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Marcinkiewicz-Fejér means of double conjugate Walsh-Kaczmarz-Fourier series and Hardy spaces

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Abstract

In the present paper we prove that for any $0 < p \leq 2/3$ there exists a martingale f in H_p such that the Marcinkiewicz-Fejér means of double conjugate Walsh-Kaczmarz-Fourier series of the martingale f is not uniformly bounded in the space L_p.

Key Words: Walsh-Kaczmarz system, Fejér means, Marcinkiewicz means, Martingale-Hardy space

In 1939 for the two-dimensional trigonometric Fourier series Marcinkiewicz [6] has proved for $f \in L\log L([0,2\pi]^2)$ that the means

$$\mathcal{M}_n f = \frac{1}{n}\sum_{j=1}^{n-1} S_{j,j}(f)$$

converge a.e. to f as $n \to \infty$. Zhizhiashvili [16] improved this result for $f \in L([0,2\pi]^2)$.

For the two-dimensional Walsh-Fourier series Weisz [12] proved that the maximal operator $\mathcal{M}^{w,*}f = \sup_{n\geq 1}|\mathcal{M}_n^w(f)|$ is bounded from the two-dimensional dyadic martingale Hardy space H_p to the space L_p for $p > 2/3$ and is of weak type (1,1). The first author [5] proved that the assumption $p > 2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^{w,*}$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.

First, we give a brief introduction to the theory of dyadic analysis [8]. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0,1\}\,(k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G,$$

$$I_n(x) := I_n(x_0, ..., x_{n-1}) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1}, ...)\},$$

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$(x \in G, n \in \mathbf{N})$. These sets are called dyadic intervals.

Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, , $I_n := I_n(0)$ $(n \in \mathbf{N})$. Set $e_n := (0, ..., 0, 1, 0, ...) \in G$, the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$.

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the kth Rademacher function. If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0, 1\}$ $(i \in \mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as a sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-k-1}}.$$

For $A \in \mathbf{N}$ define the transformation $\tau_A : G \to G$ by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_0, x_A, x_{A+1}, ...).$$

By the definition of τ_A (see [11]), we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

The space $L_p(G^2), 0 < p \leq \infty$ with norms or quasi-norms $\|\cdot\|_p$ is defined in the usual way.

The Dirichlet kernels are defined by

$$D_n^{\alpha}(x) := \sum_{k=0}^{n-1} \alpha_k(x),$$

where $\alpha_k = w_k$ or κ_k. Recall that (see e.g. [8])

$$D_{2^n}(x) := D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \notin I_n(0). \end{cases} \tag{1}$$

The two-dimensional dyadic cubes are of the form

$$I_n(x, y) := I_n(x) \times I_n(y).$$

The σ-algebra generated by the dyadic cubes $\{I_n(x, y) : (x, y) \in G \times G\}$ is denoted by $\mathcal{F}_n$.

Denote by $f=\left(f^{(n)},n\in\mathbf{N}\right)$ a martingale with respect to $(\mathcal{F}_n,n\in\mathbf{N})$ (for details see, e.g. [14]). The maximal function of a martingale f is defined by

$$f^{*}=\sup_{n\in\mathbf{N}}\left|f^{(n)}\right|.$$

In case $f\in L_1\left(G^2\right)$, the maximal function can also be given by

$$f^{*}\left(x,y\right)=\sup_{n\in\mathbf{N}}\frac{1}{\mu\left(I_n(x,y)\right)}\left|\int\limits_{I_n(x,y)}f\left(u,v\right)d\mu\left(u,v\right)\right|,\quad (x,y)\in G\times G.$$

For $0<p<\infty$ the Hardy martingale space $H_p(G^2)$ consists of all martingales for which

$$\|f\|_{H_p}:=\|f^{*}\|_p<\infty.$$

The Kronecker product $(\alpha_{n,m}:n,m\in\mathbf{N})$ of two Walsh(-Kaczmarz) system is said to be the two-dimensional Walsh(-Kaczmarz) system. That is,

$$\alpha_{n,m}\left(x,y\right)=\alpha_n\left(x\right)\alpha_m\left(y\right).$$

If $f\in L_1\left(G^2\right)$, then the number $\widehat{f}^{\alpha}\left(n,m\right):=\int\limits_{G^2}f\alpha_{n,m}\quad(n,m\in\mathbf{N})$ is said to be the (n,m)th Walsh-(Kaczmarz)-Fourier coefficient of f. We can extend this definition to martingales in the usual way (see [13, 14]).

