Turkish Journal of Mathematics

Volume 36 | Number 1 Article 13

1-1-2012

Invariant parametrizations and complete systems of global invariants of curves in the pseudo-Euclidean geometry

ÖMER PEKŞEN

DJAVVAT KHADJIEV

İDRİS ÖREN

Turk J Math
36 (2012) , 147 – 160.
© TÜBİTAK
doi:10.3906/mat-0911-145

Invariant parametrizations and complete systems of global invariants of curves in the pseudo-Euclidean geometry

Ömer Pekşen, Djavvat Khadjiev, İdris Ören

Abstract

Let $M(n,p)$ be the group of all transformations of an n-dimensional pseudo-Euclidean space E_p^n of index p generated by all pseudo-orthogonal transformations and parallel translations of E_p^n. Definitions of a pseudo-Euclidean type of a curve, an invariant parametrization of a curve and an $M(n,p)$-equivalence of curves are introduced. All possible invariant parametrizations of a curve with a fixed pseudo-Euclidean type are described. The problem of the $M(n,p)$-equivalence of curves is reduced to that of paths. Global conditions of the $M(n,p)$-equivalence of curves are given in terms of the pseudo-Euclidean type of a curve and the system of polynomial differential $M(n,p)$-invariants of a curve $x(s)$.

Key Words: Curve, pseudo-Euclidean geometry, invariant parametrization

1. Introduction

Let R be the field of real numbers, n and p are integers such that $0 \leq p < n$. The n-dimensional pseudo-Euclidean space of index p (that is the space R^n with the scalar product $< x,y >= -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_ny_n$) will be denoted by E_p^n. E_1^4 is the Minkowski spacetime. The group of all pseudo-orthogonal transformations of E_p^n (that is the set of all linear transformations $g : E_p^n \to E_p^n$ such that $< gx, gy >=< x,y >$ for all $x, y \in E_p^n$) is denoted by $O(n,p)$. Put $M(n,p)=\{F : E_p^n \to E_p^n \mid Fx = gx + b$, $g \in O(n,p),\ b \in E_p^n\}$ and $SM(n,p) = \{F \in M(n,p) : \det g = 1\}$.

The Frenet-Serret formalism for both time-like and space-like curves in spaces E_1^3 and E_1^4 is studied in papers [13, 21] and in the thesis [14]. In papers [2, 5, 6, 9, 20], the Frenet-Serret curve analysis is extended from non-null curves in E_1^4 to null (lightlike, isotropic) curves. For arbitrary n, this theory is extended to the Lorentz space E_1^n and to the space E_2^n in papers [3, 18] and in the book ([10], pp. 52–76). The Frenet-Serret theory for degenerate curves in spaces E_1^n and E_2^n is investigated in [11–12]. The Frenet-Serret theory of curves in E_p^n for arbitrary n and index p is considered in papers [4, 7, 8]. In [7], the fundamental theorem of a naturally-parametrized curve in E_p^n for arbitrary n and index p is obtained. It is found necessary and sufficient conditions under which given real-valued functions $\varphi_1, \ldots, \varphi_{n-1}$, $n \geq 2$, on an interval I of the real

2000 *AMS Mathematics Subject Classification:* 53A35.
This work was supported by the Research Fund of TUBITAK. Project number:107T049.

axis are the successive curvatures of a naturally-parametrized curve in E_p^n which is defined by them uniquely up to congruence for a given distribution of unit and pseudounit vectors in a Frenet $(n-1)$-frame of the curve.

The Frenet-Serret equations for a curve in an Euclidean space E_0^n provide curvature functions $k_1(s),\ldots,k_{n-1}(s)$ of a curve. The curvatures $k_1(s),\ldots,k_{n-2}(s)$ are $M(n,0)$-invariant. But the curvature $k_{n-1}(s)$ is not $M(n,0)$-invariant, it is $SM(n,0)$-invariant. For example, the torsion of a curve in E_0^3 is $SM(3,0)$-invariant, but it is not $M(3,0)$-invariant. Therefore the system $k_1(s),\ldots,k_{n-1}(s)$ gives a solution of the problem of the G-equivalence of curves only for $G=SM(n,0)$ ([19], p.p. 61–64). Besides, the method of moving frames essentially gives only conditions of a local G-equivalence of curves. A similar situation is valid for an arbitrary index p.

In the present paper we use an invariant-theoretic approach to the theory of curves in the pseudo-Euclidean geometry. We give a solution of the problem of global G-equivalence of curves for groups $G = M(n,p), SM(n,p)$ in terms of invariants of a curve.

This paper is organized as follows. In Section 1, the definitions of the pseudo-Euclidean type and an invariant parametrization of a curve are given. The pseudo-Euclidean type of a curve is $M(n,p)$-invariant and it has the following forms: $(0,l)$, where $0 < l \leq \infty$, $(-\infty,0)$ and $(-\infty,+\infty)$. All possible invariant parametrizations of a curve with a fixed pseudo-Euclidean type are described. In Theorem 1, the problems of the $M(n,p)$-equivalence and the $SM(n,p)$-equivalence of curves are reduced to that of paths. In Section 2, the conditions of the global G-equivalence of curves are given in terms of the pseudo-Euclidean type and the system of polynomial differential G- invariant functions.

A description of a complete system of correlations between the elements of the complete system of differential invariants of a curve in E_p^n will be considered in our next paper. The theory of regular curves in E_p^n given in the present paper contains also some class of null curves (look at the Remarks 2–3 and Example 4 below). More detailed theory of invariants of null curves in E_p^n will be considered also in our next paper.

2. Invariant parametrizations of a curve

Let $J=(a,b)$ be an open interval of R.

Definition 1 *(see [16, 17]). A C^∞-mapping $x: J \to E_p^n$ will be called a J-path (shortly, a path) in E_p^n.*

Definition 2 *(see [16, 17]). A J_1-path $x(t)$ and a J_2-path $y(r)$ in E_p^n will be called D-equivalent if a C^∞-diffeomorphism $\varphi: J_2 \to J_1$ exists such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in J_2$. A class of D-equivalent paths in E_p^n will be called a curve in E_p^n. A path $x \in \alpha$ will be called a parametrization of a curve α.*

If $x(t)$ is a J-path then $Fx(t)$ is a J-path in E_p^n for any $F \in M(n,p)$. Let G be a subgroup of $M(n,p)$.

