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Note on Hilbert-type inequalities

Predrag Vuković

Abstract

The main objective of this paper is to prove Hilbert-type and Hardy-Hilbert-type inequalities with a general homogeneous kernel, thus generalizing a result obtained in [Namita Das and Srinibas Sahoo, A generalization of multiple Hardy-Hilbert's integral inequality, Journal of Mathematical Inequalities, 3(1), (2009), 139–154].

1. Introduction

Bicheng Yang in [4] proved a Hilbert-type inequality for conjugate parameters and with the kernel $K(x, y) = (u(x) + u(y))^{-s}$, $s > 0$. His result is contained in the following theorem.

Theorem A *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r > 0$ ($r = p, q$), $\phi_p + \phi_q = s$, $u(t)$ is a differentiable strict increasing function in (a, b) ($-\infty \leq a < b \leq \infty$), such that $u(a+) = 0$ and $u(b-) = \infty$, and $f, g \geq 0$ satisfy $0 < \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx < \infty$ and $0 < \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x) dx < \infty$, then*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + u(y))^s} dx dy \tag{1.1}$$

$$< B(\phi_p, \phi_q) \left(\int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x) dx \right)^{\frac{1}{q}},$$

where the constant factor $B(\phi_p, \phi_q)$ is the best possible. If $p < 1$ ($p \neq 0$), $\{s; \phi_r > 0$ ($r = p, q$), $\phi_p + \phi_q = s\} \neq \emptyset$, with the above assumption, one has the reverse of (1.1), and the constant is still the best possible.

Recently, Namita Das et al. [1] gave a generalization of Yang's result:

Theorem B *Let $n \in \mathbb{N} \setminus \{1\}$, $p_i > 1$, ($i = 1, 2, \dots, n$), $\sum_{i=1}^n \frac{1}{p_i} = 1$, $s > 0$, $\lambda_i > 0$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \lambda_i = s$. Suppose for every $i = 1, \dots, n$; $u_i : (a_i, b_i) \rightarrow (0, \infty)$, is a strictly increasing differentiable function such that $u_i(a_i) = 0$ and $u_i(b_i) = \infty$. If $f_j \geq 0$ ($j = 1, 2, \dots, n$), satisfy*

$$0 < \int_{a_j}^{b_j} (u_j(x))^{p_j(1-\lambda_j)-1} (u'_j(x))^{1-p_j} f_j^{p_j}(x) dx < \infty,$$

then

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \frac{1}{(\sum_{i=1}^n u_i(x_i))^s} \prod_{j=1}^n f_j(x_j) dx_1 \cdots dx_n < \frac{1}{\Gamma(s)} \prod_{j=1}^n \Gamma(\lambda_j) \left(\int_{a_j}^{b_j} (u_j(x_j))^{p_j(1-\lambda_j)-1} (u'_j(x_j))^{1-p_j} f_j^{p_j}(x_j) dx_j \right)^{\frac{1}{p_j}},$$

where the constant factors $\frac{1}{\Gamma(s)} \prod_{j=1}^n \Gamma(\lambda_j)$ is the best possible.

Our main objective is to emphasize the previous theorem. Our generalization will include a general homogeneous kernel. In what follows we suppose that $K(x_1, \dots, x_n)$ is non-negative measurable homogeneous function of degree $-s$, $s > 0$. To obtain the main results we define the function $k(\beta_1, \dots, \beta_{n-1})$ by

$$k(\beta_1, \dots, \beta_{n-1}) := \int_{(0, \infty)^{n-1}} K(1, t_1, \dots, t_{n-1}) t_1^{\beta_1} \cdots t_{n-1}^{\beta_{n-1}} dt_1 \cdots dt_{n-1}, \tag{1.2}$$

where we suppose that $k(\beta_1, \dots, \beta_{n-1}) < \infty$ for $\beta_1, \dots, \beta_{n-1} > -1$ and $\beta_1 + \cdots + \beta_{n-1} + n < s + 1$.

