Geodesicity and isoclinity properties for the tangent bundle of the Heisenberg manifold with Sasaki metric

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Abstract

We prove that the horizontal and vertical distributions of the tangent bundle with the Sasaki metric are isocline, the distributions given by the kernels of the horizontal and vertical lifts of the contact form ω on the Heisenberg manifold (H_3, g) to (TH_3, g^S) are not totally geodesic, and the distributions $F^H = L(E_1^H, E_2^H)$ and $F^V = L(E_1^V, E_2^V)$ are totally geodesic, but they are not isocline. We obtain that the horizontal and natural lifts of the curves from the Heisenberg manifold (H_3, g), are geodesics on the tangent bundle endowed with the Sasaki metric (TH_3, g^s), if and only if the curves considered on the base manifold are geodesics. Then, we get two particular examples of geodesics on (TH_3, g^s), which are not horizontal or natural lifts of geodesics from the base manifold (H_3, g).

Key Words: Tangent bundle, Sasaki metric, Heisenberg metric, geodesics

1. Introduction

The tangent bundle TM of a Riemannian manifold splits into the vertical and horizontal distributions, defined by the Levi Civita connection of the metric g from the base manifold (see [33]).

In the study of the differential geometry of the tangent bundle of a Riemannian manifold, one uses several (pseudo) Riemannian metrics, induced by the Riemannian metric from the base manifold, and constructed on the horizontal and vertical distributions.

Maybe the best known Riemannian metric on the tangent bundle is that introduced by Sasaki in 1958, in the paper [30]. The results from [13]–[15], concerning the natural lifts, allowed the extension of the Sasaki metric, to the metrics of natural diagonal lift type (see [24]) and general natural lifted metrics (see [23], [31]), leading to interesting geometric structures studied in the last years years (see [1], [19] – [25]), and to interesting relations with some problems in Lagrangian and Hamiltonian mechanics (see [2], [17], [18]).

The aim of the first section from this paper is to find some geometric properties of the horizontal and vertical distributions of the tangent bundle TM of a Riemannian manifold (M, g), endowed with the Sasaki metric g^s. More precisely, we shall study the property of the the two distributions of being isocline (and implicitly totally geodesic), with respect to the Sasaki metric.

The notion of *isocline distribution* was introduced by Lutz, which conducted in [16] a metric study of the contact structures, measuring, with the help of a metric g, the evolution of a contact structure F along the geodesics of g. One of the metric characters of a totally geodesic field F, considered by Lutz in [16], is the evolution of its angle along an arbitrary geodesic γ. When the angle between a totally geodesic distribution F and $\dot\gamma$ is constant along the geodesic γ, the totally geodesic distribution is called *isocline*.

The second author studied in her Ph.D. thesis [26], the property of being isocline for the contact structures on hyper-surfaces of $\mathbb{R}^{2n+2}$. In the third section of the present paper we shall prove that the horizontal and the vertical distribution of the tangent bundle of a Riemannian manifold are always isocline with respect to the Sasaki metric g^s, and we shall construct some examples of distributions on the tangent bundle of the Heisenberg manifold, which have or have not the properties of being totally geodesic or isocline. To this aim, we shall consider the horizontal and vertical lifts of the contact form ω from the Heisenberg manifold (H_3, g) to the tangent bundle (TH_3, g^s), and we shall prove that the distributions given by $F = Ker(\omega^H)$ (or by $F = Ker(\omega^V)$) are not totally geodesic, but the distributions $F^H = L(E_1^H, E_2^H)$ and $F^V = L(E_1^V, E_2^V)$ are totally geodesic and not isocline.

An important geometric problem is to find the geodesics on the smooth manifolds with respect to the Riemannian metrics (see [4]–[10], [22], [27]–[29], [33]). In [33], Yano and Ishihara proved that the curves on the tangent bundles of Riemannian manifolds are geodesics with respect to certain lifts of the metric from the base manifold, if and only if the curves are obtained as certain types of lifts of the geodesics from the base manifold. In two very recent papers, Salimov and his collaborators studied the analogous problem for the geodesics on the tangent bundles endowed with Cheeger-Gromoll metrics (see [29]), and on the tensor bundles with Sasakian metrics (see [28]) .