Definition 3 *Two J-paths $x(t)$ and $y(t)$ in E_p^n are called G-equivalent if there exists $F \in G$ such that $y(t) = Fx(t)$. This being the case, we write $x(t) \overset{G}{\sim} y(t)$*

Let $\alpha = \{h_\tau, \tau \in Q\}$ be a curve in E_p^n, where h_τ is a parametrization of α. Then $F\alpha = \{Fh_\tau, \tau \in Q\}$ is a curve in E_p^n for any $F \in M(n,p)$.

Definition 4 *(see [16, 17]) Two curves* α *and* β *in* E_p^n *are called* G*-equivalent if* $\beta = F\alpha$ *for some* $F \in G$*. This being the case, we write* $\alpha \overset{G}{\sim} \beta$*.*

Let $x(t) = (x_1(t), \ldots, x_n(t))$ be a J-path in E_p^n, $x^{'}(t) = (x_1^{'}(t), \ldots, x_n^{'}(t))$ is its first derivative and $x^{(k)}(t)$ is its k-th derivative. Denote the determinant of vectors $x^{'}(t), x^{(2)}(t), \ldots,\ x^{(n)}(t)$ by $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right]$.

Definition 5 *A* J*-path* $x(t)$ *in* E_p^n *will be called pseudo-euclidean regular (regular, for short) if one of the following conditions hold:*
(5_1). $< x^{'}(t), x^{'}(t) > \neq 0$ *for all* $t \in J$*;*
(5_2). $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] \neq 0$ *for all* $t \in J$*;*
(5_3). $\left|< x^{'}(t), x^{'}(t) >\right| + \left|\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right]\right| \neq 0$ *for all* $t \in J$*.*
A curve α *will be called regular if it contains a regular path.*

Remark 1 It is obvious that $(5_1) \to (5_3)$ and $(5_2) \to (5_3)$. The following examples 1-3 below show that $(5_1) \not\to (5_2)$, $(5_2) \not\to (5_1)$, $(5_3) \not\to (5_1)$, $(5_3) \not\to (5_2)$ and $(5_3) \not\to (5_1) \cup (5_1)$.

Example 1 Consider the J-path $x(t) = (\frac{1}{2}t^2, \frac{1}{3}t^3)$ in E_1^2, where $J = (0, 2)$. Then $< x^{'}(t), x^{'}(t) >= 0$ for $t = 1$, but $\left[x^{'}(t)x^{(2)}(t)\right] \neq 0$ for all $t \in J$. Hence $(5_2) \not\to (5_1)$. In the case $p = 0$, it is easy to see that $(5_2) \to (5_1)$.

Example 2 Consider the J-path $x(t) = (\frac{1}{3}t^3, \frac{2}{3}t^3)$ in E_1^2, where $J = (0, 2)$. Then $\left[x^{'}(t)x^{(2)}(t)\right] = 0$ for all $t \in J$, but $< x^{'}(t), x^{'}(t) > \neq 0$ for all $t \in J$. Hence $(5_1) \not\to (5_2)$.

Example 3 Consider the J-path $x(t) = (t, \frac{1}{2}t^2, \frac{1}{4}t^4)$ in E_1^3, where $J = (-\frac{1}{2}, 2)$. Then $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right] = 6t$ and $< x^{'}(t), x^{'}(t) >= 1 + t^2 - t^6$ for all $t \in J$. The equality $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right] = 6t$ implies that $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right] = 0$ only for $t = t_1 = 0$. There exists unique $t = t_2 \in J$ such that $< x^{'}(t), x^{'}(t) >= 0$. It is easy to see that $1 < t_2 < 2$. Then $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right] = 0$ for some $t = t_1 \in J$ and $< x^{'}(t), x^{'}(t) >= 0$ for some $t = t_2 \in J$, where $t_1 \neq t_2$, but $\left|< x^{'}(t), x^{'}(t) >\right| + \left|\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right]\right| \neq 0$ for all $t \in J$. Hence $(5_3) \not\to (5_1) \cup (5_2)$. In particularly, $(5_3) \not\to (5_1)$ and $(5_3) \not\to (5_2)$.

Definition 6 *(see [2]) A* J*-path* $x(t)$ *is called null if* $< x^{'}(t), x^{'}(t) >= 0$ *for all* $t \in J$*.*

Remark 2 There exists a null J-path such that $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] \neq 0$ for all $t \in J$.

Example 4 Consider the J-path

$$x(t) = (t, \frac{1}{2}t^2, \int_0^1 \sqrt{1+t^2}dt)$$

in E_1^3, where $J=(0,1)$. Then $<x^{'}(t),x^{'}(t)>=0$ for all $t\in J$ and $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right]=(1+t^2)^{-\frac{3}{2}}\neq 0$ for all $t\in J$.

Hence there exists a regular null J-path in E_p^n. Therefore the theory of regular curves in E_p^n given below contains also some class of null curves.

Now we define invariant parametrizations of regular curves in E_p^n. Let $x(t)$ be a regular J-path in E_p^n. We put

$$l_x(c,d)=\int_c^d |<x'(t),x'(t)>|^{\frac{1}{2}}\,dt.$$

in case (5_1) of Definition 5. If (5_1) doesn't hold and case (5_2) holds, we put

$$l_x(c,d)=\int_c^d \left|\left[x^{'}(t)x^{(2)}(t)\dots x^{(n)}(t)\right]\right|^{\frac{2}{n(n+1)}}\,dt.$$

If the cases (5_1) and (5_2) don't hold and the case (5_3) holds, we put

$$l_x(c,d)=\int_c^d |<x'(t),x'(t)>|^{\frac{1}{2}}\,dt+\int_c^d \left|\left[x^{'}(t)x^{(2)}(t)\dots x^{(n)}(t)\right]\right|^{\frac{2}{n(n+1)}}\,dt.$$

The limits $l_x(a,d)=\lim_{c\to a}l_x(c,d)\leq +\infty$ and $l_x(c,b)=\lim_{d\to b}l_x(c,d)\leq +\infty$ exist. There are only four possibilities:

$$(T_1).l_x(a,d)<+\infty, l_x(c,b)<+\infty;\quad (T_2).l_x(a,d)<+\infty, l_x(c,b)=+\infty;$$
$$(T_3).l_x(a,d)=+\infty, l_x(c,b)<+\infty;\quad (T_4).l_x(a,d)=+\infty, l_x(c,b)=+\infty.$$

Suppose that the case (T_1) or (T_2) holds for some $c,d\in J$. Then $l=l_x(a,d)+l_x(c,b)-l_x(c,d)$, where $0\leq l\leq+\infty$, does not depend on $c,d\in J$. In this case we say that x belongs to the pseudo-euclidean type of $(0,l)$. The cases (T_3) and (T_4) do not depend on c,d. In these cases, we say that x belongs to the pseudo-euclidean types of $(-\infty,0)$ and $(-\infty,+\infty)$, respectively. There exist paths of all types $(0,l)$, where $l<+\infty$, $(0,+\infty)$, $(-\infty,0)$ and $(-\infty,+\infty)$. The pseudo-euclidean type of a path x will be denoted by $L(x)$. It is obvious that:

(i) if $x\overset{M(n,p)}{\sim}y$ then $L(x)=L(y)$;

(ii) if x,y is parametrizations of a curve α then $L(x)=L(y)$.

