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SARAH B. HART

PETER J. ROWLEY

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Sarah B. Hart, Peter J. Rowley

Abstract

In this paper the notion of Coxeter length for a subset of a Coxeter group, as introduced in [9], is investigated for various subsets of a Coxeter group. Mostly cosets of various subgroups are examined as well as the associated idea of X-posets, which is a vast generalization of the Bruhat order.

Key Words: Coxeter group, Bruhat order, cosets, partially ordered sets

1. Introduction

In [9] the authors introduced the notion of length for a subset of a Coxeter group which generalizes the well known length function on elements of a Coxeter group. A number of properties of this generalized length function were obtained there. For a survey of results in this area see [6].

The purpose of the present paper is to investigate the lengths of cosets of various subgroups of the Coxeter group W with particular emphasis on certain partial orders. These partial orders amount to an extensive generalization of the Bruhat order [1], [3], [8] and, indeed, of the Bruhat order defined by Deodhar on the cosets of a standard parabolic subgroup of W [5]. For W a Coxeter group and X a subset of W, we define

$$N(X) = \{\alpha \in \Phi^+ | w \cdot \alpha \in \Phi^- \text{ for some } w \in X\},$$

where Φ^+ and Φ^- are, respectively, the positive and negative roots of the root system Φ of W. So $N(X)$ consists of all the positive roots taken negative by some element of X. Now, from [9], the Coxeter length of X, $l(X)$, is defined to be the cardinality of $N(X)$. If $X = \{w\}$, then $l(X)$ is just the length of w in the traditional sense. Let $\mathrm{Ref}(W)$ be the set of reflections of W. For $w, w' \in W$ write $w \longrightarrow w'$ if $w = w't$ for some $t \in \mathrm{Ref}(W)$ and $l(w') < l(w)$. The Bruhat order on W, denoted by $<$, is defined as follows. For $w, w' \in W$, $w' < w$ provided that there are elements $w_1, \dots, w_m$ of W such that $w \longrightarrow w_1 \longrightarrow \cdots \longrightarrow w_m \longrightarrow w'$. There is a related partial order on W, referred to as the weak Bruhat order, which is defined in the same way as the Bruhat order but with $\mathrm{Ref}(W)$ replaced by the set of fundamental reflections of W. We shall use $<_w$ to denote the weak Bruhat order on W. In [5], Deodhar extended the Bruhat order to a partial order on the (right) cosets of a standard parabolic subgroup X of W. The essential fact allowing this to be done is that every (right)

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coset of X has a unique element whose length is minimal in that coset. For cosets Xg, Xh, this partial order is defined by $Xg < Xh$ if $g < h$ (where g and h are the elements of minimal length in their cosets). We shall write $\mathcal{B}(X)$ for this partial order on the (right) cosets of X in W.

One of the early impetuses for introducing and studying the Bruhat order was its connection with inclusions among closures of Bruhat cells for a corresponding semisimple algebraic group. In recent times it has become interwoven with other aspects of algebraic groups and groups of Lie type such as, for example, Kazdhan Lusztig polynomials. For standard parabolic subgroups X of W, a further generalization of $\mathcal{B}(X)$, called a generalized quotient, was given by Björner and Wachs [2]. This poset shares many of its properties with $\mathcal{B}(X)$. For example the Möbius function of generalized quotients (and $\mathcal{B}(X)$) takes values $-1, 0$ or 1 (see [Corollary 3.6; 2]). However the Möbius function for the posets to be introduced here does not necessarily take values in $\{-1, 0, 1\}$ (see [7]) and so these posets are different from generalized quotients.

Now we come to the poset which is the main subject of this paper. For a subset X of W we shall use the notation $X \leq W$ to indicate that X is a subgroup of W.

Definition 1.1 *Suppose that $X \leq W$.*

(i) For right cosets Xg and Xh of X, we write $Xg \sim Xh$ whenever $Xgt = Xh$ for some $t \in \mathrm{Ref}(W)$ and $l(Xg) = l(Xh)$. Let $\approx$ be the equivalence relation generated by $\sim$ on the set of right cosets of X in W and let $\mathfrak{X}$ be the set of $\approx$ equivalence classes.

(ii) Let $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}$. We write $\mathbf{x} \rightsquigarrow \mathbf{x}'$ if there is a right coset Xg in $\mathbf{x}$ and a right coset Xh in $\mathbf{x}'$ such that $Xgt = Xh$ for some $t \in \mathrm{Ref}(W)$ and $l(Xg) \leq l(Xh)$. The partial order $\preccurlyeq$ on $\mathfrak{X}$ is defined by $\mathbf{x} \preccurlyeq \mathbf{x}'$ if and only if there exist $\mathbf{x_1}, \ldots, \mathbf{x_m} \in \mathfrak{X}$ such that $\mathbf{x} \rightsquigarrow \mathbf{x_1} \rightsquigarrow \ldots \rightsquigarrow \mathbf{x_m} \rightsquigarrow \mathbf{x}'$. We shall call $\mathfrak{X}$ the X-poset (of W).

(iii) If, in (i) and (ii), we use the set of fundamental reflections instead of $\mathrm{Ref}(W)$, we may define, analogously, the weak X-poset (of W) denoted by $\mathfrak{X}_w$ with ordering $\preccurlyeq_w$.

There are two main reasons for studying X-posets. Firstly, as will be seen, they represent a generalization of the Bruhat order on right cosets of standard parabolic subgroups [5]. Secondly, Theorem 1.5 will show that certain X-posets are ranked and graded. We believe that many other X-posets may have similar properties, and that they represent a source of new and interesting examples of posets.

For the coset Xg we use $[Xg]$, respectively $[Xg]_w$, to denote the $\approx$ equivalence class, respectively the $\approx_w$ equivalence class containing Xg. We may now state our first result.

Theorem 1.2 *Suppose $X \leq W$ and let $\mathfrak{X}$, respectively $\mathfrak{X}_w$, denote the X-poset, respectively weak X-poset. Then both $\mathfrak{X}$ and $\mathfrak{X}_w$ have a unique minimal element, namely the $\approx$ (respectively, $\approx_w$) equivalence class containing X.*

Theorem 1.3 *Suppose that W is finite, $X \leq W$ and let w_0 be the unique longest element of W.*

(i) For any $g \in W$, $l(Xgw_0) = l(w_0) + l(X) - l(Xg)$.

(ii) Let $g \in W$ and $s \in \mathrm{Ref}(W)$. Then there exists $t \in \mathrm{Ref}(W)$ for which $l(t) = l(s)$ and $l(Xgs) - l(Xg) = l(Xgw_0) - l(Xgw_0t)$.

(iii) Both the weak X-poset and the X-poset are symmetric. That is, if $[Xg] \preccurlyeq [Xh]$, respectively $[Xg]_w \preccurlyeq_w [Xh]_w$, then $[Xhw_0] \preccurlyeq [Xgw_0]$, respectively $[Xhw_0]_w \preccurlyeq_w [Xgw_0]_w$.

So when W is finite, for $X \leq W$, $\mathfrak{X}$ and $\mathfrak{X}_w$ have a unique maximal and unique minimal element by Theorems 1.2 and 1.3(*iii*). We remark that Theorems 1.2 and 1.3 were stated but not proved in the survey paper [6]. A standard parabolic subgroup of W is a subgroup generated by some subset I of R and is denoted by W_I. Any conjugate of a standard parabolic subgroup is known simply as a parabolic subgroup of W. It is well known that standard parabolic subgroups are Coxeter groups in their own right.

The following is a corollary to Theorem 3.8.