Let A_{ij} , $i, j = 1, \dots, n$, be the real numbers satisfying

$$\sum_{i=1}^n A_{ij} = 0, \quad j = 1, 2, \dots, n. \tag{1.3}$$

We also define

$$\alpha_i = \sum_{j=1}^n A_{ij}, \quad i = 1, 2, \dots, n. \tag{1.4}$$

Our results will be based on the following result of Perić and Vuković from [2].

Theorem C Let p_1, \dots, p_n be conjugate parameters such that $p_i > 1$, $i = 1, \dots, n$, and let $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. Let $K : (0, \infty)^n \rightarrow \mathbb{R}$ be non-negative measurable homogeneous function of degree $-s$, $s > 0$, and let A_{ij} , $i, j = 1, \dots, n$, and α_i , $i = 1, \dots, n$ be real parameters satisfying (1.3) and (1.4). If $f_i : (0, \infty) \rightarrow \mathbb{R}$, $f_i \neq 0$, $i = 1, \dots, n$ are non-negative measurable functions, then the following inequalities hold and are equivalent:

$$\int_{(0, \infty)^n} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n < L \prod_{i=1}^n \left(\int_0^\infty x_i^{n-s-1+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}} \tag{1.5}$$

and

$$\int_0^\infty x_n^{(1-q)(n-1-s)-q\alpha_n} \left(\int_{(0,\infty)^{n-1}} K(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right)^q dx_n < L^q \prod_{i=1}^{n-1} \left(\int_0^\infty x_i^{n-1-s+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{q}{p_i}}, \tag{1.6}$$

where

$$L = k(p_1 A_{12}, \dots, p_1 A_{1n})^{\frac{1}{p_1}} \cdot k(s - n - p_2(\alpha_2 - A_{22}), p_2 A_{23}, \dots, p_2 A_{2n})^{\frac{1}{p_2}} \cdots k(p_n A_{n2}, \dots, p_n A_{n,n-1}, s - n - p_n(\alpha_n - A_{nn}))^{\frac{1}{p_n}}, \tag{1.7}$$

and $p_i A_{ij} > -1, i \neq j, p_i(A_{ii} - \alpha_i) > n - s - 1$.

In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions.

2. Main results

By applying Theorem C we get the following theorem.

Theorem 1 *Let $K : (0, \infty)^n \rightarrow \mathbb{R}$ and $A_{ij}, i, j = 1, \dots, n$, be as in Theorem C. Suppose for every $i = 1, \dots, n$; $u_i : (a_i, b_i) \rightarrow (0, \infty)$, is a strictly increasing differentiable function such that $u_i(a_i) = 0$ and $u_i(b_i) = \infty$. If $f_i : (0, \infty) \rightarrow \mathbb{R}, f_i \neq 0, i = 1, \dots, n$ are non-negative measurable functions, then the following inequalities hold and are equivalent*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} K(u_1(t_1), \dots, u_n(t_n)) \prod_{i=1}^n f_i(t_i) dt_1 \dots dt_n < L \prod_{i=1}^n \left(\int_{a_i}^{b_i} (u_i(t_i))^{n-s-1+p_i\alpha_i} (u_i'(t_i))^{1-p_i} f_i^{p_i}(t_i) dt_i \right)^{\frac{1}{p_i}} \tag{2.1}$$

and

$$\int_{a_n}^{b_n} (u_n(t_n))^{(1-q)(n-1-s)-q\alpha_n} \left(\int_{a_1}^{b_1} \cdots \int_{a_{n-1}}^{b_{n-1}} K(u_1(t_1), \dots, u_n(t_n)) \prod_{i=1}^{n-1} f_i(t_i) dt_1 \dots dt_{n-1} \right)^q dt_n < L^q \prod_{i=1}^{n-1} \left(\int_{a_i}^{b_i} (u_i(t_i))^{n-s-1+p_i\alpha_i} (u_i'(t_i))^{1-p_i} f_i^{p_i}(t_i) dt_i \right)^{\frac{q}{p_i}}, \tag{2.2}$$

where the constant L is defined by (1.7).