The pseudo-euclidean type of a path $x\in\alpha$, will be called the pseudo-euclidean type of the curve α and denoted by $L(\alpha)$. $L(\alpha)$ is an $M(n,p)$-invariant of a curve α.

Now we define an invariant parametrization of a regular curve in E_p^n. Let $J=(a,b)$ and $x(t)$ be a regular J-path in E_p^n. We define the pseudo-euclidean arc length function $s_x(t)$ for each pseudo-euclidean type as follows. We put $s_x(t)=l_x(a,t)$ for the case $L(x)=(0,l)$, where $l\leq+\infty$, and $s_x(t)=-l_x(t,b)$ for the case $L(x)=(-\infty,0)$. Let $L(x)=(-\infty,+\infty)$. We choose a fixed point in every interval $J=(a,b)$ of R and denote it by a_J. Let $a_J=0$ for $J=(-\infty,+\infty)$. We set $s_x(t)=l_x(a_J,t)$.

Since $s_x^{'}(t)>0$ for all $t\in J$, the inverse function of $s_x(t)$ exists. Let us denote it by $t_x(s)$. The domain of $t_x(s)$ is $L(x)$ and $t_x^{'}(s)>0$ for all $s\in L(x)$.

Proposition 1 *Let $I=(a,b)$ and x be a regular I-path in E_p^n. Then*

(*i*) $s_{Fx}(t)=s_x(t)$ *and* $t_{Fx}(s)=t_x(s)$ *for all* $F\in M(n,p)$;

(*ii*) *the equalities* $s_{x(\varphi)}(r)=s_x(\varphi(r))+s_0$ *and* $\varphi(t_{x(\varphi)}(s+s_0))=t_x(s)$ *hold for any* C^∞*-diffeomorphism* $\varphi:J=(c,d)\to I$ *such that* $\varphi'(r)>0$ *for all* $r\in J$, *where* $s_0=0$ *for* $L(x)\neq(-\infty,+\infty)$ *and* $s_0=l_x(\varphi(a_J),a_I)$ *for* $L(x)=(-\infty,+\infty)$.

Proof. The proof of statement (i) is obvious. We prove statement (ii) for case (5_3) in Definition 5. Let $L(x)=(-\infty,+\infty)$. Then we have $s_{x(\varphi)}(r)=$

$$\int_{a_J}^{r}\left(\left|<\frac{d}{dr}x(\varphi(r)),\frac{d}{dr}x(\varphi(r))>\right|^{\frac{1}{2}}+\left|\left[\frac{d}{dr}x(\varphi(r))\ldots\frac{d^n}{dr^n}x(\varphi(r))\right]\right|^{\frac{2}{n(n+1)}}\right)dr=$$

$$\int_{a_J}^{r}\frac{d\varphi}{dr}\left(\left|<\frac{d}{d\varphi}x(\varphi(r)),\frac{d}{d\varphi}x(\varphi(r))>\right|^{\frac{1}{2}}+\left|\left[\frac{d}{d\varphi}x(\varphi(r))\ldots\frac{d^n}{d\varphi^n}x(\varphi(r))\right]\right|^{\frac{2}{n(n+1)}}\right)dr=$$

$$l_x(\varphi(a_J),\varphi(r))=l_x(a_I,\varphi(r))+l_x(\varphi(a_J),a_I).$$

So $s_{x(\varphi)}(r)=s_x(\varphi(r))+s_0$, where $s_0=l_x(\varphi(a_J),a_I)$. This implies $\varphi(t_{x(\varphi)}(s+s_0))=t_x(s)$. For $L(x)\neq(-\infty,+\infty)$, it is easy to see that $s_0=0$.

Proofs of statement (ii) for cases (5_1) and (5_2) in Definition 5 are similar. □

Let α be a regular curve, $x\in\alpha$. Then $x(t_x(s))$ is a parametrization of α.

Definition 7 *The parametrization $x(t_x(s))$ of a regular curve α will be called an invariant parametrization of α.*

We denote the set of all invariant parametrizations of α by $Ip(\alpha)$. Every $y\in Ip(\alpha)$ is a J-path, where $J=L(\alpha)$.

Proposition 2 *Let α be a regular curve, $x\in\alpha$ and x be a J-path, where $J=L(\alpha)$. Assume that the condition (5_1) in Definition 5 holds for x. Then the following conditions are equivalent:*

(*i*) *x is an invariant parametrization of α;*

(*ii*) $|<x'(t),x'(t)>|=1$ *for all* $s\in L(\alpha)$;

(*iii*) $s_x(s)=s$ *for all* $s\in L(\alpha)$.

Proof. $(i)\to(ii)$. Let $x\in Ip(\alpha)$. Then there exists $y\in\alpha$ such that $x(s)=y(t_y(s))$. By Proposition 1, $s_x(s)=s_{y(t_y)}(s)=s_y(t_y(s))+s_0=s+s_0$, where s_0 is as in Proposition 1. Since s_0 does not depend on s, we have $\frac{ds_x(s)}{ds}=|<x'(t),x'(t)>|^{\frac{1}{2}}=1$. Hence $|<x'(t),x'(t)>|=1$ for all $s\in L(\alpha)$.

$(ii)\to(iii)$. Let $|<x'(t),x'(t)>|=1$ for all $s\in L(\alpha)$. Using the definition of $s_x(t)$, we get $\frac{ds_x(s)}{ds}=|<x'(t),x'(t)>|^{\frac{1}{2}}=1$. Therefore $s_x(s)=s+c$ for some $c\in R$. In the case $L(x)\neq(-\infty,+\infty)$, conditions $s_x(s) = s + c$ and $s_x(s) \in L(\alpha)$ for all $s \in L(\alpha)$ implies $c = 0$, that is, $s_x(s) = s$. In the case $L(\alpha) = (-\infty, +\infty)$, equalities $s_x(s) = l_x(a_J, s) = l_x(0, s) = s + c$ implies $0 = l_x(0, 0) = c$, that is , $s_x(s) = s$.