In the last section of the present paper, we are interested in finding some concrete examples of geodesics on the tangent bundle (TH_3, g^s), of the Heisenberg manifold (H_3, g) with respect to the Sasaki metric g^s.

We prove that if C is a curve in the Heisenberg manifold (H_3, g), then its horizontal and natural lifts, $\widetilde{C}$ and $\widehat{C}$, passing through the origin, such that $\dot{\widetilde{C}}(0) = \dot{\widehat{C}}(0) = (u, v, w, 0, 0, 0)$, are geodesics on (TH_3, g^s), if and only if the curve considered on the base manifold is a geodesic.

Working in a more general context, we look for some classes of geodesics on (TH_3, g^s), which are not obtained as horizontal or natural lifts of the geodesics from the base manifold.

2. Preliminary results.

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Just to fix the notation, recall some basic things about TM. It has a structure of $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from usual local charts on M. If $(U, \varphi) = (U, x^1, \ldots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)$ (see [33] for further details).

Denote by ∇ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM$$

of the tangent bundle to TM into the vertical distribution $VTM = \text{Ker } \tau_*$ and the horizontal distribution HTM defined by ∇.

The set set of vector fields $\left\{ \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j} \right\}_{i,j=\overline{1,n}}$ defines a local frame on TM, adapted to the direct sum decomposition (2). Notice that

$$\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i} \right)^V, \quad \frac{\delta}{\delta x^j} = \left(\frac{\partial}{\partial x^j} \right)^H = \frac{\partial}{\partial x^j} - \Gamma_j^h \frac{\partial}{\partial y^h}, \quad \Gamma_j^h = y^k \Gamma_{kj}^h,$$

where $X^V \in VTM$ and $X^H \in HTM$ denote the vertical and horizontal lift of the vector field X on M, respectively, and $\Gamma_{kj}^h(x)$ are the Christoffel symbols of g.

The Sasaki metric g^s on the tangent bundle TM is defined by the relations

$$\begin{cases} g^S(X^H, Y^H) = g^S(X^V, Y^V) = g(X, Y) \circ \tau \\ g^S(X^H, Y^V) = g^S(X^V, Y^H) = 0 \end{cases} \quad \forall X, Y \in \mathcal{T}_0^1(M).$$

If the metric g from the base manifold M has the components g_{ij} in a coordinate neighborhood, then the Sasaki metric g^s on the tangent bundle may be defined as the Riemannian metric which has the expression

$$g^s = g_{ij} dx^i dx^j + g_{ij} Dy^i Dy^j, \quad \forall i, j = \overline{1, n}$$

where $\left\{ Dy^i, dx^j \right\}_{i,j=\overline{1,n}}$ is the dual frame of $\left\{ \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j} \right\}_{i,j=\overline{1,n}}$. The covariant derivative of y^i with respect to the Levi-Civita connection of the metric g is given by

$$Dy^i = dy^i + \Gamma_j^i dx^j, \qquad \Gamma_j^i = \Gamma_{hj}^i y^h.$$

In particular, if the base manifold is the Heisenberg manifold (H_3, g), where

$$g = (dx^1)^2 + (dx^2)^2 + (dx^3 + x^2 dx^1 - x^1 dx^2)^2, \tag{1}$$

then the Sasaki metric on TH_3 is

$$g^s = (dx^1)^2 + (dx^2)^2 + (dx^3 + x^2 dx^1 - x^1 dx^2)^2 \tag{2}$$
$$+ (Dy^1)^2 + (Dy^2)^2 + (Dy^3 + x^2 Dy^1 - x^1 Dy^2)^2.$$

3. Totally geodesic and isocline distributions on the tangent bundle of a Riemannian manifold

It is well known that the horizontal distribution HTM of the tangent bundle of an n-dimensional Riemannian manifold (M, g), is integrable if and only if the manifold M is flat. In this section we shall prove that the horizontal and the vertical distributions of TM are always isocline with respect to the Sasaki metric g^s.