Denote by $S_{n,m}^{\alpha}$ the (n,m)th rectangular partial sum of the Walsh-(Kaczmarz)-Fourier series of a martingale f. Namely,

$$S_{n,m}^{\alpha}(f;x,y):=\sum_{k=0}^{n-1}\sum_{i=0}^{m-1}\widehat{f}^{\alpha}(k,i)\alpha_{k,i}(x,y).$$

The Marcinkiewicz-Fejér means of a martingale f are defined by

$$\mathcal{M}_n^{\alpha}\left(f;x,y\right):=\frac{1}{n}\sum_{k=0}^{n-1}S_{k,k}^{\alpha}(f;x,y).$$

The 2-dimensional Dirichlet kernels and Marcinkiewicz-Fejér kernels are defined by

$$D_{k,l}^{\alpha}(x,y):=D_k^{\alpha}(x)D_l^{\alpha}(y),\quad K_n^{\alpha}(x,y):=\frac{1}{n}\sum_{k=0}^{n-1}D_{k,k}^{\alpha}(x,y).$$

For a martingale

$$f\sim\sum_{n=1}^{\infty}\left(f^{(n)}-f^{(n-1)}\right),$$

the conjugate transforms are defined by

$$\widetilde{f^{(t)}}\sim\sum_{n=1}^{\infty}r_n\left(t\right)\left(f^{(n)}-f^{(n-1)}\right),$$

where $t \in G$ is fixed. Note that $\widetilde{f}^{(0)} = f$. As it is well-known, if f is an integrable function, then conjugate transforms $\widetilde{f}^{(t)}$ do exist almost everywhere, but they are not integrable in general. It is to see that $S_{2^n,2^n} f = f_n$.

Let

$$\rho_{0,0} := r_0, \quad \rho_{k,l} := r_j$$

if

$$\begin{aligned}(k,l) \in &\{2^{j-1}, 2^{j-1}+1, ..., 2^j-1\} \times \{2^{j-1}, 2^{j-1}+1, ..., 2^j-1\}\\ &\cup \{2^{j-1}, 2^{j-1}+1, ..., 2^j-1\} \times \{0, 1, ..., 2^{j-1}-1\}\\ &\cup \{0, 1, ..., 2^{j-1}-1\} \times \{2^{j-1}, 2^{j-1}+1, ..., 2^j-1\}.\end{aligned}$$

The (n,m)th rectangular partial sum of the conjugate transforms is

$$\tilde{S}_{n,m}^{\alpha,(t)}(f;x,y) := \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \rho_{k,i}(t)\hat{f}^{\alpha}(k,i)\alpha_{k,i}(x,y) = S_{n,m}^{\alpha}(\tilde{f}^{(t)};x,y)$$

$(t \in G)$. The Marcinkiewicz-Fejér means of the double conjugate Walsh(-Kaczmarz)-Fourier series are defined by

$$\tilde{\mathcal{M}}_n^{\alpha,(t)}(f;x,y) := \frac{1}{n}\sum_{k=0}^{n-1} \tilde{S}_{k,k}^{\alpha,(t)}(f;x,y).$$

It is evident that $\tilde{\mathcal{M}}_n^{\alpha,(0)}(f;x,y) = \mathcal{M}_n^{\alpha}(f;x,y)$.

For the martingale f, we consider the maximal operators

$$\mathcal{M}^{\alpha*}f(x,y) = \sup_n |\mathcal{M}_n^{\alpha}(f;x,y)|, \quad \tilde{\mathcal{M}}^{\alpha,(t)*}f(x,y) = \sup_n |\tilde{\mathcal{M}}_n^{\alpha,(t)}(f,x,y)|$$

In 1974 Schipp [9] and Young [15] proved that the Walsh-Kaczmarz system is a convergence system. In 1981 Skvortsov [11] showed that the Walsh-Kaczmarz-Fejér means converge uniformly to f for any continuous function f. For any integrable functions, Gát [1] proved, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere. Gát's result was extended by Simon [10] to H_p spaces. Namely, he proved that the maximal operator of Fejér means of one-dimensional Fourier series is bounded from Hardy space $H_p(G)$ into the space $L_p(G)$ for $p > 1/2$.

For any integrable functions, the second author [7] proved, that the Marcinkiewicz-Fejér means with respect to the two dimensional Walsh-Kaczmarz system converge almost everywhere to the function itself. This Theorem was extended in [2, 3]. Namely, we proved that the following are true.