$(iii) \to (i)$. Since $s_x(s) = s$ implies $t_x(s) = s$, we get $x(s) = x(t_x(s)) \in Ip(\alpha)$. □

Similar results are true for conditions (5_2) and (5_3) in Definition 5.

Remark 3 In papers [2–9, 11, 18, 20, 21], in the thesis [14] and in the book [10], essentially the parametrization in the 5_1 of Definition 5 is used and it is used only for curves of the type $(0, l)$, where $0 < l < \infty$. By remark 2 and Examples 1–3, parametrizations in the cases 5_2 and 5_3 are independent of the parametrization in the case 5_1. Hence the class of curves which investigated in the present paper is essentially wider then in the mentioned papers. By Remark 2 and Example 4, parametrizations in the cases 5_2 and 5_3 contain also parametrizations of some class of null curves.

Proposition 3 *Let α be a regular curve and $L(\alpha) \neq (-\infty, +\infty)$. Then there exists the unique invariant parametrization of α.*

Proof. A proof is similar to the proof of Proposition 4 in [16]. □

Let α be a regular curve and $L(\alpha) = (-\infty, +\infty)$. Then it is easy to see that the set $Ip(\alpha)$ is infinite and it is not countable.

Proposition 4 *Let α be a regular curve, $L(\alpha) = (-\infty, +\infty)$ and $x \in Ip(\alpha)$. Then $Ip(\alpha) = \{y : y(s) = x(s + c), c \in (-\infty, +\infty)\}$.*

Proof. A proof is similar to the proof of Proposition 5 in [16]. □

Theorem 1 *Let α, β be regular curves and $x \in Ip(\alpha), y \in Ip(\beta)$. Then:*

(i) for $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{M(n,p)}{\sim} \beta$ if and only if $x \overset{M(n,p)}{\sim} y$;

(ii) for $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{M(n,p)}{\sim} \beta$ if and only if $x \overset{M(n,p)}{\sim} y(\psi_c)$ for some $c \in (-\infty, +\infty)$, where $\psi_c(s) = s + c$.

Proof. (i). Let $\alpha \overset{M(n,p)}{\sim} \beta$ and $h \in \alpha$. Then there exists $F \in M(n,p)$ such that $\beta = F\alpha$. This implies $Fh \in \beta$. Using Propositions 1-3, we get $x(s) = h(t_h(s)), y(s) = (Fh)(t_{Fh}(s))$ and $Fx(s) = F(h(t_h(s))) = (Fh)(t_h(s)) = (Fh)(t_{Fh}(s)) = y(s)$. Thus $x \overset{M(n,p)}{\sim} y$. Conversely, let $x \overset{M(n,p)}{\sim} y$, that is, there exists $F \in M(n,p)$ such that $Fx = y$. Then $\alpha \overset{M(n,p)}{\sim} \beta$.

(ii). Let $\alpha \overset{M(n,p)}{\sim} \beta$. Then there exist J-paths $h \in \alpha, k \in \beta$ and $F \in M(n,p)$ such that $k(t) = Fh(t)$. We have $k(t_k(s)) = k(t_{Fh}(s)) = k(t_h(s)) = (Fh)(t_h(s))$. By Proposition 4, $x(s) = k(t_k(s + s_1)), y(s) = h(t_h(s + s_2))$ for some $s_1, s_2 \in (-\infty, +\infty)$. Therefore $x(s - s_1) = Fy(s - s_2)$. This implies that $x \overset{M(n,p)}{\sim} y(\psi_c)$, where $\psi_c(s) = s + c$ and $c = s_1 - s_2$. Conversely, let $x \overset{M(n,p)}{\sim} y(\psi_c)$ for some $c \in (-\infty, +\infty)$, where $\psi_c = s + c$. Then there exists $F \in M(n,p)$ such that $y(s + c) = Fx(s)$. Since $y(s + c) \in \beta$, then $\alpha \overset{M(n,p)}{\sim} \beta$. □

Theorem 1 reduces the problems of the G-equivalence of regular curves for groups $G = M(n,p), SM(n,p)$ to that of paths only for the case $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$. Let H be a subgroup of $M(n,p)$.

Definition 8 *J-paths $x(t)$ and $y(t)$ will be called $[H, (-\infty, +\infty)]$-equivalent, if there exist $h \in H$ and $d \in (-\infty, +\infty)$ such that $y(t) = hx(t+d)$ for all $t \in J$.*

Theorem 1 reduces the problem of the H-equivalence of curves to $[H, (-\infty, +\infty)]$-equivalence of paths for the case $L(\alpha) = L(\beta) = (-\infty, +\infty)$.

3. Conditions of G-equivalence of paths and curves

Below we use some notations and facts from the differential algebra and the theory of differential invariants of a paths. They may be found in [1, 15, 16, 17].

Definition 9 *A J-path $x(t)$ in E_p^n will be called non-singular if $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] \neq 0$ for all $t \in J$. A curve α will be called non-sigular if it contains a non-singular path.*

Let G be a subgroup of $M(n,p)$.

Definition 10 *(see [1], Definition 8). A differential polynomial function $f\{x\}$ of a path $x(t)$ is called G-invariant if $f\{gx\} = f\{x\}$ for all $g \in G$.*

Let $x(t)$ and $y(t)$ be J-paths in E_p^n such that $x \overset{M(n,p)}{\sim} y$. Then $f\{x\} = f\{y\}$ for any $M(n,p)$-invariant differential polynomial $f\{x\}$. The converse statement (that is conditions of $M(n,p)$-equivalence of J-paths) is true in the following form.

Theorem 2 *Assume that $x(t)$ and $y(t)$ be non-singular J-paths in E_p^n such that*

$$< x^{(i)}(t), x^{(i)}(t) >=< y^{(i)}(t), y^{(i)}(t) > \tag{1}$$

for all $t \in J$ and $1 \leq i \leq n$. Then $x \overset{M(n,p)}{\sim} y$.