A regular distribution F defined on a connected Riemannian manifold (M, g) is called totally geodesic if every geodesic tangent to F in one point is tangent to the distribution in all the points.

A distribution F is totally geodesic ([32]) if and only if the distribution D normal to F is Riemannian (i.e. if $\nabla_X Y + \nabla_Y X \in C^\infty(F)$ for every $X, Y \in C^\infty(F)$).

A totally geodesic distribution forms a constant angle with the integral geodesic curve. More over, if the field of planes makes a constant angle with the tangent vector field $\dot{\gamma}$ along an arbitrary geodesic curve γ, we say that the structure is isocline.

Let F be a totally geodesic distribution and N a unitary vector field, normal to F.

Definition 3.1 *The totally geodesic contact structure F is called isocline if for every geodesic curve parameterized by the arc length, the angle between F and the tangent vector field $\dot{\gamma}(s)$ is constant along the geodesic.*

Proposition 3.2 *[16] If ∇ is the Levi-Civita connection associated to the metric g on M, a totally geodesic distribution F is isocline if and only if for every vector field N normal to F, the vector field $\nabla_N N$ is normal to F.*

If $\{X_i, N_\alpha\}$, $i, = \overline{1, p}, \alpha = \overline{1, q}, p + q = n$, is an orthonormal frame of (M, g) adapted to the distribution F ($X_i \in C^\infty(F)$ and $N_\alpha \in C^\infty(F^\perp)$) then F is isocline if and only if

$$g\left(\nabla_{X_i} X_j + \nabla_{X_j} X_i, N_\alpha\right) = 0 \qquad \text{geodesicity} \tag{3}$$

$$g\left(\nabla_{N_\alpha} N_\beta + \nabla_{N_\beta} N_\alpha, X_i\right) = 0 \tag{4}$$

where $i, j = \overline{1, p}; \alpha, \beta = \overline{1, q}$.

Proposition 3.3 *The distributions HTM and VTM are isocline.*

Proof.   Let us consider the connection $\bar{\nabla}$ of (TM, g^s) and the Levi-Civita connection ∇ of (M, g), which are related by the formulas (see [3])

$$\begin{cases} (\bar{\nabla}_{X^H} Y^H)_Z = (\nabla_X Y)_Z^H - \frac{1}{2}(R_{XY} Z)^V \\ (\bar{\nabla}_{X^H} Y^V)_Z = (\nabla_X Y)_Z^V - \frac{1}{2}(R_{YZ} X)^H \\ (\bar{\nabla}_{X^V} Y^H)_Z = -\frac{1}{2}(R_{XY} Z)^V \\ (\bar{\nabla}_{X^V} Y^V) = 0, \end{cases}$$

where X, Y are tangent to M.

Now, we may easily verify the conditions (3) and (4) for HTM and VTM to be totally geodesic and isocline:

$$g^s(\bar{\nabla}_{X^H} Y^H + \bar{\nabla}_{Y^H} X^H, X^V) = 0$$

$$g^s(\bar{\nabla}_{X^V} Y^V + \bar{\nabla}_{Y^V} X^V, Y^H) = 0.$$

$\square$

In the sequel, we shall focus our attention on the geometry of the Heisenberg manifold, usually known as Heisenberg group H_3. A first remark is that its contact distribution furnishes an example of totally geodesic and isocline distribution (see [12], [16]).

We are interested in finding examples of distributions on the tangent bundle of the Heisenberg manifold, TH_3, endowed with the Sasaki metric expressed by (2), which are isocline, or which are only totally geodesic, without being isocline.