Theorem GGN [Gát, Goginava and Nagy [2]] *Let $p > 2/3$. Then there exists a constant $c_p > 0$ such that*

$$\|\mathcal{M}^{\kappa,*}f\|_p \le c_p\|f\|_{H_p}.$$

Theorem GN [Goginava and Nagy [3]] Let $0 < p \le 2/3$. Then there exists a martingale $f \in H_p(G^2)$ such that

$$\|\mathcal{M}^{\kappa*}f\|_p = +\infty.$$

Since,

$$\left\| \widetilde{f}^{(t)} \right\|_{H_p} = \|f\|_{H_p}, \ \ 0 < p < \infty$$

and

$$\|f\|_{H_p}^p \backsim \int\limits_G \left\| \widetilde{f}^{(t)} \right\|_p^p dt,$$

from Theorem GGN we obtain that $(p > 2/3)$

$$\begin{aligned} \left\| \widetilde{M}_n^{\kappa,(t)} f \right\|_{H_p}^p &= \|M_n^\kappa f\|_{H_p}^p \leq c_p \int\limits_G \left\| \widetilde{M}_n^{\kappa,(t)} f \right\|_p^p dt \\ &= c_p \int\limits_G \left\| M_n^\kappa \widetilde{f}^{(t)} \right\|_p^p dt \leq c_p \int\limits_G \left\| \widetilde{f}^{(t)} \right\|_{H_p}^p dt \\ &= c_p \|f\|_{H_p}^p . \end{aligned}$$

Hence we proved that the following is valid.

Theorem 1 *Let $p > 2/3$. Then there exists a constant $c_p > 0$ such that*

$$\left\| \tilde{\mathcal{M}}_n^{\kappa(t)} f \right\|_{H_p} \leq c_p \|f\|_{H_p} \quad (f \in H_p, t \in G) .$$

In the present paper we prove that in Theorem 1 the assumption $p > 2/3$ is essential. Moreover, the following are true.

Theorem 2 *Let $0 < p \leq 2/3$. Then there exists a martingale $f \in H_p(G \times G)$ such that*

$$\sup_n \|\tilde{\mathcal{M}}_n^{\kappa,(t)} f\|_p = +\infty, \quad t \in G.$$

Corollary 1 *Let $0 < p \leq 2/3$. Then there exists a martingale $f \in H_p(G \times G)$ such that*

$$\sup_n \|\mathcal{M}_n^\kappa f\|_p = +\infty.$$

For Walsh system the analogue of Theorem 1 is proved in [12, 14] and the analogue of Theorem 2 is discussed in [4].

A bounded measurable function a is a p-atom, if there exists a dyadic 2-dimensional cube $I \times I$, such that

a) $\int\limits_{I \times I} a d\mu = 0$;

b) $\|a\|_\infty \leq \mu(I \times I)^{-1/p}$;

c) $\operatorname{supp} a \subset I \times I$.

The basic result of atomic decomposition is due to Weisz.

Theorem W [Weisz [14]] *A martingale* $f = \left(f^{(n)} : n \in \mathbf{N}\right)$ *is in* $H_p\,(0 < p \le 1)$ *if and only if there exists a sequence* $(a_k, k \in \mathbf{N})$ *of* p*-atoms and a sequence* $(\mu_k, k \in \mathbf{N})$ *of real numbers such that for every* $n \in \mathbf{N}$*,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n,2^n} a_k = f^{(n)}, \tag{2}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$

During the proof of Theorem 1 we will use the following Lemma [4]:

Lemma 1 (Goginava [4]) *Let* $n_A := 2^{4A} + 2^{4A-4} + ... + 2^4 + 2^0$*,*

$$x \in I_{4A}(0, ..., 0, x_{4m} = 1, 0, ..., 0, x_{4l} = 1, x_{4l+1}, ..., x_{4A-1})$$

and

$$y \in I_{4A}(0, ..., 0, y_{4l} = 1, x_{4l+1}, ..., x_{4q-1}, 1 - x_{4q}, y_{4q+1}, ..., y_{4A-1})$$

for some $m < l < q$*. Then*

$$n_{A-1}|K^w_{n_{A-1}}(x, y)| \ge 2^{4q+4l+4m-3}.$$

Proof of Theorem 2: Let $\{A_k : k \in \mathbf{N}\}$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{1}{A_k^p} < \infty, \tag{3}$$

