{C}}^{n\perp}$  ( $\mathcal{I}_{\mathcal{C}}^n$ ) is immediate consequence of existing literature.

**Theorem 3.13** *Let  $R$  be a noetherian ring,  $\mathcal{C}$  be a semidualizing  $R$ -module. Then  $\mathcal{I}_{\mathcal{C}}^n$  is covering,  $\mathcal{I}_{\mathcal{C}}^{n\perp}$  is enveloping and special preenveloping.*

**Proof.** By [25, Theorem 2.11(b)], we have that  $M \in \mathcal{I}_{\mathcal{C}}^n \Leftrightarrow \mathcal{C} \otimes_R M \in \mathcal{I}^n$ . From this equivalence and the fact that  $\mathcal{I}^n$  is closed under pure submodules and pure quotients, it is easy to show that  $\mathcal{I}_{\mathcal{C}}^n$  is closed under pure submodules and pure quotients. Since  $\mathcal{I}_{\mathcal{C}}^n$  is closed under direct sums, therefore  $\mathcal{I}_{\mathcal{C}}^n$  is covering by [16, Theorem 2.5]. Moreover,  $\mathcal{I}_{\mathcal{C}}^n$  is a Kaplansky class by [16, Proposition 3.2], and  $\mathcal{I}_{\mathcal{C}}^n$  is closed under direct limits and extensions, thus  $\mathcal{I}_{\mathcal{C}}^{n\perp}$  is enveloping by [8, Theorem 2.8]. Finally, the special  $\mathcal{I}_{\mathcal{C}}^{n\perp}$ -preenveloping of every module is from Wakamatsu's Lemma.  $\square$

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