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On *CISE*-normal subgroups of finite groups*

Yong Xu, Tao Zhao and Xianhua Li

Abstract

A generalized normality *CISE* of subgroups of a finite group is introduced. Let \mathcal{F} be a saturated formation containing the class of all supersolvable groups. We give a characterization of \mathcal{F} by using *CISE*-normality of subgroups.

Key Words: Formation, supersolvable subgroup, *CISE*-normal subgroup

1. Introduction

In this paper, all groups considered are finite and G stands for a finite group. We write $N \text{ Char } G$ to mean that N is a characteristic subgroup of G . Let $\pi(G)$ stand for the set of all prime divisors of $|G|$. Let \mathcal{F} denote a saturated formation and \mathcal{U} the class of supersolvable groups. Let $G^{\mathcal{F}} = \cap\{N \trianglelefteq G \mid G/N \in \mathcal{F}\}$, say the \mathcal{F} -residual of G . Write $Z_{\mathcal{F}}(G)$ for the \mathcal{F} -hypercentre of G (see[9]). The other notations and terminologies are standard (see[10]).

It is always a question of particular interest in the theory of groups to study the structure of a group G by using a certain generalized normality of some subgroups of G . Kegel in [12] introduced the concept of s -quasinormal subgroups. A subgroup H of a group G is said to be s -permutable, s -quasinormal, or π -quasinormal in G if $PH = HP$ for all Sylow subgroups P of G . Ballester-Bolinches and Pedraza-Aguilera in [4] introduced the notion of s -quasinormal embedding. A subgroup H of a group G is said to be S -quasinormally embedded or π -quasinormally embedded in G if for each prime number p in $\pi(H)$, A Sylow p -subgroup of H is also a Sylow p -subgroup of a certain s -quasinormal subgroup of G . In [23], Wang Yanming introduced the concept of c -normal subgroups. A subgroup H of a group G is said to be c -normal in G if G has a normal subgroup T such that $HT = G$ and $H \cap T \leq H_G$, where $H_G = \cap_{x \in G} H^x$ is the core of H in G . In [25], Wei and Wang integrated these terminologies into the concept of c^* -normal subgroups. A subgroup H of a group G is said to be c^* -normal in G if G has a normal subgroup T such that $HT = G$ and $H \cap T$ is π -quasinormally embedded in G . In [21], Skiba introduced that: a subgroup H of G is said to be weakly s -permutable in G if G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s -permutable subgroup of G contained in H . Many authors have investigated the structure of a finite

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group G under the assumption that some subgroups of prime power order of G or some maximal subgroups of Sylow subgroups of G have those generalized normality in G and have obtained many results (see [3], [5], [6], [15], [17], [18], [21], [23], [24] and [25]). Now, we introduce another generalized normality of subgroups of a finite group and generalize further the concept of c^* -normal. The aim is to improve and extend these works mentioned before.

Definition 1.1 *Let H be a subgroup of a group G .*

(1) *If there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K$ is π -quasinormally embedded in G , H is called an *ISE-normal* subgroup of G .*

(2) *If H has a supersolvable supplement in G or H is *ISE-normal* in G , H is called a *CISE-normal* subgroup of G .*

It is clear that the concept of *ISE-normal* is obtained by replacing the word “normal subgroup” by “subnormal subgroup” in the definition of c^* -normal and the concept of *CISE-normal* is a generalization of *ISE-normal*. The following example shows that the generalization is proper.

Let $N = L_2(8)$, $G = \text{Aut}(N)$, then $G \cong N : 3$. We identify G with $N : 3$. Let $P \in \text{Syl}_2(N)$ and $Q \in \text{Syl}_3(N)$. Let $H = N_G(P)$ and $K = N_G(Q)$. By [7], $N_N(P) \cong 2^3 : 7$, $H \cong 2^3 : 7 : 3$ and $K \cong 9 : 6$. So K is supersolvable and $G = HK$, which shows that H is *CISE-normal* in G . But it is clear that the subnormal subgroup containing the Sylow 3-subgroup of H must be G . Hence H is not an *ISE-normal* subgroup. This shows that H is not *ISE-normal* in G .