Corollary 1.4 *Suppose that $X \leq Y \leq W$ where X is finite and Y is a standard parabolic subgroup of W. If $N(X) = N(Y)$, then the X-poset $\mathfrak{X}$ is poset isomorphic to $\mathcal{B}(Y)$.*

As a consequence of Corollary 1.4, when X is a finite standard parabolic subgroup, we recover the Bruhat order given in [5]. Recall that a *ranked* poset is a poset P such that for each $p \in P$ all maximal chains in $\{q \in P | q \leq p\}$ have the same finite length, called the rank of p. A *graded* poset is a ranked poset with a minimum and a maximum element (see [1]). In Section 4 explicit descriptions of certain X-posets are given. Among the results obtained there we single out this theorem

Theorem 1.5 *Suppose that W is the Coxeter group of type A_n and $X = \langle t \rangle$ where $t \in \mathrm{Ref}(W)$ is the longest reflection in W. Then the X-poset $\mathfrak{X}$ is a ranked, graded poset.*

Before stating some further results, we recall certain basic facts and notation about Coxeter groups. First, by definition, our Coxeter group W has presentation

$$W = \langle R \mid (rs)^{m_{rs}} = 1, r, s \in R \rangle,$$

where $m_{rs} \in \mathbb{N} \cup \{\infty\}$, $m_{rr} = 1$ and for $r, s \in R$, $r \neq s$, $m_{rs} = m_{sr} \geq 2$. The set R consists of the fundamental reflections of W. Let V be a real vector space with basis $\Pi = \{\alpha_r | r \in R\}$ and upon V define the symmetric bilinear form $(\ ,\ )$ by

$$(\alpha_r, \alpha_s) = \begin{cases} -\cos\left(\frac{\pi}{m_{rs}}\right) & \text{if } m_{rs} \neq \infty, \\ -1 & \text{otherwise,} \end{cases}$$

where $r, s \in R$. For $r, s \in R$ let

$$r \cdot \alpha_s = \alpha_s - 2(\alpha_r, \alpha_s)\alpha_r.$$

Now this extends in the natural way to an action of W on V which is not only faithful but also respects $(\ ,\ )$. The root system Φ which frequently goes hand in glove with W is the following subset of V:

$$\Phi = \{w \cdot \alpha_r | r \in R, w \in W\}.$$

Setting $\Phi^+ = \{\sum_{r \in R} \lambda_r \alpha_r \in \Phi | \lambda_r \geq 0 \text{ for all } r \in R\}$ and $\Phi^- = -\Phi^+$ gives us, respectively, the sets of positive and negative roots of Φ. As is well-known, $\Phi = \Phi^+ \dot{\cup} \Phi^-$.

A fundamental, and often useful, fact about Coxeter groups is that, for $r \in R$ and $w \in W$,

$$l(wr) = \begin{cases} l(w) + 1 & \text{if } w \cdot \alpha_r \in \Phi^+; \\ l(w) - 1 & \text{if } w \cdot \alpha_r \in \Phi^-, \end{cases}$$

where α_r is the fundamental root corresponding to r. One of our results in Section 2 may be seen as a generalization of this fact.

Proposition 1.6 *If X is a finite subset of W and $r \in R$, then*

$$l(Xr) = \begin{cases} l(X)+1 & \text{if } \alpha_r \notin N(X); \\ l(X)-1 & \text{if } \alpha_r \in N(x) \text{ for all } x \in X; \\ l(X) & \text{otherwise.} \end{cases}$$

If, in addition, we demand that X be a subgroup of W then we have

Proposition 1.7 *Let $g = r_1 \cdots r_k \in W$ be a reduced expression for g and let X be a finite subgroup of W. Then*

$$l(X) \leq l(Xr_1) \leq \cdots \leq l(Xr_1 \cdots r_k) = l(Xg).$$

In particular, for all $g \in W$, $l(Xg) \geq l(X)$.

Remark 1.8 The 'left handed' analogues of Propositions 1.6 and 1.7 do not hold. Counterexamples will be given in Section 2. This lack of symmetry is due to the fact that we are acting on the left of Φ.

The following proposition lists several well known facts which will be of use. For proofs of these see, for example, [8].

Proposition 1.9 *(i) For $w \in W$, $|N(w)| = l(w)$.*

(ii) For $w, v \in W$, $N(wv) = N(v) \backslash (-v^{-1}N(w)) \,\dot{\cup}\, v^{-1}(N(w) \backslash N(v^{-1}))$.

(iii) Let $s \in \mathrm{Ref}(W)$ and $w \in W$. Then $l(ws) > l(w)$ if $w \cdot \alpha_s \in \Phi^+$ and $l(ws) < l(w)$ if $w \cdot \alpha_s \in \Phi^-$.

(iv) The Exchange Condition: Let $w = r_1 \cdots r_k$, $r_i \in R$, be a not necessarily reduced expression for w. Suppose that $s \in \mathrm{Ref}(W)$ satisfies $l(ws) < l(w)$. Then there is an index i for which $ws = r_1 \cdots \hat{r}_i \cdots r_k$ (omitting r_i). If the expression for w is reduced, then i is unique.

Our next proposition gives some properties of the Bruhat order $\leq$. Let $I \subseteq R$. We write D_I for the set of distinguished coset representatives of minimal length for W_I (the standard parabolic subgroup generated by I). We have

Proposition 1.10 *(i) If W is finite and w_0 is the longest element of W, then $g \leq h$ if and only if $hw_0 \leq gw_0$.*

(ii) Let $g = r_1 \cdots r_k$ be any reduced expression for g. Then $h \leq g$ if and only if h can be written as a subexpression of $r_1 \cdots r_k$.

(iii) Suppose $g < h$. Then there exists a sequence $g = g_0 < g_1 < \cdots < g_n = h$ such that $l(g_i) = l(g_{i-1}) + 1$ for $1 \leq i \leq n$.

(iv) If $I \subseteq R$, then the Bruhat ordering of the Coxeter group W_I agrees with the restriction to W_I of the Bruhat ordering of W.

(v) Let $I \subseteq R$. Suppose that $g, h \in W_I$ and w, v in D_I are such that $gw < hv$. Then $w \leq v$.

Part (i) of Proposition 1.10 is mirrored in Theorem 1.3(iii). The analogue of Proposition 1.10 (iv) is our next result. For $I \subseteq R$ and X a subgroup of W_I, the ordering $\preccurlyeq$ of the X-poset (of W_I) will be denoted by $\preccurlyeq_I$.

Proposition 1.11 *Let $I \subseteq R$, and $X \leq W_I$. Then $\preccurlyeq_I$ agrees with the restriction to W_I of $\preccurlyeq$.*

Proposition 1.12 *([9], Proposition 1.1) Let X be a finite standard parabolic subgroup of W and Y be a conjugate of X. Then $l(X) \leq l(Y)$, with equality if and only if Y is also a standard parabolic subgroup of W.*

2. Lengths of Cosets

Lemma 2.1 *Let $I \subseteq R$. Then $N(D_I)\dot{\cup}N(W_I) = \Phi^+$.*

Proof. Suppose $\alpha \notin N(W_I)$. Let $w \in W$ have minimal length subject to $w \cdot \alpha \in \Phi^- \backslash \Phi_I$. (Such a w certainly exists, because $s_\alpha \cdot \alpha = -\alpha \in \Phi^- \backslash \Phi_I$.) Now suppose, for a contradiction, that there is some $s \in I$ such that $l(sw) < l(w)$. If $sw \cdot \alpha \in \Phi_I$, then, as Φ_I is s-invariant, we get $w \cdot \alpha \in \Phi_I$, a contradiction. So $sw \cdot \alpha \notin \Phi_I$. Consequently, by the minimal choice of w, $sw \cdot \alpha \in \Phi^+$. From $s \cdot (w \cdot \alpha) \in \Phi^+$, as $w \cdot \alpha \in \Phi^-$, we deduce that $w \cdot \alpha = -\alpha_s \in \Phi_I$, a contradiction. Hence $w \in D_I$, as required. $\square$