Proof. For a proof of this theorem, we use several lemmas. □

Lemma 1 *Assume that $1 \leq i, j, i+j \leq 2n+1$. Then, for each differential polynomial $< x^{(i)}, x^{(j)} >$, a differential polynomial $P_{ij}\{y_1, ..., y_k\}$ exists such that*

$$< x^{(i)}, x^{(j)} >= P_{ij}\left\{< x^{'}, x^{'} >, ..., < x^{(k)}, x^{(k)} >\right\},$$

where $k = \left[\frac{i+j}{2}\right]$.

Proof. A proof is similar to the proof of Proposition 6 in [1]. □

Lemma 2 *The equality*

$$(-1)^p[y_1 \dots y_n][z_1 \dots z_n] = det|| < y_i, z_j > ||_{i,j=1,2,\dots,n}$$

holds for all vectors $y_1, \dots, y_n, z_1, \dots, z_n$ *in* E_p^n.

Proof. Let $Y = \|y_1 \dots y_n\|$ and $Z = \|z_1 \dots z_n\|$ be $n \times n$-matrices of systems $\{y_1, \dots, y_n\}$ and $\{z_1, \dots, z_n\}$ of column vectors $y_1, \dots, y_n, z_1, \dots, z_n \in E_p^n$ and $I_p = \|b_{ij}\|$ be the diagonal $n \times n$-matrix such that $b_{ii} = -1$ for all $i = 1, \dots, p$ and $b_{jj} = 1$ for all $j = p+1, \dots, n$. Then we have $Y^\top I_p Z = \|< y_i, z_j >\|_{i,j=1,2,\dots,n}$, where $Y^\top$ is the transpose matrix of Y. Passing on to determinants, we obtain the desired equality. □

Denote the determinant $det \left\|< x^{(i)}, x^{(j)} >\right\|_{i,j=1,2,\dots,n}$ by $\Delta = \Delta_x$. Equation (1) and Lemma 1 implies that $< x^{(i)}(t), x^{(j)}(t) >=< y^{(i)}(t), y^{(j)}(t) >$ for all $t \in J$ and all $1 \leq i \leq j \leq n$. Using these equalities, we get $\Delta_x(t) = \Delta_y(t)$ for all $t \in J$. Since x, y are non-singular J-paths, we have $\Delta_x(t) \neq 0, \Delta_y(t) \neq 0$ for all $t \in J$. Hence $\Delta_x(t)^{-1} = \Delta_y(t)^{-1}$. Denote the system $\left\{< x', x' >, ..., < x^{(n)}, x^{(n)} >\right\}$ of differential polynomials by V. Denote the differential R-algebra generated by elements of the system V and the function Δ^{-1} by $R\left\{V, \Delta^{-1}\right\}$. Let $f\{x\} \in R\left\{V, \Delta^{-1}\right\}$. Then, using Equation (1) and $\Delta_x(t)^{-1} = \Delta_y(t)^{-1}$, we obtain

$$f\{x(t)\} = f\{y(t)\} \tag{2}$$

for all $t \in J$.

Denote the matrix $\left\|x'(t)x^{(2)}(t) \dots x^{(n)}(t)\right\|$ by $A(x(t))$, where we consider $x^{(i)}(t)$ as a column-vector. We let $\frac{d}{dt}A(x(t)) = \left\|x^{(2)}(t)x^{(3)}(t) \dots x^{(n+1)}(t)\right\|$. Since $x(t)$ is non-singular, we have $detA(x(t)) = \left[x'(t) \dots x^{(n)}(t)\right] \neq 0$ for all $t \in J$. Hence the matrix $A^{-1}(x(t))$ exists for all $t \in J$. We consider the matrix $A^{-1}(x(t))\frac{d}{dt}A(x(t)) = ||c_{ij}^x(t)||$. It is easy to see that

(a) $c_{j+1j}^x(t) = 1$ for all $t \in J$ and $1 \leq j \leq n-1$;

(b) $c_{ij}^x(t) = 0$ for all $t \in J$ and $j \neq n$, $i \neq j+1$, $1 \leq i \leq n$;

(c) $c_{in}^x(t) = \frac{\left[x'(t) \dots x^{(i-1)}(t)x^{(n+1)}(t)x^{(i+1)}(t) \dots x^{(n)}(t)\right]}{\left[x'(t) \dots x^{(n)}(t)\right]}$

for all $t \in J$ and $1 \leq i \leq n$.

Lemma 3 $c_{ij}^x(t) = c_{ij}^y(t)$ *for all* $t \in J$ *and* $1 \leq i \leq j \leq n$.

Proof. The above equality (a) implies $c_{j+1j}^x(t) = c_{j+1j}^y(t)$ for all $1 \leq j \leq n-1$ and the equality (b) implies $c_{ij}^x(t) = c_{ij}^y(t)$ for all $j \neq n$, $i \neq j+1$, $1 \leq i \leq n$. Prove $c_{in}^x(t) = c_{in}^y(t)$ for all $1 \leq i \leq n$. Using Lemma 2 to vectors $y_i = x^{(i)}(t), z_j = x^{(j)}(t)$ $(i, j = 1, \dots, n)$, we obtain

$$(-1)^p \left[x'(t) \dots x^{(n)}(t)\right]^2 = det|| < x^{(i)}(t), x^{(j)}(t) > ||. \tag{3}$$

Similarly, using Lemma 2 to vectors $x^{'},\ldots,x^{(i-1)},x^{(n+1)},x^{(i+1)},\ldots,x^{(n)},x^{'},\ldots,x^{(n)}$, we have

$$(-1)^{p}\left[x^{'}\ldots x^{(i-1)}x^{(n+1)}x^{(i+1)}\ldots x^{(n)}\right]\left[x^{'}\ldots x^{(n)}\right]=det||<x^{(k)},x^{(l)}>||, \tag{4}$$

where $k=1,\ldots,i-1,n+1,i+1,\ldots,n; l=1,2,\ldots,n$. From Equation (3), Equation (4), Equation (1), Lemma 1 and the equality $c_{in}^{x}(t)=$

$$\frac{\left[x^{'}\ldots x^{(i-1)}x^{(n+1)}x^{(i+1)}\ldots x^{(n)}\right]}{\left[x^{'}\ldots x^{(n)}\right]}=\frac{(-1)^{p}\left[x^{'}\ldots x^{(i-1)}x^{(n+1)}x^{(i+1)}\ldots x^{(n)}\right]\left[x^{'}\ldots x^{(n)}\right]}{(-1)^{p}\left[x^{'}\ldots x^{(n)}\right]^{2}},$$