Proof. The proof follows directly from Theorem C setting the functions $g_i : [0, \infty) \rightarrow \mathbb{R}$, $i = 1, \dots, n$, such that $f_i(t_i) = g_i(u_i(t_i))u_i'(t_i)$. Namely, the inequality (1.5) with the functions g_i defined above, becomes

$$\int_{(0, \infty)^n} K(x_1, \dots, x_n) \prod_{i=1}^n g_i(x_i) dx_1 \dots dx_n \tag{2.3}$$

$$< L \prod_{i=1}^n \left(x_i^{n-s-1+p_i\alpha_i} g_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}},$$

where the constant L is defined by (1.7). Now, let I and J denote the left-hand and right-hand side of the inequalities (2.3) respectively. By using the substitution $x_i = u_i(t_i)$, $i = 1, \dots, n$, we obtain

$$I = \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} K(u_1(t_1), \dots, u_n(t_n)) \prod_{i=1}^n (g_i(u_i(t_i))u_i'(t_i)) dt_1 \dots dt_n, \tag{2.4}$$

where we used the facts $u_i(a_i) = 0$ and $u_i(b_i) = \infty$.

Similarly, we get

$$J = L \prod_{i=1}^n \left(\int_{a_i}^{b_i} (u_i(t_i))^{n-s-1+p_i\alpha_i} g_i^{p_i}(u_i(t_i))u_i'(t_i) dt_i \right)^{\frac{1}{p_i}}$$

$$= L \prod_{i=1}^n \left(\int_{a_i}^{b_i} (u_i(t_i))^{n-s-1+p_i\alpha_i} (u_i'(t_i))^{1-p_i} g_i^{p_i}(u_i(t_i))(u_i'(t_i))^{p_i} dt_i \right)^{\frac{1}{p_i}}. \tag{2.5}$$

Now, from (2.3), (2.4), (2.5) and the fact $f_i(t_i) = g_i(u_i(t_i))u_i'(t_i)$ follows the inequality (2.1). The second inequality (2.2) can be proved by applying (1.6) from Theorem C. □

To obtain a case of the best possible inequality it is natural to impose the following conditions on the parameters A_{ij} :

$$p_j A_{ji} = s - n - p_i(\alpha_i - A_{ii}), \quad i, j = 1, 2, \dots, n, \quad i \neq j. \tag{2.6}$$

In that case the constant L from Theorem 1 is simplified to the form:

$$L^* = k(\tilde{A}_2, \dots, \tilde{A}_n), \tag{2.7}$$

where

$$\tilde{A}_i = p_j A_{ji}, \quad i, j = 1, 2, \dots, n, \quad i \neq j. \tag{2.8}$$

It is easy to see that the parameters \tilde{A}_i satisfy the relation

$$\sum_{i=1}^n \tilde{A}_i = s - n. \tag{2.9}$$

By using (1.3) and (2.8) we have

$$\begin{aligned}
 A_{ii} &= -A_{1i} - A_{2i} - \dots - A_{i-1,i} - A_{i+1,i} - \dots - A_{ni} \\
 &= -\frac{\tilde{A}_i}{p_1} - \frac{\tilde{A}_i}{p_2} - \dots - \frac{\tilde{A}_i}{p_{i-1}} - \frac{\tilde{A}_i}{p_{i+1}} - \dots - \frac{\tilde{A}_i}{p_n} \\
 &= \tilde{A}_i \left(\frac{1}{p_i} - 1 \right).
 \end{aligned}
 \tag{2.10}$$