Proof of Proposition 1.6 By definition, $N(Xr) = \bigcup_{x \in X} N(xr)$. It is a special case of Proposition 1.9(ii) that if $\alpha_r \notin N(x)$, then $N(xr) = rN(x)\dot{\cup}\{\alpha_r\}$ and if $\alpha_r \in N(x)$, then $N(xr) = r(N(x)\backslash\{\alpha_r\})$. Suppose $\alpha_r \notin N(X)$. Then for all $x \in X$, $\alpha_r \notin N(x)$. Hence

$$\begin{aligned} N(Xr) = \bigcup_{x \in X} N(xr) &= \bigcup_{x \in X} rN(x)\dot{\cup}\{\alpha_r\} \\ &= rN(X)\dot{\cup}\{\alpha_r\}. \end{aligned}$$

Thus $l(Xr) = l(X) + 1$. Now assume that $\alpha_r \in N(x)$ for all $x \in X$. Then

$$\begin{aligned} N(Xr) &= \bigcup_{x \in X} r(N(x)\backslash\{\alpha_r\}) \\ &= r(N(X)\backslash\{\alpha_r\}), \end{aligned}$$

and so $l(Xr) = l(X) - 1$. Finally, if $\alpha_r \in N(X)$ and for some $x \in X$ we have $\alpha_r \notin N(x)$ then we see that $N(Xr) = r(N(X)\backslash\{\alpha_r\})\dot{\cup}\{\alpha_r\}$, hence $l(Xr) = l(X)$. $\square$

Proof of Proposition 1.7 At each stage $(0 \leq i < k)$, $r_1 \cdots r_i \in Xr_1 \cdots r_i$. Also, $r_1 \cdots r_i r_{i+1}$ is a reduced expression and thus $\alpha_{r_{i+1}} \notin N(r_1 \cdots r_i)$. Thus it is not the case that $\alpha_{r_{i+1}} \in N(x)$ for all $x \in Xr_1 \cdots r_i$. Therefore, by Proposition 1.6, $l(Xr_1 \cdots r_{i+1}) \geq l(Xr_1 \cdots r_i)$. $\square$

It is sometimes useful to know $N(Xg)$ explicitly and this is the content of the next lemma.

Lemma 2.2 *Suppose that X is a subgroup of W. Then for $g \in W$,*

$$N(Xg) = N(g)\dot{\cup} g^{-1}(N(X)\backslash N(g^{-1}))$$

and hence

$$l(Xg) = l(g) + l(X) - |N(X) \cap N(g^{-1})|.$$

Proof. Let $X \leq W$ and $g \in W$. Again we use Proposition 1.9(ii) to calculate $N(Xg)$ as follows.

$$\begin{aligned} N(Xg) &= \bigcup_{x \in X} N(xg) = N(g) \cup \left(\bigcup_{1 \neq x \in X} N(xg) \right) \\ &= N(g) \cup \left(\bigcup_{1 \neq x \in X} N(g)\backslash(-g^{-1}N(x)) \,\dot{\cup}\, g^{-1}(N(x)\backslash N(g^{-1})) \right) \\ &= N(g) \cup \left(\bigcup_{1 \neq x \in X} g^{-1}(N(x)\backslash N(g^{-1})) \right) \\ &= N(g) \cup g^{-1}(N(X)\backslash N(g^{-1})). \end{aligned}$$

Now if $\alpha \in N(g) \cap g^{-1}(N(X)\backslash N(g^{-1}))$, then $g \cdot \alpha \in gN(g) \cap N(X) \subseteq \Phi^- \cap \Phi^+$, which is empty. Thus $N(Xg)$ really is the disjoint union of $N(g)$ and $g^{-1}(N(X)\backslash N(g^{-1}))$. □

Remark 2.3 The analogous statements comparing the length of gX with X are not true. For example, let W be the Coxeter group of type A_4 (so $W \cong \mathrm{Sym}(5)$). Then if $r = (12)$ and $X = \{(132), (12)(34)\}$ we have $N(X) = \{\alpha_{(12)}, \alpha_{(23)}, \alpha_{(34)}, \alpha_{(13)}\}$, thus $l(X) = 4$ but $l(rX) = 2 < l(X) - 1$. If we take X to be the parabolic subgroup generated by (13) and (45), then $l(X) = 4$ but $l((12)X) = 3 < l(X)$. However, if X is a standard parabolic subgroup, the next lemma, which follows from Proposition 1.12, does give such an analogue.

Lemma 2.4 *Let X be a standard parabolic subgroup of W and let $g \in W$. Then $l(gX) \geq l(X)$.*

Proof. We have

$$l(gX) = l((gXg^{-1})g) \geq l(gXg^{-1})$$

by Proposition 1.7, and then $l(gXg^{-1}) \geq l(X)$ by Proposition 1.12. □

3. X-posets of W

Proof of Theorem 1.2 Let $g \in W$ have a reduced expression $g = r_1 r_2 \cdots r_k$ where $r_i \in R$. From Proposition 1.7

$$[X] \preccurlyeq [Xr_1] \preccurlyeq \cdots \preccurlyeq [Xr_1 \cdots r_k] = [Xg],$$

and so $\mathbf{x} \preccurlyeq \mathbf{x}'$ where $\mathbf{x}$ and $\mathbf{x}'$ are, respectively, the $\approx$ equivalence class of X and the $\approx$ equivalence class of Xg. Therefore $\mathbf{x}$ is the unique minimal element of $\mathfrak{X}$, and by a similar argument of $\mathfrak{X}_w$. □

Proof of Theorem 1.3

(i) By Lemma 2.2,

$$l(Xg) = l(g) + l(X) - |N(X) \cap N(g^{-1})|$$

and

$$N(Xgw_0) = N(gw_0)\dot{\cup} w_0 g^{-1}(N(X)\backslash N(w_0 g^{-1})).$$

Now it follows from Proposition 1.9 (ii) and the fact that $N(w_0) = \Phi^+$ that $l(gw_0) = l(w_0) - l(g)$ and that $N(w_0 g^{-1}) = \Phi^+ \backslash N(g^{-1})$. Hence

$$N(X)\backslash N(w_0 g^{-1}) = N(X)\backslash(\Phi^+ \backslash N(g^{-1})) = N(X) \cap N(g^{-1}).$$

So

$$\begin{aligned} l(Xgw_0) &= l(gw_0) + |N(X)\backslash N(w_0 g^{-1})| \\ &= l(w_0) - l(g) + |N(X) \cap N(g^{-1})| \\ &= l(w_0) - l(g) + (l(g) + l(X) - l(Xg)) \\ &= l(w_0) + l(X) - l(Xg). \end{aligned}$$

(ii) Let $s \in \mathrm{Ref}(W)$. Then set $t = s^{w_0}$, so that $w_0 t = s w_0$. A quick calculation shows that $l(t) = l(w_0 s w_0) = l(w_0) - l(sw_0) = l(w_0) - (l(w_0) - l(s)) = l(s)$. Furthermore, by (i),

$$\begin{aligned} l(Xgw_0) - l(Xgw_0 t) &= l(Xgw_0) - l(Xgsw_0) \\ &= l(w_0) + l(X) - l(Xg) - (l(w_0) + l(X) - l(Xgs)) \\ &= l(Xgs) - l(Xg). \end{aligned}$$

(iii) Follows immediately from part (ii).