The metric g on H_3, given by (1), is invariant with respect to the left translations and with respect to the rotations around the z axis. We shall use the invariant orthonormal coframe

$$\theta^1 = dx^1, \quad \theta^2 = dx^2, \quad \theta^3 = dx^3 + x^2 dx^1 - x^1 dx^2$$

and the dual basis

$$E_1 = \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^3}, \quad E_2 = \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3}, \quad E_3 = \frac{\partial}{\partial x^3}.$$

The Levi-Civita connection of the metric g is given by

$$\begin{cases} \nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = E_3, \quad \nabla_{E_1} E_3 = -E_2 \\[2mm] \nabla_{E_2} E_1 = -E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = E_1 \\[2mm] \nabla_{E_3} E_1 = -E_2, \quad \nabla_{E_3} E_2 = E_1, \quad \nabla_{E_3} E_3 = 0. \end{cases}$$

The non vanishing components of the curvature tensor field

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and of the Riemann-Christoffel curvature

$$R(X,Y,Z,W) = g(R(X,Y)W,Z)$$

are, respectively,

$$\begin{cases} R^2_{112} = 3, \quad R^3_{113} = -1, \quad R^1_{212} = -3 \\[2mm] R^1_{313} = 1, \quad R^3_{223} = -1, \quad R^2_{323} = 1, \end{cases}$$

and

$$R_{1212} = -3, \quad R_{1313} = R_{2323} = 1,$$

where we used the notations

$$R(E_a, E_b)E_c = R^i_{cab} E_i, \quad R(E_a, E_b, E_c, E_d) = R_{abcd}.$$

We may ask what happens with the distributions determined by the kernels of the horizontal and vertical lift of the contact form $\omega = dx^3 + x^2 dx^1 - x^1 dx^2$. In this sense, we may prove the following proposition.

Proposition 3.4 *If ω^H (ω^V) is the horizontal (vertical) lift of the contact form ω from the Heisenberg manifold H_3, then the distribution F of codimension 1, defined by $F = Ker(\omega^H)$ ($F = Ker(\omega^V)$) is not totally geodesic.*

Proof. If the distribution F is given by $Ker(\omega^H)$, then one can choose a basis given by the vector fields $\{E_1^H, E_2^H, E_1^V, E_2^V, E_3^V\}$, and we may easily verify that

$$\begin{aligned} g^s_{(p,y)} \left(\widetilde{\nabla}_{E_1^H} E_1^V + \widetilde{\nabla}_{E_1^V} E_1^H, E_3^H \right) &= -\frac{1}{2} g^s_{(p,y)} \left((R(E_1, y)E_1)^H, E_3^H \right) \\ &= -\frac{1}{2} g_{\tau(y)} (R(E_1, y)E_1, E_3) \neq 0, \end{aligned}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of the Sasaki metric on TH_3, and y represents a tangent vector from TH_3.

Analogously, if the distribution is defined by $F = Ker(\omega^V)$, then a basis for F is given by the vector fields $\{E_1^H, E_2^H, E_3^H, E_1^V, E_2^V\}$, and it may be verified that

$$g_{(p,y)}^s\left(\widetilde{\nabla}_{E_1^H} E_2^V + \widetilde{\nabla}_{E_2^V} E_1^H, E_3^V\right) = g_{(p,y)}^s\left((\nabla_{E_1} E_2)^V, E_3^V\right) -$$

$$-\frac{1}{2} g_{(p,y)}^s\left((R(E_2, y)E_1)^H - (R(E_2, E_1)y)^V, E_3^V\right)$$

$$= g_{\tau(y)}(\nabla_{E_1} E_2, E_3) = 1 \neq 0.$$

Thus the proposition is proved. □

Now we give an example of totally geodesic distributions on TH_3, which are not isocline.

Proposition 3.5 *The distributions $F^H = L(E_1^H, E_2^H)$ and $F^V = L(E_1^V, E_2^V)$ are totally geodesic and they are not isocline.*

Proof.   Since the Levi-Civita connection of the Sasaki metric g^s from the tangent bundle of a Riemannian manifold (M, g) has the expressions (5), and the Levi-Civita connection from (H_3, g) satisfies the relation (3), we may easily prove that the Levi-Civita connection $\widetilde{\nabla}$ from TH_3 verify the relations

$$\widetilde{\nabla}_{E_1^H} E_2^H + \widetilde{\nabla}_{E_2^H} E_1^H = 0$$

$$\widetilde{\nabla}_{E_1^V} E_2^V + \widetilde{\nabla}_{E_2^V} E_1^V = 0$$

and thus both distributions F^H, F^V are totally geodesic.