Further, by using (2.7) and (2.8) and (2.10), the inequalities (2.1) and (2.2) with the parameters A_{ij} , satisfying the relation (2.6), become

$$\begin{aligned}
 &\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} K(u_1(t_1), \dots, u_n(t_n)) \prod_{i=1}^n f_i(t_i) dt_1 \dots dt_n \\
 &< L^* \prod_{i=1}^n \left(\int_{a_i}^{b_i} (u_i(t_i))^{-1-p_i \tilde{A}_i} (u_i'(t_i))^{1-p_i} f_i^{p_i}(t_i) dt_i \right)^{\frac{1}{p_i}}
 \end{aligned}
 \tag{2.11}$$

and

$$\begin{aligned}
 &\int_{a_n}^{b_n} (u_n(t_n))^{(1-q)(-1-p_n \tilde{A}_n)} \left(\int_{a_1}^{b_1} \dots \int_{a_{n-1}}^{b_{n-1}} K(u_1(t_1), \dots, u_n(t_n)) \prod_{i=1}^{n-1} f_i(t_i) dt_1 \dots dt_{n-1} \right)^q dt_n \\
 &< (L^*)^q \prod_{i=1}^{n-1} \left(\int_{a_i}^{b_i} (u_i(t_i))^{-1-p_i \tilde{A}_i} (u_i'(t_i))^{1-p_i} f_i^{p_i}(t_i) dt_i \right)^{\frac{q}{p_i}}.
 \end{aligned}
 \tag{2.12}$$

In the following theorem we show that, if the parameters A_{ij} satisfy condition (2.6), then one obtains the best possible constant.

Theorem 2 *If the parameters A_{ij} , $i, j = 1, \dots, n$, satisfy conditions (1.3) and (2.6), then the constants L^* and $(L^*)^q$ are the best possible in inequalities (2.11) and (2.12).*

Proof. As in the proof of Theorem 1, let $g_i : [0, \infty) \rightarrow \mathbb{R}$, $i = 1, \dots, n$ be the functions such that $f_i(t_i) = g_i(u_i(t_i))u_i'(t_i)$. The inequality (1.5) with the functions g_i defined above, becomes

$$\begin{aligned}
 &\int_{(0, \infty)^n} K(x_1, \dots, x_n) \prod_{i=1}^n g_i(x_i) dx_1 \dots dx_n \\
 &< L^* \prod_{i=1}^n \left(\int_0^\infty x_i^{-1-p_i \tilde{A}_i} g_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}},
 \end{aligned}
 \tag{2.13}$$

where the constant L^* is defined by (2.7).

Now, let's suppose that the constant factor L^* is not the best possible. Then, there exists a positive constant L_1 , smaller than L^* such that the inequality (2.13) is still valid if we replace L^* by L_1 . For this purpose, set

$$\tilde{g}_i(x_i) = \begin{cases} 0 & x \in (0, 1) \\ x_i \tilde{A}_i - \frac{\varepsilon}{p_i} & x \in [1, \infty) \end{cases}, \quad i = 1, \dots, n,$$

where $0 < \varepsilon < \min_{1 \leq i \leq n} \{p_i + p_i \tilde{A}_i\}$. If we put these functions in the inequality (2.13), then the right-hand side of the inequality becomes $\frac{L_1}{\varepsilon}$, since

$$\prod_{i=1}^n \left[\int_0^\infty x_i^{-1-p_i \tilde{A}_i} \tilde{g}_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}} = \frac{1}{\varepsilon}. \tag{2.14}$$

Further, let J denotes the left-hand side of the inequality (2.13), for above choice of the functions \tilde{g}_i . By using substitution $u_i = \frac{x_i}{x_1}$, $i = 2, \dots, n$ in J , we find that

$$J = \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{\frac{1}{x_1}}^\infty \dots \int_{\frac{1}{x_1}}^\infty K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n \right] dx_1. \tag{2.15}$$