□

Lemma 3.1 *Let $w \in W$, $s \in \mathrm{Ref}(W)$ and $\alpha \in N(s)$. If $\{\alpha, -s \cdot \alpha\} \subseteq N(w)$, then $\alpha_s \in N(w)$.*

Proof. Since $\alpha \in N(s)$, $s \cdot \alpha = \alpha - 2(\alpha, \alpha_s)\alpha_s \in \Phi^-$. Set $\lambda = 2(\alpha, \alpha_s)$. Then $\lambda > 0$ and

$$\alpha_s = \frac{1}{\lambda}\left((-s \cdot \alpha) + \alpha\right).$$

Hence

$$w \cdot \alpha_s = \frac{1}{\lambda}\left(w \cdot (-s \cdot \alpha) + w \cdot \alpha\right),$$

which, as a positive linear combination of negative roots, must also be a negative root. Therefore $\alpha_s \in N(w)$, and the lemma holds. □

Lemma 3.2 *Suppose that X is a finite subset of W and that $s \in \mathrm{Ref}(W)$. Then*

$$l(Xs) - l(X) = l(s) - |N(s) \cap \left(\bigcap_{x \in X} N(x) \right)| - |N(s) \cap N(X)|.$$

Proof. By definition, and using Proposition 1.9(ii), we have

$$\begin{aligned} N(Xs) &= \bigcup_{x \in X} N(xs) \\ &= \bigcup_{x \in X} \left[N(s) \backslash (-sN(x)) \;\dot{\cup}\; s(N(x) \backslash N(s)) \right]. \end{aligned}$$

Note that for $x \neq x' \in X$, we have $N(s) \backslash (-sN(x)) \;\cap\; s(N(x') \backslash N(s)) = \emptyset$, because $N(s) \cap s\Phi^+$ is empty. So

$$N(Xs) = \left[N(s) \backslash - s \left(\bigcap_{x \in X} N(x) \right) \right] \dot{\bigcup} \left[s\left(N(X) \backslash N(s)\right) \right]$$

and hence

$$l(Xs) = l(s) - |N(s) \cap \left(\bigcap_{x \in X} N(x) \right)| + |N(X)| - |N(s) \cap N(X)|.$$

Thus

$$l(Xs) - l(X) = l(s) - |N(s) \cap \left(\bigcap_{x \in X} N(x) \right)| - |N(s) \cap N(X)|,$$

so verifying the lemma. □

Our next proposition, which makes use of Lemma 3.2, reveals a connection between the partial order $\preccurlyeq$ and the Bruhat order.

Proposition 3.3 *Let X be a finite subset of W and let $s \in \mathrm{Ref}(W)$. If $l(Xs) \leq l(X)$, then there exists $x \in X$ such that $l(xs) < l(x)$.*

Proof. Suppose, for a contradiction, that for each $x \in X$, $l(xs) > l(x)$. Then, by Proposition 1.9(iii), $x \cdot \alpha_s \in \Phi^+$ for all $x \in X$.

(3.3.1) For all $x \in X$ and $\alpha \in N(s)$, $|\{\alpha, -s \cdot \alpha\} \cap N(x)| \leq 1$.

If (3.3.1) is false, then $\{\alpha, -s \cdot \alpha\} \subseteq N(x)$ for some $x \in X$. Hence $\alpha_s \in N(x)$ by Lemma 3.1 and thus $x \cdot \alpha_s \in \Phi^-$. But $x \cdot \alpha_s \in \Phi^+$, a contradiction. So (3.3.1) holds.

Suppose that $\alpha \in N(s) \cap \left(\bigcap_{x \in X} N(x) \right)$. Then, by (3.3.1), for each $x \in X$, $-s \cdot \alpha \notin N(x)$. In particular $-s \cdot \alpha \notin N(s) \cap N(X)$. Also note that α_s is not in $N(x)$ for any $x \in X$. Therefore

$$|N(s) \cap N(X)| + |N(s) \cap \left(\bigcap_{x \in X} N(x) \right)| \leq l(s) - 1.$$

Combining this with Lemma 3.2 we get

$$\begin{aligned} l(Xs) - l(X) &= l(s) - |N(s) \cap \left(\bigcap_{x \in X} N(x) \right)| - |N(s) \cap N(X)| \\ &\geq l(s) - (l(s) - 1) = 1, \end{aligned}$$

which contradicts the hypothesis that $l(Xs) \leq l(X)$. The result now follows. $\square$

We will mainly make use of Proposition 3.3 in the case when X is a right coset of some subgroup of W. Our next result is a generalization of Proposition 1.9(iii).

Proposition 3.4 *Let X be a finite subset of W and $s = s_\alpha \in \mathrm{Ref}(W)$. If $\alpha \notin N(X)$, then $l(Xs) > l(X)$. If $\alpha \in N(x)$ for all $x \in X$, then $l(Xs) < l(X)$. Otherwise*

$$l(X) - \tfrac{l(s)-1}{2} \leq l(Xs) \leq l(X) + \tfrac{l(s)-1}{2}.$$

Proof. Suppose that $\alpha \notin N(X)$. Then for each $x \in X$, $l(xs) > l(x)$ by Proposition 1.9(iii). Hence, by Proposition 3.3, $l(Xs) > l(X)$. Suppose that $\alpha \in N(x)$ for all $x \in X$. Then for each $xs \in Xs$, $l((xs)s) > l(xs)$, so by Proposition 3.3 again, $l(X) > l(Xs)$. So assume that there exist $x, x' \in X$ such that $\alpha \in N(x) \backslash N(x')$. Now for any $g \in W$, by Proposition 1.9(ii), $l(gs) = l(g) + l(s) - 2|N(g) \cap N(s)|$. Thus, by Proposition 1.9(iii), $\frac{l(s)+1}{2} \leq |N(x) \cap N(s)| \leq l(s)$ and $0 \leq |N(x') \cap N(s)| \leq \frac{l(s)-1}{2}$. From Proposition 3.2,

$$l(Xs) - l(X) = l(s) - |N(s) \cap \left(\bigcap_{x \in X} N(x) \right)| - |N(s) \cap N(X)|.$$

Hence

$$l(X) + l(s) - \tfrac{l(s)-1}{2} - l(s) \leq l(Xs) \leq l(X) + l(s) - 0 - \tfrac{l(s)+1}{2},$$

so giving

$$l(X) - \tfrac{l(s)-1}{2} \leq l(Xs) \leq l(X) + \tfrac{l(s)-1}{2}.$$

$\square$

Let $I \subseteq R$. For X and Y subgroups of W_I, we write $X \approx_I Y$ if $X \approx Y$ and the equivalence is achieved via a sequence of reflections in W_I.

Lemma 3.5 *Suppose that $X \leq W_I$ for some $I \subseteq R$. Then each coset of X is of the form Xgw for some $g \in W_I$ and unique $w \in D_I$. We have*

$$N(Xgw) = N(w) \dot{\cup} w^{-1} N(Xg)$$

and hence $l(Xgw) = l(Xg) + l(w)$.