2. Preliminary results

Lemma 2.1 ([4, Lemma 1]) *Suppose that U is π -quasinormally embedded in a group G , $H \leq G$ and K a normal subgroup of G .*

- (a) *If $U \leq H$, then U is π -quasinormally embedded in H .*
- (b) *UK is π -quasinormally embedded in G and UK/K is π -quasinormally embedded in G/K .*
- (c) *Let $K \leq H$ such that H/K is π -quasinormally embedded in G/K , then H is π -quasinormally embedded in G .*

Lemma 2.2 *Let H be a subgroup of a group G .*

- (1) *If H is *ISE-normal* (*CISE-normal*) in G and $H \leq M \leq G$, then H is *ISE-normal* (respectively, *CISE-normal*) in M .*
- (2) *Let $N \triangleleft G$ and $N \leq H$. Then H is *ISE-normal* in G if and only if H/N is *ISE-normal* in G/N ; if H is *CISE-normal* in G , then H/N is *CISE-normal* in G/N .*
- (3) *Let π be a set of primes, H a π -subgroup of G , and N a normal π' -subgroup of G . If H is *ISE-normal* (respectively, *CISE-normal*) in G , then HN/N is *ISE-normal* (respectively, *CISE-normal*) in G/N .*

(4) Suppose that H is a p -subgroup for some prime p and H is not π -quasinormally embedded in G but ISE -normal in G , then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.

Proof. (1) Suppose that H is ISE -normal in G . By the hypothesis, there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T$ is π -quasinormally embedded in G . Then $M = M \cap HT = H(M \cap T)$, $M \cap T \triangleleft \triangleleft M$ by [8, Chap A, Lemma 14.1(a)], and $H \cap (M \cap T) = H \cap T$ is π -quasinormally embedded in M by Lemma 2.1(a). So H is ISE -normal in M . If H has a supersolvable supplement L in G , then $G = HL$, $M = M \cap HL = H(M \cap L)$. Obviously, $M \cap L$ is also a supersolvable subgroup of M . Hence if H is $CISE$ -normal in G , then H is $CISE$ -normal in M . \square

(2) Suppose that H is ISE -normal in G . Then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T$ is π -quasinormally embedded in G . Thus $G/N = HT/N = (H/N) \cdot (TN/N)$, $TN/N \triangleleft \triangleleft G/N$ by [8, Chap A, Lemma 14.1(b)], and $(H/N) \cap (TN/N) = (H \cap TN)/N = (H \cap T)N/N$ is π -quasinormally embedded in G/N by Lemma 2.1(b). So H/N is ISE -normal in G/N . Conversely, if H/N is ISE -normal in G/N , then there exists a subnormal subgroup T/N of G/N such that $G/N = (H/N) \cdot (T/N)$ and $(H/N) \cap (T/N) = (H \cap T)/N$ is π -quasinormally embedded in G/N . Then $G = HT$, T is subnormal G , and $H \cap T$ is π -quasinormally embedded in G by Lemma 2.1(c). So H is ISE -normal in G . The second part is clear. \square

(3) Suppose that H is ISE -normal in G , then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T$ is π -quasinormally embedded in G . Then $G/N = HT/N = (HN/N) \cdot (TN/N)$ and $HN/N \cap TN/N = (HN \cap TN)/N$. By [8, Chap A, Lemma 14.1(b)], TN/N is subnormal in G/N . Since H is a π -subgroup of G and $G = HT$, $\pi'(G) \subseteq \pi(T)$. Since T is subnormal in G , we have $O_{\pi'}(G) \leq T$, so $N \leq O_{\pi'}(G) \leq T$, thus by Lemma 2.1(b), $(HN \cap TN)/N = (HN \cap T)/N = (H \cap T)N/N$ is π -quasinormally embedded in G/N , hence HN/N is ISE -normal in G/N . If H has a supersolvable supplement L in G , then $G = HL$, so $G/N = HL/N = (HN/N) \cdot (LN/N)$. Obviously $LN/N \cong L/(L \cap N)$ is supersolvable. Thus HN/N also has a supersolvable supplement LN/N in G/N . Hence we have claim (3).