It is easy to see that the following inequality holds

$$\begin{aligned} J &\geq \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{(0, \infty)^{n-1}} K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n \right] dx_1 \\ &\quad - \int_1^\infty x_1^{-1-\varepsilon} \sum_{j=2}^n I_j(x_1) dx_1 \\ &= \frac{1}{\varepsilon} k \left(\tilde{A}_2 - \frac{\varepsilon}{p_2}, \dots, \tilde{A}_n - \frac{\varepsilon}{p_n} \right) - \int_1^\infty x_1^{-1-\varepsilon} \sum_{j=2}^n I_j(x_1) dx_1, \end{aligned} \tag{2.16}$$

where for $j = 2, \dots, n$, $I_j(x_1)$ is defined by

$$I_j(x_1) = \int_{D_j} K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n,$$

where $D_j = \{(u_2, u_3, \dots, u_n); 0 < u_j \leq \frac{1}{x_1}, 0 < u_k < \infty, k \neq j\}$. Without losing generality, we only estimate the integral $I_2(x_1)$. In fact, since $1 - u_2^\varepsilon \rightarrow 1$ ($u_2 \rightarrow 0^+$), there exists $M \geq 0$ such that $1 - u_2^\varepsilon \leq M$ ($u_2 \in (0, 1]$),

and by Fubini's theorem, it follows that

$$\begin{aligned}
 0 &\leq \varepsilon \int_1^\infty x_1^{-1-\varepsilon} I_2(x_1) dx_1 \\
 &= \varepsilon \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{(0,\infty)^{n-2}} \int_0^{\frac{1}{x_1}} K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n \right] dx_1 \\
 &= \varepsilon \int_{(0,\infty)^{n-2}} \int_0^1 K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} \left(\int_1^{\frac{1}{u_2}} x_1^{-1-\varepsilon} dx_1 \right) du_2 \dots du_n \\
 &= \varepsilon \int_{(0,\infty)^{n-2}} \int_0^1 K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} \left(\frac{1}{\varepsilon} (1 - u_2^\varepsilon) \right) du_2 \dots du_n \\
 &\leq M \int_{(0,\infty)^{n-2}} \int_0^1 K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n \\
 &\leq M \int_{(0,\infty)^{n-1}} K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n \\
 &= M \cdot k \left(\tilde{A}_2 - \frac{\varepsilon}{p_2}, \dots, \tilde{A}_n - \frac{\varepsilon}{p_n} \right) < \infty.
 \end{aligned}$$

Hence by (2.16), we have that

$$J \geq \frac{1}{\varepsilon} k \left(\tilde{A}_2 - \frac{\varepsilon}{p_2}, \dots, \tilde{A}_n - \frac{\varepsilon}{p_n} \right) - 0(1). \tag{2.17}$$

We conclude, by using (2.14) and (2.17), that $L^* \leq L_1$ when $\varepsilon \rightarrow 0^+$, which is an obvious contradiction. It follows that the constant L^* in (2.11) is the best possible.

Further, since the equivalence keeps the best possible constant, the proof is completed. □

3. Some applications

To obtain the following results we need some lemmas.

Lemma 1 (see [3]) *If $n \in \mathbb{N}$, $r_i > 0$, $i = 1, \dots, n$, then*

$$\int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{r_i-1}}{\left(1 + \sum_{i=1}^{n-1} u_i\right)^{\sum_{i=1}^n r_i}} du_1 \dots du_{n-1} = \frac{\prod_{i=1}^n \Gamma(r_i)}{\Gamma(\sum_{i=1}^n r_i)}. \tag{3.1}$$

By using Lemma 1 we have

Lemma 2 If $n \in \mathbb{N}$, $s, \lambda > 0$, $\beta_i > -1$, $i = 1, \dots, n-1$, and $\sum_{i=1}^{n-1} \beta_i < \lambda s - n + 1$, then

$$\int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{\beta_i}}{\left(1 + \sum_{i=1}^{n-1} t_i^\lambda\right)^s} dt_1 \dots dt_{n-1} \tag{3.2}$$

$$= \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=1}^{n-1} \Gamma\left(\frac{\beta_i + 1}{\lambda}\right)\right) \Gamma\left(s - \frac{1}{\lambda} \sum_{i=1}^{n-1} (\beta_i + 1)\right).$$