for $1\leq i\leq n$, we obtain

$$\frac{\left[x^{'}\ldots x^{(i-1)}x^{(n+1)}x^{(i+1)}\ldots x^{(n)}\right]}{\left[x^{'}\ldots x^{(n)}\right]}=\frac{\left[y^{'}\ldots y^{(i-1)}y^{(n+1)}y^{(i+1)}\ldots y^{(n)}\right]}{\left[y^{'}\ldots y^{(n)}\right]}$$

for all $i=1,\ldots,n$. The lemma is proved. □

Equation (1) and Lemma 3 implies $A^{-1}(x(t))\frac{d}{dt}A(x(t))=A^{-1}(y(t))\frac{d}{dt}A(y(t))$ for all $t\in J$. The last equality implies

$$\frac{\partial}{\partial t}(A(y)A(x)^{-1})=(\frac{\partial}{\partial t}A(y))A(x)^{-1}+A(y)\frac{\partial}{\partial t}(A(x)^{-1})=(\frac{\partial}{\partial t}A(y))A(x)^{-1}-$$

$$A(y)A(x)^{-1}(\frac{\partial}{\partial t}A(x))A(x)^{-1}=A(y)(A(y)^{-1}\frac{\partial}{\partial t}A(y)-A(x)^{-1}\frac{\partial}{\partial t}A(x))A(x)^{-1}=0.$$

for all $t\in J$. Using this equality and connectedness of J, we obtain that $A(y(t))A(x(t))^{-1}$ does not depend on $t\in J$. Put $F=A(y)A(x)^{-1}$. According to $detA(x(t))\neq 0$ and $detA(y(t))\neq 0$ for all $t\in J$, we have $detF\neq 0$ and $A(y(t))=FA(x(t))$ for all $t\in J$. We prove that $F\in O(n,p)$.

Let $A(x)^{\top}$ be the transpose matrix of $A(x)$. Let $I_p=\|b_{ij}\|$ be the diagonal $n\times n$-matrix such that $b_{ii}=-1$ for all $i=1,\ldots,p$ and $b_{jj}=1$ for all $j=p+1,\ldots,n$. Using the equality $A(x)^{\top}I_pA(x)=\|<x^{(i)},x^{(j)}>\|_{i,j=1,2,\ldots,n}$, Lemma 1 and Equation (1), we obtain that $A(x)^{\top}I_pA(x)=A(y)^{\top}I_pA(y)$. This equality and the equality $A(y)=FA(x)$ imply that $F^{\top}I_pF=I_p$. Hence $F\in O(n,p)$.

The equality $Ay(t)=FAx(t)$ implies $\frac{\partial}{\partial t}y(t)=F\frac{\partial}{\partial t}x(t)$ for all $t\in J$. Then there exists a constant vector $b\in E_p^n$ such that $y(t)=Fx(t)+b$ for all $t\in J$. The theorem is completed. □

Corollary 1 *Let α,β be non-singular curves in E_p^n and $x\in Ip(\alpha),y\in Ip(\beta)$. Assume that x,y satisfy the condition (5_1) in Definition 5. Then*

(i) *in the case $L(\alpha)=L(\beta)\neq(-\infty,+\infty)$, $\alpha\overset{M(n,p)}{\sim}\beta$ if and only if*

$$sgn<x^{'}(s),x^{'}(s)>=sgn<y^{'}(s),y^{'}(s)>, \tag{5}$$

$$<x^{(i)}(s),x^{(i)}(s)>=<y^{(i)}(s),y^{(i)}(s)> \tag{6}$$

for all $s\in L(\alpha)$ and $i=2,\ldots,n$;

(ii) in the case $L(\alpha)=L(\beta)=(-\infty,+\infty)$, $\alpha \overset{M(n,p)}{\sim} \beta$ *if and only if*

$$sgn < x^{'}(s), x^{'}(s) >= sgn < y^{'}(s), y^{'}(s) >,$$
$$< x^{(i)}(s), x^{(i)}(s) >=< y^{(i)}(s+s_1), y^{(i)}(s+s_1) >$$

for some $s_1 \in (-\infty,+\infty)$, *all* $s \in L(\alpha)$ *and* $i = 2,\ldots,n$;

Proof. Let $\alpha \overset{M(n,p)}{\sim} \beta$. Then it is obvious that Equation (5) and Equation (6) hold. Conversely, assume that Equation (5) and Equation (6) hold. By Proposition 2, $\left|< x^{'}(s), x^{'}(s) >\right| = \left|< y^{'}(s), y^{'}(s) >\right| = 1$ for all $s \in L(\alpha)$. This equality and Equation (5) imply that $< x^{'}(s), x^{'}(s) >=< y^{'}(s), y^{'}(s) >$ for all $s \in L(\alpha)$. The last equality and Equation (6), by Theorem 2, imply $x \overset{M(n,p)}{\sim} y$. Applying Theorem 1, we obtain $\alpha \overset{M(n,p)}{\sim} \beta$. Similarly, the proof of statement (ii) follows from statement (ii) of Theorem 1. □

Remark 4 Similar results are true if x, y satisfy conditions (5_2) or (5_3)) in Definition 5.

Let α be a curve and $x \in Ip(\alpha)$.

Remark 5 According to Corollary 1 the system

$$\left\{L(\alpha), sgn < x^{'}, x^{'} >, < x^{(2)}, x^{(2)} >, \ldots, < x^{(n)}, x^{(n)} >\right\}$$

is a complete system of $M(n,p)$-invariants of a curve α for the case $L(\alpha) \neq (-\infty,+\infty)$. But they are not invariants of a curve α for the case $L(\alpha) = (-\infty,+\infty)$. They depend on reparametrizations $s \to s+a$ of a curve α.

Let $\delta = \delta_x$ be the determinant of the matrix $|| < y_i, z_j > ||_{i,j=1,2,\ldots,n-1}$, where $y_1 = z_1 = x^{'}, y_2 = z_2 = x^{(2)}, \cdots, y_{n-1} = z_{n-1} = x^{(n-1)}$. Denote the system

$$\left\{< x^{'}, x^{'} >, ..., < x^{(n-1)}, x^{(n-1)} >, \left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right]\right\}$$

of differential polynomials by Z. Denote the differential R-algebra generated by elements of Z by $R\{Z\}$.

Lemma 4 $< y_i, z_j > \in R\{Z\}$ *for all* $1 \leq i, j, i+j \leq 2n-1$ *and* $\delta \in R\{Z\}$.