Proof. By Lemma 2.2, $N(Xgw) = N(gw)\dot{\cup}(gw)^{-1}(N(X)\backslash N((gw)^{-1}))$. Now, using Lemma 2.1 and Proposition 1.9 (ii), we see that

$$\begin{aligned} N(gw) &= N(w)\dot{\cup}w^{-1}N(g) \text{ and} \\ N((gw)^{-1}) &= N(g^{-1})\dot{\cup}gN(w^{-1}). \end{aligned}$$

Also note that $N(w^{-1})\cap\Phi_I=\emptyset$. So

$$\begin{aligned} N(Xgw) &= N(w)\dot{\cup}w^{-1}N(g)\dot{\cup}w^{-1}g^{-1}(N(X)\backslash\left(N(g^{-1})\dot{\cup}gN(w^{-1})\right) \\ &= N(w)\dot{\cup}w^{-1}\left(N(g)\dot{\cup}g^{-1}(N(X)\backslash(N(g^{-1})\dot{\cup}gN(w^{-1}))\right) \\ &= N(w)\dot{\cup}w^{-1}\left(N(Xg)\backslash gN(w^{-1})\right). \end{aligned}$$

Because $g\in W_I$, clearly $g^{-1}N(Xg)\subseteq\Phi_I$. Hence $N(w^{-1})\cap g^{-1}N(Xg)=\emptyset$ and so $gN(w^{-1})\cap N(Xg)=\emptyset$. Thus $N(Xgw)=N(w)\dot{\cup}w^{-1}N(Xg)$ and $l(Xgw)=l(Xg)+l(w)$. □

Proposition 3.6 *Suppose that $X\leq W_I$ and let $g,h\in W_I$, $w,v\in D_I$. Then $Xgw\approx Xhv$ if and only if $Xg\approx_I Xh$ and $w=v$.*

Proof. Suppose that $Xg\approx_I Xh$ where $g,h\in W_I$. It is enough to consider the case $Xgs=Xh$ for some $s\in\mathrm{Ref}(W_I)$, noting that $l(Xg)=l(Xh)$. Define $t=w^{-1}sw$. Then $Xgwt=Xgsw$. Since $l(Xgwt)=l(Xgs)+l(w)=l(Xh)+l(w)$ by Lemma 3.5, $Xgw\approx Xhw$. For the reverse implication, suppose that $Xgw\approx Xhv$ where $g,h\in W_I$, $w,v\in D_I$. Again, it suffices to consider the case $Xgwt=Xhv$ for some $t\in\mathrm{Ref}(W)$, where we may assume that $gwt=hv$. Then, by Proposition 3.3, there exist some $x,y\in X$ such that $l(xgw)<l(xhv)$ and $l(ygw)>l(yhv)$. Therefore $xgw<xhv$ and $ygw>yhv$. Applying Proposition 1.10(v) twice, we see that $w\leq v$ and $v\leq w$, whence $w=v$. Now we have $Xgwt=Xhw$. Let $s=wtw^{-1}$. Then $Xgs=Xh$ and $s\in g^{-1}Xh\subseteq W_I$. Therefore $Xg\approx_I Xh$, which completes the proof of Proposition 3.6. □

We require a technical lemma before giving the proof of Theorem 3.8.

Lemma 3.7 *Suppose that $X\leq Y\leq W$, where X and Y are finite, Y is a reflection subgroup of W and $N(X)=N(Y)$. For any $y\in Y, g\in W$, if $N(Xg)=N(Xyg)$ then $Xg\approx Xyg$.*

Proof. Since Y is generated by reflections, it suffices to show that for any reflection s in Y, and for any $g\in W$, if $N(Xg)=N(Xsg)$ then $Xg\approx Xsg$. But this is obvious because $l(Xg)=l(Xsg)$ and $Xsg=Xg(g^{-1}sg)$. □

Theorem 3.8 *Let $X\leq Y\leq W$ where X and Y are finite and Y is a reflection subgroup of W. Let $\mathfrak{X}$ and $\mathcal{Y}$ denote, respectively, the X-poset and the Y-poset of W. If $N(X)=N(Y)$ then $\mathfrak{X}$ is poset isomorphic to $\mathcal{Y}$. If W is finite and $\mathfrak{X}$ is poset isomorphic to $\mathcal{Y}$ then $N(X)=N(Y)$.*

Proof. Suppose that $N(X) = N(Y)$. Let $g \in W$. Then by Lemma 2.2,

$$\begin{aligned} N(Yg) &= N(g)\dot{\cup}g^{-1}\left(N(Y)\backslash N(g^{-1})\right) \\ &= N(g)\dot{\cup}g^{-1}\left(N(X)\backslash N(g^{-1})\right) \\ &= N(Xg). \end{aligned}$$

Thus, for all $g \in W$, $l(Xg) = l(Yg)$. Let $g \in W$, $s \in \mathrm{Ref}(W)$ and $h = gs$ with $[Yg] \rightsquigarrow [Ygs] = [Yh]$. Since Y is a union of X-cosets, there exists $y \in Y$ with the property that $Xgs = Xyh$. Now $N(Xyh) = N(Yyh) = N(Yh) = N(Xh)$. Therefore, by Lemma 3.7, $Xh \approx Xgs$. Hence $[Xg] \rightsquigarrow [Xgs]$. It is easy to deduce from this that the posets $\mathfrak{X}$ and $\mathcal{Y}$ are isomorphic. Suppose now that W is finite and $\mathfrak{X}$ is poset isomorphic to $\mathcal{Y}$. Since the posets are isomorphic, the maximal length k of a path $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathfrak{X}$ with $l(\mathbf{x}_i) + 1 = l(\mathbf{x}_{i+1})$ is the same as the maximal length of such a path in Y. Let w_0 be the longest element of W. The coset Yw_0 has length $|\Phi^+|$. It is clear that $k \leq |\Phi^+| - l(Y)$. Let $r_1 \cdots r_m$ be a reduced expression for w_0. Then, by Proposition 1.7, $l(X) \leq l(Xr_1) \leq \cdots \leq l(Xr_1 \cdots r_m) = |\Phi^+|$. So $k \geq |\Phi^+| - l(X)$. Combining the two bounds for k we see that $l(Y) \leq l(X)$. And since $X \leq Y$ it must be the case that $N(X) \subseteq N(Y)$, whence $l(X) \leq l(Y)$ with equality precisely when $N(X) = N(Y)$. This completes the proof of Theorem 3.8. □

We now give the

Proof of Proposition 1.11 We must show that if $X \leq W_I$ for some $I \subseteq R$, and $g, h \in W_I$, then $[Xg] \preccurlyeq [Xh]$ if and only if $[Xg] \preccurlyeq_I [Xh]$. It is obvious that if $[Xg] \preccurlyeq_I [Xh]$ then $[Xg] \preccurlyeq [Xh]$. Suppose that $[Xg] \preccurlyeq [Xh]$. Then there is a sequence $g_1w_1, \ldots, g_mw_m$ with $g_i \in W_I$ and $w_i \in D_I$, and a set $s_1, \ldots, s_{m-1}$, $s_i \in \mathrm{Ref}(W)$ such that $g = g_1w_1, h = g_mw_m$, $g_iw_is_i = g_{i+1}w_{i+1}$ and $[Xg_iw_i] \preccurlyeq [Xg_{i+1}w_{i+1}]$. This implies that $w_1 \leq \ldots \leq w_m$. But $h \in W_I$, so that $w_m = 1$, whence $w_i = 1$ for each $1 \leq i \leq m$. Thus $s_i = g_i^{-1}g_{i+1} \in W_I$ for $1 \leq i \leq m-1$ and therefore $[Xg] \preccurlyeq_I [Xh]$, as required. □

4. Some examples of X-posets

In this section we examine in detail some specific X-posets when W is of type A_n, also giving a couple of examples of weak X-posets. Before doing this we note the following general result.