Proof. Let J denotes the left-hand side of the identity (3.2). By using the substitution $u_i = t_i^\lambda$, $i = 1, \dots, n-1$, we find that

$$J = \frac{1}{\lambda^{n-1}} \int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{\frac{\beta_i+1}{\lambda}-1}}{\left(1 + \sum_{i=1}^{n-1} u_i\right)^s} du_1 \dots du_{n-1}.$$

Applying Lemma 1 we get

$$J = \frac{1}{\Gamma(s)\lambda^{n-1}} \prod_{i=1}^n \Gamma(r_i),$$

where $r_i = \frac{\beta_i+1}{\lambda}$, $i = 1, \dots, n-1$, and $r_n = s - \frac{1}{\lambda} \sum_{i=1}^{n-1} (\beta_i + 1)$. In this way we prove (3.2). □

Remark 1 It is easy to see that Theorem 2 is the generalization of Theorem B. Namely, let us define $\tilde{A}_i = \lambda_i - 1$, $i = 1, \dots, n$, and $K(x_1, \dots, x_n) = (x_1 + \dots + x_n)^{-s}$. By using Lemma 1 we have $L = k(\lambda_1, \dots, \lambda_n) = \frac{1}{\Gamma(s)} \prod_{i=1}^n \Gamma(\lambda_i)$.

We proceed with some special homogeneous function. Since the function $K(x_1, \dots, x_n) = \left(\sum_{i=1}^n x_i^{s(\lambda-1)}\right) / \left(\sum_{i=1}^n x_i^\lambda\right)^s$, $\lambda > 1$, is homogeneous of degree $-s$, by using Theorem 2 we obtain:

Corollary 1 Let $n \geq 2$ be an integer and let p_1, \dots, p_n be conjugate parameters such that $p_i > 1$, $i = 1, \dots, n$ and let $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. If $a > 1$, $f_i > 0$, $i = 1, \dots, n$, measurable functions, then the following inequalities hold and are equivalent:

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\sum_{i=1}^n a^{s(\lambda-1)t_i}}{\left(\sum_{i=1}^n a^{\lambda t_i}\right)^s} \prod_{i=1}^n f_i(t_i) dt_1 \dots dt_n$$

$$< L_1 \prod_{i=1}^n \left(\int_{-\infty}^{\infty} a^{-st_i} f_i^{p_i}(t_i) dt_i\right)^{\frac{1}{p_i}} \tag{3.3}$$

and

$$\left[\int_{-\infty}^{\infty} a^{\frac{st_n}{p_n-1} - t_n} \left(\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\sum_{i=1}^n a^{s(\lambda-1)t_i}}{\left(\sum_{i=1}^n a^{\lambda t_i}\right)^s} \prod_{i=1}^{n-1} f_i(t_i) dt_1 \dots dt_{n-1} \right)^q dt_n \right]^{\frac{1}{q}}$$

$$< L_1 \prod_{i=1}^{n-1} \left(\int_{-\infty}^{\infty} a^{-st_i} f_i^{p_i}(t_i) dt_i\right)^{\frac{1}{p_i}}, \tag{3.4}$$

where the constant

$$L_1 = \frac{(\lambda \ln a)^{1-n}}{\Gamma(s)} \sum_{j=1}^n \left[\left(\prod_{i=1, i \neq j}^n \Gamma\left(\frac{s}{p_i \lambda}\right) \right) \cdot \Gamma\left(\frac{sp_j(\lambda-1) + s}{p_j \lambda}\right) \right] \quad (3.5)$$

is the best possible in the inequalities (3.3) and (3.4).

Proof. Set $K(x_1, \dots, x_n) = \left(\sum_{i=1}^n x_i^{s(\lambda-1)}\right) / \left(\sum_{i=1}^n x_i^\lambda\right)^s$, $\lambda > 0$, and $x_i = u_i(t_i) := a^{t_i}$, $i = 1, \dots, n$, in Theorem 2. Then, using the notation of Theorem 1 we have $a_i = -\infty$ and $b_i = \infty$.