$$\sum_{l=0}^{k-1} \frac{2^{8A_l/p}}{A_l} < \frac{2^{8A_k/p}}{A_k}, \tag{4}$$

$$\frac{10 \cdot 2^{8A_{k-1}}}{A_{k-1}} < \frac{2^{A_k}}{A_k}. \tag{5}$$

We note that such an increasing sequence $\{A_k : k \in \mathbf{N}\}$ which satisfies conditions (3)–(5) can be constructed. Let

$$f^{(A)}(x, y) := \sum_{\{k : 4A_k < A\}} \lambda_k a_k (x, y), \text{ where } \lambda_k := \frac{4}{A_k}$$

and

$$a_k (x, y) := 2^{8(1/p-1)A_k-2} \left(D_{2^{4A_k+1}}(x) - D_{2^{4A_k}}(x)\right)\left(D_{2^{4A_k+1}}(y) - D_{2^{4A_k}}(y)\right).$$

The martingale $f := \left(f^{(0)}, f^{(1)}, ..., f^{(A)}, ...\right) \in H_p\left(G^2\right)$. Since,

$$S_{2^A,2^A} a_k(x,y) = \begin{cases} 0, & \text{if } A \leq 4A_k, \\ a_k(x,y), & \text{if } A > 4A_k, \end{cases}$$

$$f^{(A)}(x) = \sum_{\{k:4A_k<A\}} \lambda_k a_k(x,y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A,2^A} a_k(x,y).$$

(3) and Theorem W yield that $f \in H_p\left(G^2\right)$.

Now, we give the Fourier coefficients.

$$\widehat{f}^{\kappa}(i,j) = \begin{cases} \frac{2^{8A_k(1/p-1)}}{A_k}, & (i,j) \in \{2^{4A_k}, ..., 2^{4A_k+1}-1\} \times \{2^{4A_k}, ..., 2^{4A_k+1}-1\}, \\ 0, & (i,j) \notin \bigcup\limits_{k=1}^{\infty} \{2^{4A_k}, ..., 2^{4A_k+1}-1\} \times \{2^{4A_k}, ..., 2^{4A_k+1}-1\}. \end{cases} \tag{6}$$

We decompose the n_{A_k}th Marcinkiewicz-Fejér means of double conjugate Walsh-Kaczmarz-Fourier series as follows:

$$\begin{aligned} \tilde{\mathcal{M}}_{n_{A_k}}^{\kappa,(t)}(f;x,y) &= \frac{1}{n_{A_k}} \sum_{j=1}^{n_{A_k}-1} \tilde{S}_{j,j}^{\kappa,(t)}(f;x,y) \\ &= \frac{1}{n_{A_k}} \sum_{j=1}^{2^{4A_k}-1} \tilde{S}_{j,j}^{\kappa,(t)}(f;x,y) + \frac{1}{n_{A_k}} \sum_{j=2^{4A_k}}^{n_{A_k}-1} \tilde{S}_{j,j}^{\kappa,(t)}(f;x,y) \\ &=: I + II. \end{aligned} \tag{7}$$

Let $j \in \left\{0, 1, ..., 2^{4A_k}-1\right\}$ for some k. Then from (6) and (4), it is easy to show that

$$\begin{aligned} \left|\tilde{S}_{j,j}^{\kappa,(t)}(f;x,y)\right| &\leq \sum_{l=0}^{k-1} \left| r_{4A_l}(t) \sum_{\nu=2^{4A_l}}^{2^{4A_l+1}-1} \sum_{\mu=2^{4A_l}}^{2^{4A_l+1}-1} \widehat{f}^{\kappa}(\nu,\mu)\kappa_\nu(x)\kappa_\mu(y) \right| \\ &\leq \sum_{l=0}^{k-1} \sum_{\nu=2^{4A_l}}^{2^{4A_l+1}-1} \sum_{\mu=2^{4A_l}}^{2^{4A_l+1}-1} \left|\widehat{f}^{\kappa}(\nu,\mu)\right| \\ &\leq \sum_{l=0}^{k-1} \frac{2^{8A_l/p}}{A_l} \leq 2\frac{2^{8A_{k-1}/p}}{A_{k-1}}. \end{aligned}$$

This yields that

$$|I| \leq \frac{1}{n_{A_k}} \sum_{j=1}^{2^{4A_k}-1} \left|\tilde{S}_{j,j}^{\kappa,(t)}(f;x,y)\right| \leq 2\frac{2^{8A_{k-1}/p}}{A_{k-1}}. \tag{8}$$

Now, we discuss II.