Proposition 4.1 *Suppose X is a finite subgroup of W which is not contained in any proper parabolic subgroup of W. Then $\mathfrak{X}$ consists of one element.*

Proof. By a result of Tits, any finite subgroup of a Coxeter group W is contained in a finite parabolic subgroup of W (see Proposition 1.3 of [4]). By hypothesis, X is finite and not contained in a proper parabolic subgroup of W, and so W must be finite. Suppose for a contradiction that there exists $\alpha \in \Phi^+$ such that $\alpha \notin N(X)$. Then in particular $x \cdot \alpha \in \Phi^+$ for each $x \in X$. Set $\beta = \sum_{x \in X} x \cdot \alpha$. Then $\beta \neq 0$ and X is contained in the stabilizer of β. Now W is finite, which implies that the form $(\ ,\ )$ is positive definite. Thus the radical of W is trivial and therefore the stabilizer of β cannot be the whole of W. It is well known that the stabilizer of any non-zero vector in V is a parabolic subgroup of W. Because the stabilizer of β cannot be

W, it must be a proper parabolic subgroup of W, whereas X is not contained in a proper parabolic subgroup. Therefore $N(X) = \Phi^+$. Now, since $l(Xg) \geq l(X)$ for all $g \in W$, we see that $l(Xg) = l(X) = |\Phi^+|$ for each coset Xg. Hence $\mathfrak{X} = \{[X]\}$ and this completes the proof of Proposition 4.1. $\square$

4.1. X generated by a reflection

We now consider the case when $X = \langle t \rangle$ is generated by some $t \in \mathrm{Ref}(W)$. We need the following lemma about reflections.

Lemma 4.2 *Suppose W is irreducible and that t is the longest reflection in W.*

(i) Let $\alpha \in \Phi^+$. Then $t \cdot \alpha \in \Phi^+$ if and only if $t \cdot \alpha = \alpha$.

(ii) Let $I = \{r \in R \mid t \cdot \alpha_r = \alpha_r\}$. Then $\Phi^+ = N(t) \dot\cup \Phi_I^+$.

Proof. Let $\tilde\alpha$ be the (positive) root corresponding to t, and let $\alpha \in \Phi^+ \backslash N(t)$. Thus $t \cdot \alpha = \alpha - 2(\tilde\alpha, \alpha)\tilde\alpha \in \Phi^+$. If $(\tilde\alpha, \alpha) > 0$ then $(\tilde\alpha, \alpha) \geq 1/2$. But $\tilde\alpha$ is the highest root, so we must have $t \cdot \alpha \in \Phi^-$, a contradiction. So $(\tilde\alpha, \alpha) \leq 0$. We have $r_\alpha \cdot \tilde\alpha = \tilde\alpha + 2|(\alpha, \tilde\alpha)|\alpha$. Now, since $\tilde\alpha$ is the highest root in Φ, we deduce that $(\tilde\alpha, \alpha) = 0$. This shows that for $\alpha \in \Phi^+$, $(\tilde\alpha, \alpha) = 0$ precisely when $\alpha \notin N(t)$ and (i) follows immediately. For (ii), note that $\Phi^+ = N(t) \dot\cup \{\alpha \in \Phi^+ : t \cdot \alpha = \alpha\}$, so that clearly $\Phi_I^+ \subseteq \Phi^+ \backslash N(t)$. For the converse, suppose that $t \cdot \alpha = \alpha$ where $\alpha \in \Phi^+$. Then $\alpha = \sum_{r \in I} \lambda_r \alpha_r + \sum_{r \in R-I} \lambda_r \alpha_r$ for some non-negative λ_r, and $\alpha - 2\sum_{r \in R-I} \lambda_r (\tilde\alpha, \alpha_r)\tilde\alpha = t \cdot \alpha = \alpha$. Hence $\sum_{r \in R-I} \lambda_r (\tilde\alpha, \alpha_r)\tilde\alpha = 0$. But for each $r \in R - I$, $t \cdot \alpha_r \in \Phi^-$ by definition of I, and so we must have $(\tilde\alpha, \alpha_r) > 0$. Therefore $\lambda_r = 0$ for every $r \in R - I$ and thus $\alpha \in \Phi_I^+$. This proves (ii). $\square$

For the rest of this subsection let $t \in \mathrm{Ref}(W)$, and further assume that there exists $I \subseteq R$ such that t is the longest reflection in W_I. (This assumption certainly holds for any reflection in $W \cong A_n$.) By Lemma 4.2 there exists $J \subseteq I$ such that $\Phi_I^+ = N(t) \dot\cup \Phi_J^+$ and J is given by $J = \{r \in I \mid t \cdot \alpha_r = \alpha_r\}$. We fix t, I and J for the rest of the subsection, and write D_J^I for the set of minimal coset representatives of W_J in W_I.

Lemma 4.3 *Let $g \in W$. Then*

(i) there exist unique $u \in W_J$, $v \in D_I^J$ and $w \in D_I$ such that $g = uvw$ and $l(g) = l(u) + l(v) + l(w)$; and

(ii) $N(Xg) = N(w) \dot\cup w^{-1}N(v) \dot\cup (vw)^{-1}N(u) \dot\cup (uvw)^{-1}[N(t) \backslash uN(v^{-1})]$ and

$$l(Xg) = l(X) + l(u) + l(w).$$

Proof. Part (i) is clear. For part (ii) Lemma 3.5 gives

$$N(Xg) = N(w) \dot\cup w^{-1}N(Xuv).$$

Now we invoke Proposition 1.9(ii) to get

$$\begin{aligned} N(Xuv) &= N(uv) \cup N(tuv) \\ &= N(uv) \cup \left[N(uv) \backslash -(uv)^{-1}N(t)\right] \dot\cup (uv)^{-1} \left[N(t) \backslash N((uv)^{-1})\right] \\ &= N(uv) \dot\cup (uv)^{-1} \left(N(t) \backslash N((uv)^{-1})\right). \qquad (1) \end{aligned}$$

It is easy to check that $N((uv)^{-1}) = N(u^{-1})\dot{\cup} uN(v^{-1})$. We claim that $uN(v^{-1}) \subseteq \Phi_I^+ \backslash \Phi_J^+ = N(t)$. For, suppose that $\alpha \in N(v^{-1})$ with $u \cdot \alpha \in \Phi_J^+$. Then there exists $u' \in W_J$ with $u'(u \cdot \alpha) \in \Phi^-$. But this can only happen if $\alpha \in \Phi_J^+$, which contradicts the fact that $v \in D_J^I$. So the claim is true. Therefore

$$N(t)\backslash N((uv)^{-1}) = N(t)\backslash (N(u^{-1})\dot{\cup} uN(v^{-1})) = N(t)\backslash uN(v^{-1}),$$

and $|N(t)\backslash N((uv)^{-1})| = l(t) - l(v)$. Using (1) we find

$$\begin{aligned} N(Xg) &= N(w)\dot{\cup} w^{-1}\left(N(v)\dot{\cup} v^{-1}N(u)\right)\dot{\cup}(uvw)^{-1}\left(N(t)\backslash uN(v^{-1})\right) \\ &= N(w)\dot{\cup} w^{-1}N(v)\dot{\cup}(vw)^{-1}N(u)\dot{\cup}(uvw)^{-1}[N(t)\backslash uN(v^{-1})] \end{aligned}$$

and $l(Xg) = l(X) + l(u) + l(w)$, completing the proof of Lemma 4.3 □

We now restrict ourselves to the case when W is of type A_n. So $W \cong \mathrm{Sym}(n+1)$. Set $R = \{(12), \ldots, (n\, n{+}1)\}$. Then $t = (ij)$ for some $1 \leq i < j \leq n+1$, giving $I = \{(i\, i{+}1), (i{+}1\, i{+}2), \ldots, (j-1\, j)\}$ and $J = \{(i+1\; i+2), \ldots, (j-2\; j-1)\}$. Set $w_i = (i\; i+1)$ for $1 \leq i \leq n$. It is easily seen that elements of D_J^I are of the form xy where x is a minimal length coset representative of W_J in $W_{J\cup\{w_i\}}$ and y is a minimal length coset representative of $W_{J\cup\{w_{j-1}\}}$ in W_I. Moreover, there exist k, j with $i-1 \leq k \leq j-2$, $i \leq l \leq j$ such that $x = w_i w_{i+1} \cdots w_k$ and $y = w_{j-1}w_{j-2}\cdots w_l$.