(4) Since H is ISE -normal in G , there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K$ is π -quasinormally embedded in G . Since H is not π -quasinormally embedded in G , $H \cap K \neq H$. Hence K is a proper subnormal subgroup of G , then there exists a proper normal subgroup T of G such that $K \leq T$, so $|G/T| = |H : T \cap H| = p^i$, where i is a natural number, thus G has a normal maximal subgroup M such that $G = HM$ and $|G : M| = p$. \square

Lemma 2.3 ([25, Lemma 2.5]) *Let G be a group, K a π -quasinormally embedded subgroup of G and P a Sylow p -subgroup of K , where p is a prime. If either $P \leq O_p(G)$ or $K_G = 1$, then P is π -quasinormal in G .*

Lemma 2.4 *Let N be an elementary abelian normal p -subgroup of G . Assume that N has a subgroup U with $1 < |U| < |N|$ such that every subgroup H of N of order $|U|$ is ISE -normal in G . Then N is not a minimal normal subgroup of G .*

Proof. Suppose that this lemma is false, then N is a minimal normal subgroup of G . If some subgroup H of N satisfying $|H| = |U|$ is not π -quasinormally embedded in G , then by Lemma 2.2 (4), there exists a normal subgroup of G , M such that $|G : M| = p$ and $G = HM$. It follows that $NM = G$, $p = |NM : M| = |N : N \cap M|$. So $M \cap N$ is a maximal subgroup of N and $N \cap M \triangleleft G$. By the minimality of

N , we get $N \cap M = 1$. By $HM = NM$, we get $|N| = |H| < |N|$, a contradiction. Hence every subgroup H of N satisfying $|H| = |U|$ is π -quasinormally embedded in G , and $H \leq N \leq O_p(G)$. By Lemma 2.3, these subgroups H are π -quasinormal in G . By [16, Lemma 2.2], we have $O^p(G) \leq N_G(H)$, hence $|G : N_G(H)| = p^{j_H}$, where $j_H \geq 0$. Since $|H| < |N|$ and N is a minimal normal subgroup of G , we have $N_G(H) \neq G$. So $j_H > 0$ and j_H are natural numbers. Let Ω be the set of all subgroups with order $|U|$ of N . Then G acts on Ω by conjugation and we can obtain a partition of Ω into orbits. For $K \in \Omega$, let the G -orbit of K be $\{K_1, K_2, \dots, K_s\}$, then $s = |G : N_G(K)| = p^{j_K}$. Hence p divides $|\Omega|$. On the other hand, by ([10, III 8.5(d)]), we know $|\Omega| \equiv 1 \pmod{p}$, a contradiction. \square

Lemma 2.5 *Let \mathcal{F} be a saturated formation containing the class of all nilpotent groups \mathcal{N} , G be a group. Suppose that $G^{\mathcal{F}}$ is soluble and every maximal subgroup of G not containing $G^{\mathcal{F}}$ belongs to \mathcal{F} . If every cyclic subgroup of $G^{\mathcal{F}}$ with prime order or order 4 (if $G^{\mathcal{F}}$ is a non-abelian 2-group) is CISE-normal in G , then $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.*

Proof. Let $R = G^{\mathcal{F}}$. By [19, VI Theorem 24.2], R is a p -subgroup, $\exp R = p$ or 4 (if R is a non-abelian 2-group) and $R/\Phi(R)$ is a minimal normal subgroup of $G/\Phi(R)$. Suppose that there exists $x \in R$ such that $\langle x \rangle$ has a supersolvable supplement M in G . If $G = M$, then $G \in \mathcal{F}$ and $G^{\mathcal{F}} = 1$, a contradiction. So we may assume that $M \neq G$, then $M\Phi(R) \neq G$ by $\Phi(R) \leq \Phi(G)$. And $G = \langle x \rangle M = RM$, so $(M\Phi(R)/\Phi(R))(\langle x \rangle\Phi(R)/\Phi(R)) = G/\Phi(R)$. Hence $|G/\Phi(R) : M\Phi(R)/\Phi(R)| = p$ and $G/\Phi(R) = (M\Phi(R)/\Phi(R))(R/\Phi(R))$. Since $R/\Phi(R)$ is a minimal normal subgroup of $G/\Phi(R)$, $R/\Phi(R) \cap M\Phi(R)/\Phi(R) = \bar{1}$, so $|(G/\Phi(R)) : (M\Phi(R)/\Phi(R))| = |R/\Phi(R)| = p$. If there exists $y \in R$ with $y \notin \Phi(R)$ such that $y \in M_G$, since $R/\Phi(R)$ is a minimal normal subgroup of $G/\Phi(R)$, we have $R \leq M_G$, contrary to $G = RM$. So for any $y \in R$ with $y \notin \Phi(R)$, there exists $g \in G$ such that $y \notin M^g$. Thus $G = \langle y \rangle M^g$ and $\langle y \rangle$ has a supersolvable supplement. Hence $|R/\Phi(R)| = p$ and the result is true. If for any $y \in R$ with $y \notin \Phi(R)$, $\langle y \rangle$ has no supersolvable supplement, then $\langle y \rangle$ is ISE-normal in G by the hypothesis. By Lemma 2.2 (2), every cyclic subgroup of $R/\Phi(R)$ of order prime is ISE-normal in $G/\Phi(R)$. Since $R/\Phi(R)$ is a minimal normal subgroup of $G/\Phi(R)$, by Lemma 2.4, $R/\Phi(R)$ has no a proper cyclic subgroup of prime order, hence we conclude that $|R/\Phi(R)| = p$. Thus we have Lemma 2.5. \square