We may easily prove that $\widetilde{\nabla}$ fulfills also the relations

$$g_{(p,y)}^s\left(\widetilde{\nabla}_{E_3^H} E_1^V + \widetilde{\nabla}_{E_1^V} E_3^H, E_1^H\right) = -\frac{1}{2} g_{(p,y)}^s\left(\left(R(E_1, y)E_3\right)^H, E_1^H\right) \neq 0$$

$$g_{(p,y)}^s\left(\widetilde{\nabla}_{E_3^V} E_1^H + \widetilde{\nabla}_{E_1^H} E_3^V, E_1^V\right) = -\frac{1}{2} g_{(p,y)}^s\left(\left(R(E_3, E_1)y\right)^V, E_1^V\right) \neq 0$$

which prove that the distributions F^H and F^V are not isocline. □

4.   Geodesics in the tangent bundle $(\mathbf{TH_3}, \mathbf{g^s})$.

Let M be an n-dimensional Riemannian manifold and $C : I \to M$ a curve parametrized on it, expressed locally by

$$C(t) = \{x^1(t), \cdots, x^n(t)\},$$

and let X be a vector field along a curve C. Then, in the tangent bundle TM, a curve $\widetilde{C}$ may be defined by

$$\widetilde{C}(t) = \{x^1(t), \cdots, x^n(t), X^1(t), \cdots, X^n(t)\}.$$

where $X^j(t)$ denotes the components of X in a natural basis. The curve $\widetilde{C}$ is called *horizontal lift* of the curve C in M, if X is parallel along C. When X is the vector field $\frac{dC}{dt}$ (tangent to C), the curve $\widetilde{C}$ in the tangent bundle is called *the natural lift* of C.

Let us consider a curve C in H_3 expressed locally by $x^h = x^h(t)$ and $Y = y^j(t)\frac{\partial}{\partial x^j}$ a vector field along C. Then, in the tangent bundle TH_3, we define a curve $\widetilde{C}$ by

$$x^h = x^h(t), \quad y^h = y^h(t), \qquad h = 1, 2, 3.$$

A curve $\gamma(t) = (x^1(t), x^2(t), x^3(t), y^1(t), y^2(t), y^3(t))$ on (TH_3, g^s) is a geodesic if and only if

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = 0,$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of the Sasaki metric on TH_3, and

$$\dot\gamma = \sum_{i=1}^{3} \frac{dx^i}{dt}\frac{\delta}{\delta x^i} + \frac{Dy^i}{dt}\frac{\partial}{\partial y^i}.$$

Combining the relations above, taking into account the expressions of Levi-Civita connection for the Sasaki metric, and then identifying the horizontal and vertical components, we obtain that γ is a geodesic on (TH_3, g^s) if and only if

$$\begin{cases} \frac{D^2 x^h}{dt^2} + R^h_{kji} y^k \frac{Dy^j}{dt}\frac{dx^i}{dt} = 0 \\[2mm] \frac{D^2 y^h}{dt^2} = 0, \end{cases} \qquad k, i, j, h = 1, 2, 3$$

where R^h_{kji} is the curvature of the Heisenberg manifold. We have denoted

$$\frac{Dy^i}{dt} = \frac{dy^i}{dt} + \Gamma^i_{kj} y^k \frac{dx^j}{dt}, \qquad i, j, k = 1, 2, 3.$$