Proposition 4.4 *Suppose that W is of type A_n. Let $s \in \mathrm{Ref}(W)$, $u_1, u_2 \in W_J$, $v_1, v_2 \in D_J^I$ and $z_1, z_2 \in D_I$. Then $Xu_1v_1z_1 \approx Xu_1v_1z_1s = Xu_2v_2z_2$ if and only if $z_1 = z_2$, $l(u_1) = l(u_2)$ and $u_2^{-1}u_1$ is a subinterval of some $v \in D_J^I = D_J$.*

Proof. By Proposition 3.6, $Xu_1v_1z_1 \approx Xu_2v_2z_2$ if and only if $z_1 = z_2$ and $Xu_1v_1 \approx_I Xu_2v_2$. So it suffices to prove the result in the case $W_I = W$ (so we must show that $Xu_1v_1 \approx Xu_1v_1s = Xu_2v_2$ if and only if $l(u_1) = l(u_2)$ and $u_2^{-1}u_1$ is a subinterval of some $v \in D_J^I = D_J$). Suppose first that $Xu_1v_1 \approx Xu_1v_1s = Xu_2v_2$. Then, without loss of generality, $u_1v_1 = u_2v_2s$ and $l(u_1v_1) < l(u_2v_2)$. Certainly $l(u_1) = l(u_2)$ because, using Lemma 2.2, $l(X) + l(u_1) = l(Xu_1v_1) = l(Xu_2v_2) = l(X) + l(u_2)$. Now u_2v_2s is obtained from u_2v_2 by removing one element from a reduced expression for u_2v_2 (and then cancelling using the deletion condition if necessary). Suppose for a contradiction that the element is removed from u_2 so that $u_1v_1 = \hat{u}_2v_2$. Then, because minimal coset representatives are unique and $\hat{u}_2 \in W_J$, it follows that $u_1 = \hat{u}_2$ and $v_1 = v_2$. Therefore $l(X) + l(\hat{u}_2) = l(Xu_1v_1)$. But by assumption $l(Xu_1v_1) = l(Xu_2v_2) = l(X) + l(u_2)$, implying that $l(\hat{u}_2) = l(u_2)$, which is impossible. Therefore $u_1v_1 = u_2v_2s$ is obtained from u_2v_2 by removing an element from v_2, so that $u_1v_1 = u_2\hat{v}_2$. Let $u_3 = u_2^{-1}u_1$. Then $u_3v_1 = \hat{v}_2$. Now $v_2 = xy$ with $x = w_iw_{i+1}\cdots w_k$ and $y = w_{j-1}w_{j-2}\cdots w_l$ as described above, so $\hat{v}_2$ is either $\hat{x}y$ or $x\hat{y}$. Suppose that $\hat{v}_2 = \hat{x}y$. Then there exists m such that $u_3v_1 = \hat{v}_2 = w_i\cdots w_{m-1}w_{m+1}\cdots w_ky = (w_{m+1}\cdots w_k)w_i\cdots w_{m-1}y$. Observe that $w_{m+1}\cdots w_k \in W_J$ and $w_i\cdots w_{m-1}y \in D_J^I$, so that $u_3 = w_{m+1}\cdots w_k$ (and $v_1 = w_i \cdots w_{m-1}y$). Therefore $u_3 = u_2^{-1}u_1$ is a subinterval of some $v \in D_J^I$ (namely v_2). If $\hat{v}_2 = x\hat{y}$, then a similar calculation shows that again u_3 is a subinterval of $v_2 \in D_J^I$.

It remains to show the reverse implication. So we assume that $u_3 = u_2^{-1}u_1$ is a subinterval of some $v \in D_J^I$ and that $l(u_1) = l(u_2)$. Write $v = xy$ as before. If x is non-trivial it begins with w_i, and if y is non-trivial it begins with w_{j-1}, neither of which can appear in u_3. Thus u_3 must be a proper subinterval either of some $x \neq 1$ or some $y \neq 1$. Suppose the former. Now, since every right truncation of minimal left coset representative is still a minimal left coset representative, we may write $x = w_i \cdots w_k u_3$ where $k \geq i$. Set $g = w_i \cdots w_{k-1}$ and $r = w_m$. Note that $gu_3 = u_3 g$. If u_3 is a proper subinterval of some $y \neq 1$, then again we may write $y = w_{j-1} \cdots w_l u_3$ where $l \leq j-1$. This time set $g = w_{j-1} \cdots w_{l+1}$, $r = w_l$, and again note that g commutes with u_3. Now we let $v_1 = g$, $v_2 = gru_3$ and $s = u_3^{-1} r u_3$. Since g is a (right) truncation of v, $g \in D_J^I$ and also, by construction, $gru_3 \in D_J^I$. We have $u_1 v_1 s = u_1 g u_3^{-1} r u_3 = u_1 u_3^{-1} g r u_3 = u_2 v_2$ and $l(Xu_1v_1) = l(X) + l(u_1) = l(X) + l(u_2) = l(Xu_2v_2)$. Hence $Xu_1v_1 \approx Xu_2v_2$, and we have proved Proposition 4.4. $\square$

We may now give the

Proof of Theorem 1.5 Since t is the longest reflection, $I = R$ and each coset may be written Xuv where $u \in W_J$, $v \in D_J = D_J^I$. Suppose then that $Xuv \prec Xuvs = Xu'v'$ for some $s \in \mathrm{Ref}(W)$. Then without loss of generality $uv < u'v'$. So either $uv = \hat{u}'v'$ or $uv = u'\hat{v}'$. Note also that $l(u) = l(Xuv) - l(X) < l(Xu'v') - l(X) = l(u')$. Now if $uv = \hat{u}'v'$, then $u = \hat{u}'$ and $v = v'$. Therefore $u < u'$ in the usual Bruhat order, and so there exist $s_1, \ldots, s_d \in \mathrm{Ref}(W)$ with $l(us_1 \cdots s_i) = l(us_1 \cdots s_{i-1}) + 1$ for each $1 \leq i \leq d$ and $us_1 \cdots s_d = u'$. Moreover,

$$Xuv \approx Xu \prec Xus_1 \prec \cdots \prec Xu' \approx Xu'v'$$

with $l(Xus_1 \cdots s_i) - l(Xus_1 \cdots s_{i-1}) = 1$ for $1 \leq i \leq d$. Hence Theorem 1.5 holds in the case $uv = \hat{u}'v'$. Assume now that $uv = u'\hat{v}'$. Let $v' = xy$ where $x = w_i w_{i+1} \cdots w_k$ and $y = w_{j-1} w_{j-2} \cdots w_l$ as before. If $\hat{v}' = \hat{x}y$, then there is an m such that

$$v' = w_i \cdots w_{m-1} w_{m+1} \cdots w_k y = (w_{m+1} \cdots w_k)(w_i \cdots w_{m-1} y).$$