Lemma 2.6 ([22, Lemma 1.6]) *Let P be a nilpotent normal subgroup of a group G . If $P \cap \Phi(G) = 1$, then P is the direct product of some minimal normal subgroups of G .*

Lemma 2.7 *Let G be a group and let P be a Sylow 2-subgroup of G . Suppose that there exists a maximal subgroup P_1 of P such that P_1 is CISE-normal in G . Then G is not a nonabelian simple group.*

Proof. Suppose that G is a nonabelian simple group. By the hypothesis, P_1 has a supersolvable supplement M in G or it is ISE-normal in G . If the former case is true, then $P_1M = G$ and $M \neq G$ where M is supersolvable. So $|G : M| = 2^r$ and $2^r < |P|$. By [1, Theorem 5.8], we get either M is a Hall r' -subgroup of G or G is isomorphic to A_n with $5 \leq n = 2^r$, $r \geq 2$ and $M \cong A_{n-1}$. Since $2^r < |P|$ and M is supersolvable, we have G is not a nonabelian simple group. If the later case holds, then there exists a subnormal subgroup T such that $G = P_1T$ and $P_1 \cap T$ is π -quasinormally embedded in G . Since G is simple, we have $T = G$, then

$P_1 \cap T = P_1$ is π -quasinormally embedded in G . So there exists a π -quasinormal subgroup K of G such that $P_1 \in \text{Syl}_2(K)$. Since G is simple, we get $K = G$. So $P_1 = P$, a contradiction. Then G is not a nonabelian simple group. \square

Lemma 2.8 *Let G be a group and M a subgroup of G . Then we have:*

- (1) *If M is normal in G , then $F^*(M) \leq F^*(G)$.*
- (2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \text{Soc}(F(G)C_G(F(G)))/F(G)$.*
- (3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*
- (4) *$C_G(F^*(G)) \leq F(G)$.*