Let $i \in \left\{2^{4A_k}, ..., n_{A_k} - 1\right\}$. Then from (6) we conclude that

$$\begin{aligned}
\tilde{S}_{i,i}^{\kappa,(t)}(f;x,y) &= \sum_{\nu=0}^{i-1}\sum_{\mu=0}^{i-1} \rho_{\nu,\mu}(t)\widehat{f}^{\kappa}\left(\nu,\mu\right)\kappa_{\nu}(x)\kappa_{\mu}(y) \\
&= \sum_{l=0}^{k-1} r_{4A_l}(t) \sum_{\nu=2^{4A_l}}^{2^{4A_l+1}-1} \sum_{\mu=2^{4A_l}}^{2^{4A_l+1}-1} \widehat{f}^{\kappa}\left(\nu,\mu\right)\kappa_{\nu}(x)\kappa_{\mu}(y) \\
&\quad + r_{4A_k}(t) \sum_{\nu=2^{4A_k}}^{i-1} \sum_{\mu=2^{4A_k}}^{i-1} \widehat{f}^{\kappa}\left(\nu,\mu\right)\kappa_{\nu}(x)\kappa_{\mu}(y) \\
&= \sum_{l=0}^{k-1} r_{4A_l}(t) \frac{2^{8A_l(1/p-1)}}{A_l}\left(D_{2^{4A_l+1}}(x) - D_{2^{4A_l}}(x)\right)\left(D_{2^{4A_l+1}}(y) - D_{2^{4A_l}}(y)\right) \\
&\quad + r_{4A_k}(t) \frac{2^{8A_k(1/p-1)}}{A_k}\left(D_i^{\kappa}(x) - D_{2^{4A_k}}(x)\right)\left(D_i^{\kappa}(y) - D_{2^{4A_k}}(y)\right)
\end{aligned}$$

and

$$\begin{aligned}
II &= \frac{n_{A_k-1}}{n_{A_k}} \sum_{l=0}^{k-1} r_{4A_l}(t) \frac{2^{8A_l(1/p-1)}}{A_l}\left(D_{2^{4A_l+1}}(x) - D_{2^{4A_l}}(x)\right) \times \\
&\quad \times \left(D_{2^{4A_l+1}}(y) - D_{2^{4A_l}}(y)\right) \\
&+ \quad r_{4A_k}(t) \frac{2^{8A_k(1/p-1)}}{n_{A_k}A_k} \sum_{i=2^{4A_k}}^{n_{A_k}-1} \left(D_i^{\kappa}(x) - D_{2^{4A_k}}(x)\right)\left(D_i^{\kappa}(y) - D_{2^{4A_k}}(y)\right) \\
&=: \quad II_1 + II_2.
\end{aligned}$$

By (4), (5) and $|D_{2^n}(x)| \le 2^n$, we get that

$$|II_1| \le \sum_{l=0}^{k-1} \frac{2^{8A_l(1/p-1)}}{A_l} 2^{8A_l+2} \le \frac{2^{8A_{k-1}/p+3}}{A_{k-1}}$$

and

$$\left|\tilde{\mathcal{M}}_{n_{A_k}}^{\kappa,(t)}(f;x,y)\right| \ge |II_2| - \frac{2^{A_k}}{A_k}.$$

We can write the nth Dirichlet kernel with respect to the Walsh-Kaczmarz system in the following form:

$$D_n^{\kappa}(x) = D_{2^{|n|}}(x) + r_{|n|}(x) D_{n-2^{|n|}}^{w}(\tau_{|n|}(x)). \tag{9}$$

This equation immediately implies for II_2 that

$$\begin{aligned}
II_2 &= r_{4A_k}(t) \frac{2^{8A_k(1/p-1)}}{n_{A_k}A_k} r_{4A_k}(x) r_{4A_k}(y) \sum_{i=0}^{n_{A_k-1}-1} D_i^{w}(\tau_{4A_k}(x)) D_i^{w}(\tau_{4A_k}(y)) \\
&= r_{4A_k}(t) \frac{2^{8A_k(1/p-1)}}{n_{A_k}A_k} r_{4A_k}(x) r_{4A_k}(y) n_{A_k-1} K_{n_{A_k-1}}^{w}(\tau_{4A_k}(x)), \tau_{4A_k}(y)).
\end{aligned}$$