It is easy to see that the parameters A_{ij} , $i, j = 1, \dots, n$, defined by

$$A_{ij} = \frac{s - p_j}{p_i p_j}$$

satisfy the condition (2.6). Therefrom, from the statement of Theorem 2 follows $\tilde{A}_i = \frac{s-p_i}{p_i}$, $i = 1, \dots, n$. Now, by using the definition of $u_i(t_i)$ and the parameters \tilde{A}_i we get

$$(u_i(t_i))^{-1-p_i \tilde{A}_i} (u_i'(t_i))^{1-p_i} = (\ln a)^{1-p_i} a^{-st_i}$$

and

$$(u_n(t_n))^{(1-q)(-1-p_n \tilde{A}_n)} = (a^{t_n})^{\frac{s}{p_n-1}-1}.$$

Further, it is enough to calculate the constant $L_1 = (\ln a)^{1-n} \cdot L$, where $L = k\left(\frac{s-p_2}{p_2}, \dots, \frac{s-p_n}{p_n}\right)$. Using the definition of the function $k(\alpha_1, \dots, \alpha_{n-1})$ given by (1.2) we have

$$L = \int_{(0, \infty)^{n-1}} \frac{1 + t_1^{s(\lambda-1)} + \dots + t_{n-1}^{s(\lambda-1)}}{(1 + \sum_{i=1}^n t_i^\lambda)^s} t_1^{\frac{s}{p_2}-1} \dots t_{n-1}^{\frac{s}{p_n}-1} dt_1 \dots dt_{n-1} = \sum_{k=0}^{n-1} I_k, \quad (3.6)$$

where

$$I_0 = \int_{(0, \infty)^{n-1}} \frac{t_1^{\frac{s}{p_2}-1} \dots t_{n-1}^{\frac{s}{p_n}-1}}{(1 + \sum_{i=1}^n t_i^\lambda)^s} dt_1 \dots dt_{n-1}$$

and

$$I_k = \int_{(0, \infty)^{n-1}} \frac{t_1^{\frac{s}{p_2}-1} \dots t_k^{s(\lambda-1) + \frac{s}{p_{k+1}} - 1} \dots t_{n-1}^{\frac{s}{p_n}-1}}{(1 + \sum_{i=1}^n t_i^\lambda)^s} dt_1 \dots dt_{n-1}, \text{ for } k = 1, \dots, n-1.$$

By using Lemma 2 we get

$$I_0 = \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=2}^n \Gamma\left(\frac{s}{p_i \lambda}\right) \right) \cdot \Gamma\left(\frac{sp_1(\lambda-1) + s}{p_1 \lambda}\right),$$

and similarly

$$I_k = \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=1, i \neq k+1}^n \Gamma\left(\frac{s}{p_i \lambda}\right) \right) \cdot \Gamma\left(\frac{sp_{k+1}(\lambda-1) + s}{p_{k+1} \lambda}\right),$$

for $k = 1, \dots, n-1$. Now, from (3.6) we get (3.5). □

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References

- [1] Das, N., Sahoo, S.: A generalization of multiple Hardy-Hilbert's integral inequality, *Journal of Mathematical Inequalities*, **3(1)**, 139–154, (2009).
- [2] Perić, I., Vuković, P.: Hardy-Hilbert's inequality with a general homogeneous kernel, *Math. Inequal. Appl.* **12(3)** 12–41, (2009).
- [3] Rassias, T. M., Yang, B.: On the way of weight coefficient and research for the Hilbert-type inequalities, *Math. Inequal. Appl.* **6(2)** 625–658, (2003).
- [4] Yang, B.: A relation to Hardy-Hilbert's integral inequality and Mulholland's inequality, *Journal of Inequalities in Pure and Applied Mathematics*, **6(4)**, Art. 112, (2005)

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