Proof. (1) \sim (4) can be found in [11, Chap. X, §13]. \square

3. Main results

Theorem 3.1 *Let \mathcal{F} be a saturated formation containing \mathcal{U} and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every Sylow subgroup of $F^*(N)$ is cyclic, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false and let G be a counterexample with $|G| + |N|$ minimal. Suppose that all Sylow subgroups of $F^*(N)$ are cyclic. Clearly, $F^*(N)$ is supersolvable, then $F^*(N) = F(N)$ by Lemma 2.8 (3). Since $F(N)/\Phi(N) = F(N/\Phi(N))$, all Sylow subgroups of $F(N/\Phi(N))$ are cyclic. So $(G/\Phi(N), N/\Phi(N))$ satisfies the hypothesis of Theorem 3.1. Thus if $\Phi(N) \neq 1$, then $G \in \mathcal{F}$. Hence we may assume that $\Phi(N) = 1$. Since $\Phi(F(N)) \subseteq \Phi(N)$, $\Phi(F(N)) = 1$. Let $\pi(F(N)) = \{p_i \mid 1 \leq i \leq t\}$ and $K_i \in \text{Syl}_{p_i}(F(N))$, where $p_1 < p_2 < \dots < p_t$, then $K_i \triangleleft G$ and $|K_i| = p_i$. Then $|\text{Aut}(K_i)| = p_i - 1$ and $G/C_G(K_i)$ is isomorphic to a subgroup of $\text{Aut}(K_i)$. From $\text{Aut}(K_i)$ cyclic, we get that $G/C_G(K_i)$ is cyclic. Let $U = \bigcap_{i=1}^t C_G(K_i)$, then $G/U \in \mathcal{U}$ and so $G/U \cap N \in \mathcal{F}$. It is easy to see that $F^*(U \cap N) = F^*(N)$. Hence if $U \cap N < N$, then $G \in \mathcal{F}$, a contradiction. So we may assume that $N \leq U$. It is clear that $U = C_G(F(N))$. So $N \leq C_G(F(N))$, $F(N) \leq Z(N)$. By Lemma 2.8 (4), $N = C_N(F(N)) = C_N(F^*(N)) \leq F(N)$, so $N = F(N)$ and N is cyclic. Since $(G/K_1)/(N/K_1) \cong G/N \in \mathcal{F}$, and $F(N/K_1) = N/K_1$ is cyclic, by the minimality of $|G| + |N|$, we get $G/K_1 \in \mathcal{F}$. So we may assume that $N = K_1$ and $K_1 \not\leq \Phi(G)$. Let M be a maximal subgroup of G such that $G = K_1M$, then $M \cap K_1 = 1$. Since K_1 centralizes $C_G(K_1) \cap M$ and M normalizes $C_G(K_1) \cap M$, we get $C_G(K_1) \cap M \triangleleft G$. Let $T = C_G(K_1) \cap M$. Then $K_1 \not\leq T$ and so $K_1 \cap T = 1$. By $G = C_G(K_1)M$, we have that $M/T \cong C_G(K_1)M/C_G(K_1) = G/C_G(K_1)$. Since $G/C_G(K_1)$ is cyclic and $G/T = K_1T/T \rtimes M/T$, we have $G/T \in \mathcal{U}$. Because \mathcal{F} is a formation, we have obtained that $G \cong G/K_1 \cap T \in \mathcal{F}$, a contradiction. This contradiction completes the proof of Theorem 3.1. \square

Theorem 3.2 *Let \mathcal{F} be a saturated formation containing \mathcal{U} and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every non-cyclic Sylow subgroup P of N has a subgroup U with $1 < |U| < |P|$*

such that every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is *CISE*-normal in G , then $G \in \mathcal{F}$.

Proof. Assume that the result is false and let (G, N) be a counterexample with $|G| + |N|$ minimal.

If all Sylow subgroups of N are cyclic, then all Sylow subgroups of $F(N)$ are cyclic. By Theorem 3.1, $G \in \mathcal{F}$. Hence when we want to prove $\overline{G} \in \mathcal{F}$ in the following arguments, we assume always that \overline{N} has a non-cyclic Sylow subgroup if $(\overline{G}, \overline{N})$ satisfy the hypothesis of (G, N) in Theorem 3.2.

Step 1. If T is a Hall subgroup of N , the hypothesis is true for (T, T) . In addition, if $T \triangleleft N$, then the hypothesis is also true for $(G/T, N/T)$. Especially, if T is a non-identity normal Hall subgroup of N , we may assume that $N = T$.

Let T be a Hall subgroup of N , P be a non-cyclic Sylow subgroup of T , of course, P is also a non-cyclic Sylow subgroup of N . By the hypothesis, P has a subgroup U with $1 < |U| < |P|$ such that every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is *CISE*-normal in G . By Lemma 2.2 (1), the hypothesis is true for (T, T) . In addition, if $T \triangleleft N$, then by T is a characteristic subgroup of N and $N \triangleleft G$, we get that $T \triangleleft G$. Let P^*/T be a non-cyclic Sylow subgroup of N/T , then by Shur-Zassenhaus Theorem, it is easy to prove that $P^* = T \rtimes P$, where $P \in Syl_p(P^*)$. Obviously, P is also a non-cyclic Sylow subgroup of N . By the hypothesis, P has a subgroup U such that $1 < |U| < |P|$ and every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is *CISE*-normal in G . By Lemma 2.2 (3), H^*/T is *CISE*-normal in G/T . Especially, if T is a non-identity normal Hall subgroup of N , then the hypothesis is true for $(G/T, N/T)$, so $G/T \in \mathcal{F}$. Thus the hypothesis is still true for (G, T) . By the minimality of $|G| + |N|$, we may assume that $T = N$.

Step 2. Let $p = \min \pi(N)$, then Sylow p -subgroups P of N are not cyclic. Thus by the hypothesis, P has a subgroup U such that $1 < |U| < |P|$ and every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is *CISE*-normal in G .