If $\hat{v}' = x\hat{y}$, then suppose w_m is removed from the expression for y. Simple calculations show that

$$\hat{v}' = \begin{cases} w_{m-1} \cdots w_l (w_i w_{i+1} \cdots w_k w_j w_{j-1} w_{m+1}) & \text{if } k+1 < l \\ w_{m-1} \cdots w_{l+1} (w_i w_{i+1} \cdots w_{k+1} w_j w_{j-1} w_{m+1}) & \text{if } l \leq k+1 < m \\ w_m \cdots w_{l+1} (w_i w_{i+1} \cdots w_k w_j w_{j-1} w_{m+1}) & \text{if } m \leq k+1. \end{cases}$$

In each case we may write $\hat{v}' = hv^*$ where $h \in W_J$, $v^* \in D_J^I$ and

$$h \in \{w_{m+1} \cdots w_k, w_{m-1} \cdots w_l, w_{m-1} \cdots w_{l+1}, w_m \cdots w_{l+1}\}.$$

Note that each possibility for h is a subinterval of some element of D_J^I, and each subinterval of h also has this property. We have $uv = u'\hat{v}' = (u'h)v^*$ and so $v = v^*$ and $u = u'h$. Write $h = r_1 \cdots r_M$ where $r_1, \cdots, r_M \in R$. Now set $u_0 = u'$, $u_1 = u'r_1$, $\ldots$, $u_M = u'r_1 \cdots r_M = u$. Also define $\lambda_0 = 0$ and λ_k to be the least λ such that $\lambda > \lambda_{k-1}$ and $l(u_\lambda) \leq \min\{l(u_{\lambda_l}) : 0 \leq l \leq k\}$. Because $l(u) < l(u')$ we know that $u = u_M = u_{\lambda_k}$ for some k. Now for each k either

(*i*) $l(u_{\lambda_k}) = l(u_{\lambda_{k-1}}) - 1$ (and $\lambda_k = \lambda_{k-1} + 1$) with $u_{\lambda_k} = u_{\lambda_{k-1}} r_{\lambda_k}$. Thus $Xu_{\lambda_k} \prec Xu_{\lambda_{k-1}}$ and $l(Xu_{\lambda_k}) = l(Xu_{\lambda_{k-1}}) - 1$; or *(ii)* $l(u_{\lambda_k}) = l(u_{\lambda_{k-1}})$. In this case $u_{\lambda_k} = u_{\lambda_{k-1}}(r_{\lambda_{k-1}+1}\cdots r_{\lambda_k})$ where $r_{\lambda_{k-1}+1}\cdots r_{\lambda_k}$ is a subinterval of some element of D_J^I. Therefore, by Proposition 4.4, $Xu_{\lambda_k} \approx Xu_{\lambda_{k-1}}$.

So at each stage the length decreases by at most 1. This completes the proof of Theorem 1.5. □

4.2. Two weak X-posets

Here we describe two examples by giving their Hasse diagrams. The numbers given on the right are the lengths of the elements $\mathbf{x}$ in $\mathfrak{X}_w$ on that level. By definition $l(\mathbf{x})$ is $l(Xg)$ for any Xg in $\mathbf{x}$.

For $W \cong \mathrm{Sym}(4)$ (so of type A_3) and $X = \langle(24)\rangle$, $\mathfrak{X}_w$ is given in Figure 1.

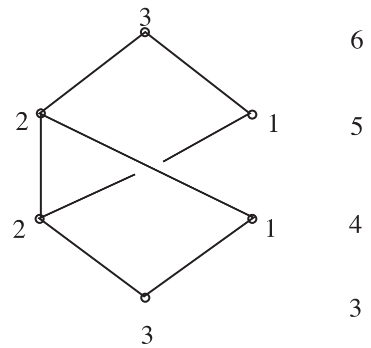

Figure 1. $W \cong \mathrm{Sym}(4), X = \langle(24)\rangle$.

The numbers next to a node indicate the number of right cosets of X in that particular element of $\mathfrak{X}_w$.

For $W \cong \mathrm{Sym}(5)$ and $X = \langle(13)(45)\rangle$ we have $|\mathfrak{X}_w| = 25$. Its Hasse diagram is displayed in Figure 2. The white circled elements consist of four right X-cosets, the squared ones of six right X-cosets while the remaining elements of $\mathfrak{X}_w$ consist of two right X-cosets. For comparison, see [7] for the Hasse diagram of the X-poset $\mathfrak{X}$ which has $|\mathfrak{X}| = 10$.

4.3. Three examples of X-posets

In these three examples we take $W \cong \mathrm{Sym}(7)$ and X to be a subgroup of order four generated by two reflections. So there are 1260 right X-cosets. As we shall see there is quite a diversity of behaviour in relation to the number of X-cosets in elements of $\mathfrak{X}$ and the length distribution of the elements of $\mathfrak{X}$.

(i) $X = \langle(14),(57)\rangle$. Then $|\mathfrak{X}| = 70$ with length distribution

$$8^1 \cdot 9^2 \cdot 10^3 \cdot 11^5 \cdot 12^7 \cdot 13^8 \cdot 14^9 \cdot 15^9 \cdot 16^8 \cdot 17^7 \cdot 18^5 \cdot 19^3 \cdot 20^2 \cdot 21^1.$$

Each element of $\mathfrak{X}$ consists of 18 X-cosets.

(ii) $X = \langle(15),(23)\rangle$. Then $|\mathfrak{X}| = 126$ with length distribution

$$8^1 \cdot 9^3 \cdot 10^6 \cdot 11^9 \cdot 12^{12} \cdot 13^{15} \cdot 14^{17} \cdot 15^{17} \cdot 16^{15} \cdot 17^{12} \cdot 18^9 \cdot 19^6 \cdot 20^3 \cdot 21^1.$$

Each element of $\mathfrak{X}$ consists of 10 X-cosets.

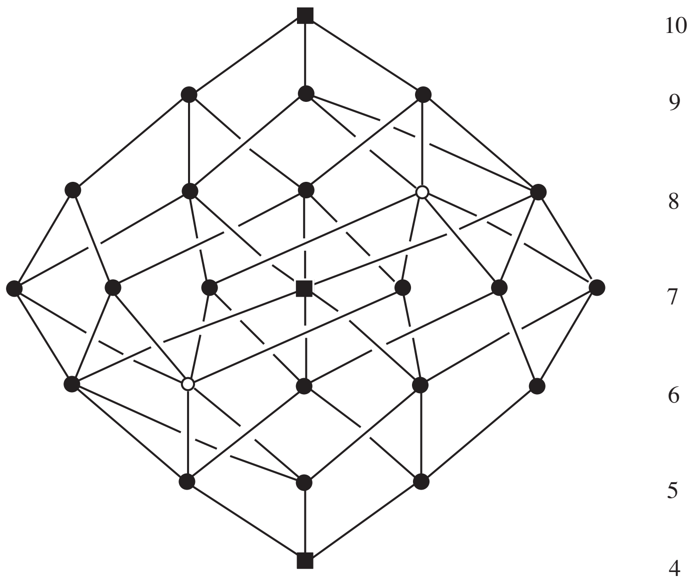

Figure 2. $W \cong \mathrm{Sym}(5), X = \langle (13)(45) \rangle$.

(iii) $X = \langle (15), (47) \rangle$. Then $|\mathfrak{X}| = 10$.

Length of $\mathbf{x}$	12	13	14	15	16	17	18	19	20	21
Number of X-cosets in $\mathbf{x}$	31	81	131	176	211	211	176	131	81	31

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Sarah B. HART
Department of Economics,
Mathematics & Statistics
Birkbeck, University of London, Malet Street
London WC1E 7HX, UNITED KINGDOM
e-mail: s.hart@bbk.ac.uk

Peter J. ROWLEY
School of Mathematics, University of Manchester
Oxford Road, Manchester M13 9PL,
UNITED KINGDOM
e-mail: peter.j.rowley@manchester.ac.uk