Proof. Using Lemma 1, we get $< x^{(i)}, x^{(j)} > \in R\{Z\}$ for all $1 \leq i, j, i+j \leq 2n-1$. Since the element $< y_i, z_j >$ of the determinant δ is the function $< x^{(i)}, x^{(j)} >$, where $1 \leq i, j \leq n-1$, we obtain that $\delta \in R\{Z\}$. □

Theorem 3 *Assume that* $x(t)$ *and* $y(t)$ *be non-singular* J*-paths in* E_p^n *such that* $\delta_x(t) \neq 0$ *and* $\delta_y(t) \neq 0$ *for all* $t \in J$. *Then equalities*

$$< x^{(i)}(t), x^{(i)}(t) >=< y^{(i)}(t), y^{(i)}(t) >, \left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] = \left[y^{'}(t)y^{(2)}(t)\ldots y^{(n)}(t)\right] \tag{7}$$

for all $t \in J$ *and* $1 \leq i \leq j \leq n, i+j \leq 2n-1$ *implies* $x \overset{SM(n)}{\sim} y$.

Proof. Let $f\{x\} \in R\{Z\}$. Then Equation (7) implies

$$f\{x(t)\} = f\{y(t)\} \tag{8}$$

for all $t \in J$. By Lemma 4, $\delta_x \in R\{Z\}$. Hence Equation (8) implies $\delta_x = \delta_y$ for all $t \in J$. By the assumption of our theorem, we have $\delta_x \neq 0$ and $\delta_y \neq 0$ for all $t \in J$. Hence the equality $\delta_x = \delta_y$ for all $t \in J$ implies $\delta_x^{-1} = \delta_y^{-1}$ for all $t \in J$. Denote the differential R-algebra generated by elements of the system Z, the functions Δ^{-1} and δ^{-1} by $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. Let $f\{x\} \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\})$. Then the equality $\delta_x^{-1} = \delta_y^{-1}$, Equation (7) and Equation (8) imply

$$f\{x(u)\} = f\{y(u)\} \tag{9}$$

for all $t \in J$.

Lemma 5 $\Delta \in R\{Z\}$.

Proof. Using Lemma 2 to vectors $y_1 = z_1 = x^{'}, y_2 = z_2 = x^{(2)}, \cdots, y_n = z_n = x^{(n)}$, we obtain

$$(-1)^p \left[x^{'} x^{(2)} \ldots x^{(n)}\right]^2 = det \left\| < y_i, z_j > \right\|_{i,j=1,2,\ldots n} = \Delta. \tag{10}$$

Since $\left[x^{'} x^{(2)} \ldots x^{(n)}\right] \in Z$, we have $\Delta \in R\{Z\}$. □

Lemma 6 $< x^{(n)}, x^{(n)} > \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ *and* $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$.

Proof. For $i = 1, 2, \ldots, n$, denote the cofactor of the element $< y_n, z_j >$ of the matrix $A = \left\| < y_i, z_j > \right\|_{i,j=1,2,\ldots n}$ in Equation (10) by D_{ni}. Then we obtain the equality

$$\Delta = < y_n, z_1 > D_{n1} + < y_n, z_2 > D_{n2} + \cdots + < y_n, z_{n-1} > D_{nn-1} + < y_n, z_n > D_{nn}.$$

Since $\delta = D_{nn} \neq 0$, this equality implies

$$\begin{aligned} < y_n, z_n > = < x^{(n)}, x^{(n)} > = \Delta\delta^{-1} - < y_n, z_1 > D_{n1}\delta^{-1} - < y_n, z_2 > D_{n2}\delta^{-1} - \\ \cdots - < y_n, z_{n-1} > D_{nn-1}\delta^{-1}. \end{aligned} \tag{11}$$

By Lemma 1, we have $< y_n, z_j > = < x^{(n)}, x^{(j)} > \in R\{Z\}$ for each $1 \leq j \leq n-1$. We prove that $D_{ns} \in R\{Z\}$ for every $1 \leq s \leq n-1$. We have

$$D_{ns} = (-1)^{n+s} det \left\| < y_i, z_j \right\|_{i=1,2,\ldots,n-1; j=1,2,\ldots,s-1,s+1,\ldots,n}.$$

Elements of D_{ns} have forms $< y_i, z_j >$, $< y_i, z_n >$, where $i, j < n$. By $< y_i, z_j > \in R\{Z\}, < y_i, z_n > = < y_n, z_i > \in R\{Z\}$, we obtain $D_{ns} \in R\{Z\}$. Hence Equation (11) implies $< y_n, z_n > \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. Using $V \subset Z \cup \{(y_n, z_n)\}$, we get $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. □

Using Equations (7), (9)–(11) and $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ in Lemma 6, we obtain Equation (1). Hence, by Theorem 2, $F \in O(n,p)$ and $b \in E_p^n$ exist such that $y(u) = Fx(u) + b$. Using this equality and $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] = \left[y^{'}(t)y^{(2)}(t)\ldots y^{(n)}(t)\right]$ in Equation (7), we get $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] = detF\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right]$. Since $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] \neq 0$ for all $t \in J$, we obtain $detF = 1$. Hence $x \overset{SM(n)}{\sim} y$. The theorem is completed. □

Corollary 2 *Let α, β be non-singular curves in E_p^n and $x \in Ip(\alpha), y \in Ip(\beta)$. Assume that x, y satisfy the condition (5_1)) in Definition 5 and conditions $\delta_x(t) \neq 0$, $\delta_y(t) \neq 0$ for all $t \in J$. Then*

(*i*) *in the case $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{SM(n,p)}{\sim} \beta$ if and only if*

$$\left[x^{'}(s)\ldots x^{(n)}(s)\right] = \left[y^{'}(s)\ldots y^{(n)}(s)\right], \tag{12}$$

$$sgn < x^{'}(s), x^{'}(s) >= sgn < y^{'}(s), y^{'}(s) >, \tag{13}$$

$$< x^{(i)}(s), x^{(i)}(s) >=< y^{(i)}(s), y^{(i)}(s) > \tag{14}$$

for all $s \in L(\alpha)$ and all $i = 2, \ldots, n-1$;

(*ii*) *in the case $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{SM(n,p)}{\sim} \beta$ if and only if*

$$\left[x^{'}(s)\ldots x^{(n)}(s)\right] = \left[y^{'}(s+s_1)\ldots y^{(n)}(s+s_1)\right],$$

$$sgn < x^{'}(s), x^{'}(s) >= sgn < y^{'}(s), y^{'}(s) >,$$

$$< x^{(i)}(s), x^{(i)}(s) >=< y^{(i)}(s+s_1), y^{(i)}(s+s_1) >$$

for some $s_1 \in (-\infty, +\infty)$, all $s \in L(\alpha)$ and $i = 2, \ldots, n-1$;

Proof. (i). Let $\alpha \overset{SM(n,p)}{\sim} \beta$. Since elements of Z and the function $sgn < x^{'}(s), x^{'}(s) >$ are $SM(n,p)$-invariant, we obtain that Equation (12)–(14) hold.