The Lagrangian of the Sasaki metric g^s given by (2) has the expression

$$L = \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt} + x^2\frac{dx^1}{dt} - x^1\frac{dx^2}{dt}\right)^2$$
$$+ \left(\frac{Dy^1}{dt}\right)^2 + \left(\frac{Dy^2}{dt}\right)^2 + \left(\frac{Dy^3}{dt} + x^2\frac{Dy^1}{dt} - x^1\frac{Dy^2}{dt}\right)^2$$

and the corresponding Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot x^i}\right) = \frac{\delta L}{\delta x^i}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot y^i}\right) = \frac{\partial L}{\partial y^i}$$

where we used the notations $\dot x^i = \frac{dx^i}{dt}$ and $\dot y^i = \frac{Dy^i}{dt}$, are

$$\frac{d}{dt}\left[\frac{dx^1}{dt} + x^2\left(\frac{dx^3}{dt} + x^2\frac{dx^1}{dt} - x^1\frac{dx^2}{dt}\right)\right] = -\frac{dx^2}{dt}\left(\frac{dx^3}{dt} + x^2\frac{dx^1}{dt} - x^1\frac{dx^2}{dt}\right) \qquad (5)$$

$$\frac{d}{dt}\Big[\frac{dx^2}{dt} - x^1\big(\frac{dx^3}{dt} + x^2\frac{dx^1}{dt} - x^1\frac{dx^2}{dt}\big)\Big] = \frac{dx^1}{dt}\Big(\frac{dx^3}{dt} + x^2\frac{dx^1}{dt} - x^1\frac{dx^2}{dt}\Big) \tag{6}$$

$$\frac{d}{dt}\Big(\frac{dx^3}{dt} + x^2\frac{dx^1}{dt} - x^1\frac{dx^2}{dt}\Big) = 0 \tag{7}$$

$$\frac{d}{dt}\Big[\frac{Dy^1}{dt} + x^2\Big(\frac{Dy^3}{dt} + x^2\frac{Dy^1}{dt} - x^1\frac{Dy^2}{dt}\Big)\Big] = 0 \tag{8}$$

$$\frac{d}{dt}\Big[\frac{Dy^2}{dt} - x^1\Big(\frac{Dy^3}{dt} + x^2\frac{Dy^1}{dt} - x^1\frac{Dy^2}{dt}\Big)\Big] = 0 \tag{9}$$

$$\frac{d}{dt}\Big(\frac{Dy^3}{dt} + x^2\frac{Dy^1}{dt} - x^1\frac{Dy^2}{dt}\Big) = 0. \tag{10}$$

For a geodesic $\gamma : I \to TH_3$, $\gamma(t) = (x^1(t), x^2(t), x^3(t), y^1(t), y^2(t), y^3(t))$ which at the instant zero passes through the origin, with the velocity $\dot{\gamma}(0) = (u, v, w, l, m, n)$, the Euler-Lagrange equations become

$$\frac{dx^3}{dt} + x^2\frac{dx^1}{dt} - x^1\frac{dx^2}{dt} = w \tag{11}$$

$$\frac{d}{dt}\Big(\frac{dx^1}{dt} + x^2 w\Big) = -\frac{dx^2}{dt}w \tag{12}$$

$$\frac{d}{dt}\Big(\frac{dx^2}{dt} - x^1 w\Big) = \frac{dx^1}{dt}w \tag{13}$$

$$\frac{Dy^3}{dt} + x^2\frac{Dy^1}{dt} - x^1\frac{Dy^2}{dt} = n \tag{14}$$

$$\frac{Dy^1}{dt} + x^2 n = l \tag{15}$$

$$\frac{Dy^2}{dt} - x^1 n = m. \tag{16}$$

Remark 4.1 *The first three Euler-Lagrange equations above are satisfied if and only if the curve $(x^1(t), x^2(t), x^3(t))$ is a geodesic on the base (H_3, g), which at the moment zero passes through the point $(0, 0, 0)$ with the velocity (u, v, w).*

Taking into account that $\dot{\gamma}(0) = (u, v, w, l, m, n)$, from the equation (13) we obtain that

$$\frac{dx^2}{dt} = 2x^1 w + v,$$

which substituted into (12) yields the equation

$$\frac{d^2 x^1}{dt^2} + 4x^1 w^2 = 0,$$

with the solution

$$x^1(t) = \frac{v}{2w}\cos(2wt) + \frac{u}{2w}\sin(2wt) - \frac{v}{2w}. \tag{17}$$