This implies

$$\left|\tilde{\mathcal{M}}_{n_{A_k}}^{\kappa,(t)}(f;x,y)\right| \geq \frac{n_{A_k-1}2^{8A_k(1/p-1)}}{n_{A_k}A_k}\left|K_{n_{A_k-1}}^{w}(\tau_{4A_k}(x)),\tau_{4A_k}(y))\right| - \frac{2^{A_k}}{A_k}.$$

For a fix A_k we give a subset of $G\times G$ as the following disjoint union:

$$G\times G \supseteq \bigcup_{m=[A_k/2]}^{A_k-3}\bigcup_{l=m+1}^{A_k-2}\bigcup_{q=l+1}^{A_k-1} J_{4A_k}^{m.l}\times L_{4A_k}^{l,q},$$

where $J_{4A_k}^{m,l} := \{x\in G : x_{4A_k-1} = ... = x_{4A_k-4m} = 0, x_{4A_k-4m-1} = 1, x_{4A_k-4m-2} = ... = x_{4A_k-4l} = 0, x_{4A_k-4l-1} = 1\}$, and $L_{4A_k}^{l,q} := \{y\in G : y_{4A_k-1} = ... = y_{4A_k-4l} = 0, y_{4A_k-4l-1} = 1, x_{4A_k-4l-2}, ..., x_{4A_k-4q}, y_{4A_k-4q-1} = 1 - x_{4A_k-4q-1}\}$.

Notice that, for any $(x,y)\in J_{4A_k}^{m.l}\times L_{4A_k}^{l,q}$, $([A_k/2]\leq m<l<q<A_k)$ by the definition of τ_{4A_k} and Lemma 1 we have

$$\left|\tilde{\mathcal{M}}_{n_{A_k}}^{\kappa,(t)}(f;x,y)\right| \geq \frac{2^{8A_k(1/p-1)}}{n_{A_k}A_k}2^{4q+4l+4m-3} - \frac{2^{A_k}}{A_k} \geq c\frac{2^{8A_k(1/p-1)}}{n_{A_k}A_k}2^{4q+4l+4m}.$$

Therefore, we write

$$\begin{aligned}
\int_{G\times G}\left|\tilde{\mathcal{M}}_{n_{A_k}}^{\kappa,(t)}(f;x,y)\right|^p d\mu(x,y) &\geq \sum_{m=[A_k/2]}^{A_k-3}\sum_{l=m+1}^{A_k-2}\sum_{q=l+1}^{A_k-1}\int_{J_{4A_k}^{m,l}\times L_{4A_k}^{l,q}}\left|\tilde{\mathcal{M}}_{n_{A_k}}^{\kappa,(t)}(f;x,y)\right|^p d\mu(x,y)\\
&\geq c\sum_{m=[A_k/2]}^{A_k-3}\sum_{l=m+1}^{A_k-2}\sum_{q=l+1}^{A_k-1}\mu(J_{4A_k}^{m,l}\times L_{4A_k}^{l,q})\frac{2^{8A_k(1-p)}}{n_{A_k}^p A_k^p}2^{p(4q+4l+4m)}\\
&= c\frac{2^{8A_k(1-p)}}{n_{A_k}^p A_k^p}\sum_{m=[A_k/2]}^{A_k-3}\sum_{l=m+1}^{A_k-2}\sum_{q=l+1}^{A_k-1}2^{-4l-4q}2^{p(4q+4l+4m)}\\
&= c\frac{2^{8A_k(1-p)}}{n_{A_k}^p A_k^p}\sum_{m=[A_k/2]}^{A_k-3}2^{4pm}\sum_{l=m+1}^{A_k-2}2^{4(p-1)l}\sum_{q=l+1}^{A_k-1}2^{4(p-1)q}\\
&= c\frac{2^{8A_k(1-p)}}{n_{A_k}^p A_k^p}\sum_{m=[A_k/2]}^{A_k-3}2^{12pm-8m}\\
&\geq c\frac{2^{4A_k(2-3p)}}{A_k^p}\sum_{m=[A_k/2]}^{A_k-3}2^{4m(3p-2)}\\
&= \begin{cases} cA_k^{1/3}, & \text{if } p=2/3,\\ c\frac{2^{2A_k(2-3p)}}{A_k^p}, & \text{if } 0<p<2/3.\end{cases}
\end{aligned}$$

The fact, that $A_k\to\infty$ and $\frac{2^{2A_k(2-3p)}}{A_k^p}\to\infty$ $(0<p<2/3)$ as $k\to\infty$, completes the proof of the main theorem. □

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