Conversely, assume that Equation (12)–(14) hold. According to Proposition 2, we get $\left|< x^{'}(s), x^{'}(s) >\right| = \left|< y^{'}(s), y^{'}(s) >\right| = 1$ for all $s \in L(\alpha)$. Then, using Equation (13), we obtain $< x^{'}(s), x^{'}(s) >= < y^{'}(s), y^{'}(s) >$ for all $s \in L(\alpha)$. The latest equality, Equation (12) and Equation (14), by Lemmas 4 and 5, imply $\delta_x = \delta_y, \Delta_x = \Delta_y$. Then, by Lemma 6, we obtain $< x^{(n)}, x^{(n)} >=< y^{(n)}, y^{(n)} >$. By this equality, Equation (12), Equation (14) and Theorem 3, there exists $F \in SM(n,p)$ such that $y(s) = Fx(s) = gx(s) + b$. The proof of statement (i) is completed. Similarly, the proof of (ii) follows from statement (ii) of Theorem 1. □

Remark 6 Similar results are true for conditions (5_2)) or (5_3)) in Definition 5.

Let α be a curve and $x \in Ip(\alpha)$.

Remark 7 According to Corollary 2, the system

$$\left\{L(\alpha), sgn < x^{'}, x^{'} >, < x^{(2)}, x^{(2)} >, \ldots, < x^{(n-1)}, x^{(n-1)} >, \left[x^{'} x^{(2)} \ldots x^{(n)}\right]\right\}$$

is a complete system of $SM(n,p)$-invariants of a curve α for the case $L(\alpha) \neq (-\infty, +\infty)$. But they are not invariants of a curve α for the case $L(\alpha) = (-\infty, +\infty)$. They depend on reparametrizations $s \to s + a$ of the curve α.

References

[1] Aripov, R. G. and Khadzhiev, D.:, The complete system of differential and integral invariants of a curve in Euclidean geometry. Russian Mathematics (Iz VUZ), **51**, No. 7, 1-14 (2007).

[2] Aslaner, R. and Boran, A. I.: On the geometry of null curves in the Minkowski 4-space. Turkish J. of Math. **32**, 1-8 (2008).

[3] Bejancu, A.: Lightlike curves in Lorentz manifolds. Publ. Math. Debrecen. **44**, 145-155 (1994).

[4] Bérard B. L. and Charuel X.: A generalization of Frenet's frame for nondegenerate quadratic forms with any index. In: Séminaire de théorie spectrale et géométrie. Année 2001-2002, St. Martin d'Héres: Université de Grenoble I, Institut Fourier, Sémin. Théor. Spectr. Géom. **20**, 101-130 (2002).

[5] Bini, D., Geralico, A. and Jantzen, R. T.: Frenet-Serret formalism for null world lines. Class. Quantum Grav. **23**, 3963-3981 (2006).

[6] Bonnor, W.: Null curves in a Minkowski spacetime. Tensor, N. S. **20**, 229-242 (1969).

[7] Borisov Yu. F.: Relaxing the a priori constraints of the fundamental theorem of space curves in E_l^n. Siberian Math. J. **38**, No. 3, 411-427 (1997).

[8] Borisov Yu. F.: On the theorem of natural equations of a curve. Siberian Math. J. **40**, No. 4, 617-621 (1999).

[9] Çöken, C. and Çiftçi, Ü.: On the Cartan curvatures of a null curve in Minkowski spacetime. Geometriae Dedicata. **114**, 71-78 (2005).

[10] Duggal, K. L. and Becancu A.: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Dordrecht, Boston, London. Kluwer Acad. Publ. 1996.

[11] Ferrández, A., Giménez, A. and Lucas, P.: Degenerate curves in pseudo-Euclidean spaces of index two. In: Mladenov, Ivailo M. (ed.) et al., Proceedings of the 3rd international conference on geometry, integrability and quantization, Varna, Bulgaria, June 14-23, 2001. Sofia: Coral Press Scientific Publishing. 209-223 (2002).

[12] Ferrández, A., Giménez, A. and Lucas, P.: s-degenerate curves in Lorentzian space forms. J. Geom. Phys. **45**, No. 1-2, 116-129 (2003).

[13] Formiga, L. B. and Romero, C.: On the differential geometry of time-like curves in Minkowski spacetime. Am. J. Phys. **74**(10), 1012-1016 (2006).

[14] Ichimura, H.: Time-like and space-like curves in Frenet-Serret formalisms. Thesis. Hadronic J. Suppl. **3**, No. 1, 1-94 (1987).

[15] Kaplansky, I.: An Introduction to Differential Algebra. Paris. Hermann 1957.

[16] Khadjiev, D. and Pekşen Ö.: The complete system of global integral and differential invariants for equi-affine curves. Differential Geometry and its Applications. **20**, 167-175 (2004).

[17] Pekşen Ö. and Khadjiev, D.: On invariants of curves in centro-affine geometry. J. Math. Kyoto Univ. (JMKYAZ). **44-3**, 603-613 (2004).

[18] Petrović-Torgašev, Ilarslan K. and Nešović E.: On partially null and pseudo-null curves in the semi-euclidean space R_2^4. J. Geom. **84**, 106-116 (2005).

[19] Spivak, M.: A Comprehensive Introduction to Differential Geometry, Vol.2. Berkeley, CA. Publ. of Perish. Inc. 1979.

[20] Urbantke H.: Local differential geometry of null curves in conformally flat space-time. J. Math. Phys. **30**(10), 2238-2245 (1989).

[21] Yılmaz, S. and Turgut, M.: On the differential geometry of curves in Minkowski space-time I. Int. J. Contemp. Math. Sciences. **3**, No. 27, 1343-1349 (2008).

Ömer PEKŞEN, Djavvat KHADJIEV, İdris ÖREN
Department of Mathematics
Karadeniz Technical University, Trabzon-TURKEY
e-mail: peksen@ktu.edu.tr; haciyev@ktu.edu.tr; oren@ktu.edu.tr

Received: 12.11.2009