Analogously, from (12) and (13) we obtain

$$x^2(t) = -\frac{u}{2w}\cos(2wt) + \frac{v}{2w}\sin(2wt) + \frac{v}{2w}. \tag{18}$$

Replacing the solutions (17) and (18) into (11), we obtain that x^3 has the expression

$$x^3(t) = wt + \frac{u^2 + v^2}{2w}t - \frac{u^2 + v^2}{2w}\sin(2wt).$$

In the case when $w = 0$, the solutions of the system obtained from the equations (11), (12), and (13) have simpler expressions

$$x^1(t) = ut, \ x^2(t) = vt, \ x^3(t) = 0. \tag{19}$$

We may state now the following result.

Theorem 4.2 *The horizontal lift $\widetilde{C}$ and the natural lift $\widehat{C}$ of a curve C from the Heisenberg manifold H_3 are geodesic in the tangent bundle endowed with the Sasaki metric (TH_3, g^s), if and only if the curve C is a geodesic in (H_3, g), and $\widetilde{C}$, $\widehat{C}$ pass through the point $(0, 0, 0, 0, 0, 0)$, such that $\dot{\widetilde{C}}(0) = \dot{\widehat{C}}(0) = (u, v, w, 0, 0, 0)$.*

Proof. If the curve $\widetilde{C}(t) = (C(t), Y(t))$ is the horizontal lift to TH_3 of the curve $C(t) = (x^1(t), x^2(t), x^3(t))$ from H_3, then Y is a parallel vector field along C, i.e.

$$\frac{Dy^1}{dt} = \frac{Dy^2}{dt} = \frac{Dy^3}{dt} = 0,$$

and in this case the last three Euler-Lagrange equations, (14), (15), and (16) reduce to $l = m = n = 0$.

If the curve $\widehat{C}(t) = (C(t), Y(t))$ on TH_3 is the natural lift of the curve $C(t) = (x^1(t), x^2(t), x^3(t))$ from H_3, then Y is the tangent vector field to C, i.e.

$$y^h = \frac{dx^h}{dt}, \qquad h = 1, 2, 3$$

from which we obtain that the covariant derivative of Y has the expression

$$\frac{Dy^h}{dt} = \frac{d^2x^h}{dt^2} + \Gamma_{ij}^h \frac{dx^i}{dt}\frac{dx^j}{dt}, \quad \forall i, j, h = \overline{1, 3}. \tag{20}$$

Taking into account Remark 4.1, it follows that the curve $C(t) = (x^1(t), \ x^2(t), x^3(t))$ is a geodesic on the Heisenberg manifold (H_3, g), and then $\frac{Dy^h}{dt}$ expressed by (20) vanishes, and the equations (14), (15), (16) reduce again to $l = m = n = 0$. Thus the theorem is proved. $\square$

In a more general context, when we search some examples of geodesics on (TH_3, g^s), which are not horizontal or natural lifts of the curves from the base manifold (H_3, g), we obtain, by taking into account the expressions of the Christoffel symbols constructed with the Heisenberg metric, that the last three Euler-Lagrange

equations, (14), (15), (16) get the forms

$$\frac{dy^1}{dt} + x^2 y^2 \frac{dx^1}{dt} + (x^2 y^1 - 2x^1 y^2 + y^3)\frac{dx^2}{dt} + y^2 \frac{dx^3}{dt} = l - vnt$$

$$\frac{dy^2}{dt} + (-2x^2 y^1 + x^1 y^2 - y^3)\frac{dx^1}{dt} + x^1 y^1 \frac{dx^2}{dt} - y^1 \frac{dx^3}{dt} = m + unt$$

$$\frac{dy^3}{dt} + [-2x^1 x^2 y^1 + (1 + (x^1)^2 - (x^2)^2)y^2 - x^1 y^3]\frac{dx^1}{dt}$$

$$+ [(1 + (x^1)^2 - (x^2)^2)y^1 + 2x^1 x^2 y^2 - x^2 y^3]\frac{dx^2}{dt}$$

$$- (x^1 y^1 + x^2 y^2)\frac{dx^3}{dt} + x^2(l - x^2 n) - x^1(m + x^1 n) = n.$$

In the case when the curve on the base manifold is a geodesic given by (19), then the above equations become

$$\frac{dy^1}{dt} + tvy^2 u + (tvy^1 - 2tuy^2 + y^3)v = l - vnt$$

$$\frac{dy^2}{dt} + (-2tvy^1 + tuy^2 - y^3)u + tuy^1 v = m + unt$$

$$\frac{dy^3}{dt} + [-2t^2 uvy^1 + (1 + t^2 u^2 - t^2 v^2)y^2 - tuy^3]u$$

$$+ [(1 + t^2 u^2 - t^2 v^2)y^1 + 2t^2 uvy^2 - tvy^3]v$$

$$+ tv(l - vnt) - tu(m + unt) = n,$$

i.e. we have the system

$$(*)\begin{cases} \frac{dy_1}{dt} + tv^2 y_1 - tuvy_2 + vy_3 = l - vnt \\ \frac{dy_2}{dt} - tuvy_1 + tu^2 y_2 - uy_3 = m + unt \\ \frac{dy_3}{dt} + v(1 - t^2 u^2 - t^2 v^2)y_1 + u(1 + t^2 u^2 + t^2 v^2)y_2 - t(u^2 + v^2)y_3 + \\ \quad + tv(l - vnt) - tu(m + unt) = n. \end{cases}$$

Remark 4.3 If $u = v = 0$, we obtain the following particular solution of the above system:

$$y_1 = lt, \ y_2 = mt, \ y_3 = nt.$$

Taking this remark into account, we may prove the next theorem.

Theorem 4.4 *If the curve from the Heisenberg manifold reduces to the origin point $(0,0,0)$, then the geodesics from the tangent bundle with the Sasaki metric (TH_3, g^s) are the curves γ passing through the origin $(0,0,0,0,0,0)$ with velocity $\dot\gamma(0) = (0,0,0,l,m,n)$, namely $\gamma(t) = (0,0,0,lt,mt,nt)$.*

Yanno and Ishihara proved that if a geodesic lies in a fiber of the tangent bundle (TM, g^s) of an n-dimensional Riemannian manifold (M, g), given by $x^h = c^h, \forall h = \overline{1,n}$, where c^h is a real constant, then the geodesic is expressed by linear equations $x^h = c^h$, $y^h = a^h t + b^h$, with respect to the induced coordinates

$\{x^h, y^h\}_{h=\overline{1,n}}$, where a^h, b^h, c^h are constants. In the case of the tangent bundle (TH_3, g^s) this result reduces to Theorem 4.4, since when x^i are constants, the expressions (4) of $\frac{Dy^i}{dt}$ become $\frac{Dy^i}{dt} = \frac{dy^i}{dt}, i = \overline{1,3}$, and the Euler-Lagrange equations (14) - (16) with the initial conditions $\frac{dy^1}{dt}(0) = l, \frac{dy^2}{dt}(0) = m, \frac{dy^3}{dt}(0) = n$ lead to $x^i = 0, i = \overline{1,3}$.

Remark 4.5 *If* $m = n = 0$*, a particular solution of the system* $(*)$ *is of the form*

$$y_1 = lt, \ y_2 = 0, \ y_3 = -lvt^2.$$

Taking into account Remark 4.5 and the solution (19) of the system obtained from the equations (5), (6), (7), we may formulate our final theorem.

Theorem 4.6 *One of the geodesics from the tangent bundle with the Sasaki metric* (TH_3, g^s)*, is a curve* $\widetilde{\gamma} : I \to TH_3$*, which at the moment zero passes through the point* $\widetilde{\gamma}(0) = (0,0,0,0,0,0)$*, with the property* $\dot{\widetilde{\gamma}}(0) = (u, v, 0, l, 0, 0)$*, namely the curve* $\widetilde{\gamma}(t) = (ut, vt, 0, lt, 0, -lvt^2)$*.*

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