If P is cyclic, then N is p -nilpotent by [10, V, 2.8], so N has a normal p' -Hall subgroup $N_{p'}$. If $N_{p'} \neq 1$, by **Step 1**, $(G/N_{p'}, N/N_{p'})$ satisfies the hypothesis, then $G \in \mathcal{F}$ by the minimality of $|G| + |N|$, a contradiction. So $N_{p'} = 1$, $N = P$. Since $G/N \in \mathcal{F}$ and P is cyclic, we get $G \in \mathcal{F}$ by Theorem 3.1, a contradiction.

Step 3. If either $N = G$ or $N = P$, then $|U| > p$.

Suppose that $N = G$, by the hypothesis, G is not a nilpotent group and so it has a p -closed Schmidt subgroup E [10, IV, 5.4]. If $|U| = p$, then E satisfies the condition of Lemma 2.5, so E is p -nilpotent, a contradiction. Hence $|U| > p$. Suppose that $N = P$, then $G/P \in \mathcal{F}$. Assume that $|U| = p$ and $R = G^{\mathcal{F}}$. Let T be an arbitrary maximal subgroup of G not containing P , then $G/P = TP/P \cong T/T \cap P$, so the hypothesis is still true for $(T, T \cap P)$, thus $T \in \mathcal{F}$ by the choice of G . So every maximal subgroup of G not containing P belongs to \mathcal{F} . Since $R \leq P$, every maximal subgroup of G not containing R does not contain P , thus every maximal subgroup of G not containing R belongs to \mathcal{F} , so $|R/\Phi(R)| = p$ by Lemma 2.5. Since $(G/\Phi(R))/(R/\Phi(R)) \cong G/R \in \mathcal{F}$, we have $G/\Phi(R) \in \mathcal{F}$ by Theorem 3.1, which implies $G \in \mathcal{F}$, a contradiction. Hence $|U| > p$.

Step 4. If $|P : U| > p$, then every subgroup H of P of order $|U|$ not having a supersoluble supplement in G is π -quasinormally embedded in G . If P is a non-abelian 2-group and $|U| = 2$, then every subgroup H of P of order 4 not having a supersoluble supplement in G is also π -quasinormally embedded in G .

Assume that P has a subgroup H of order $|U|$ and H neither has a supersoluble supplement in G nor is π -quasinormally embedded in G . Then G has a normal subgroup M such that $G = HM$ and $|G : M| = p$ by Lemma 2.2 (4), so $G/N \cap M \in \mathcal{F}$. Hence the hypothesis is still true for $(G, N \cap M)$ by $|P : U| > p$ and Lemma 2.2. If $M \cap N = N$, then $N \leq M$, so $G = HM = NM = M$, a contradiction. Thus $|N \cap M| < |N|$, so $|G| + |N \cap M| < |G| + |N|$, contrary to the minimality of $|G| + |N|$. Similarly, we can prove the second statement of **Step 4**.

Step 5. If L is a minimal normal subgroup of G and $L \leq P$, then $|L| \leq |U|$.

Suppose that $|L| > |U|$, then every subgroup H of L of order $|U|$ is *CISE*-normal in G . If H has a supersoluble supplement M in G , then $MH = G$. Since $L = L \cap HM = H(L \cap M)$ and $|L| > |U|$, $L \cap M \neq 1$. Obviously, $L \cap M \triangleleft ML = MH = G$. By the minimality of L , we get $L \cap M = L$, then $L \leq M$. So $G = MH = ML = M$ is supersoluble, a contradiction. Thus every subgroup H of L of order $|U|$ is *ISE*-normal in G . But by Lemma 2.4, L is not a minimal normal subgroup of G , a contradiction.

Step 6. If either $N = G$ or $N = P$ and L is an abelian minimal normal subgroup of G contained in N , then the hypothesis is still true for $(G/L, N/L)$ and so $G/L \in \mathcal{F}$.

Assume that $|P : U| = p$. By the hypothesis, every maximal subgroup of P is *CISE*-normal in G . Let T/L be a maximal subgroup of PL/L , then $p = |(PL/L) : (T/L)|$, and $T = PL \cap T = (P \cap T)L$. Let $P_1 = P \cap T$, then $P_1 \cap L = P \cap T \cap L = P \cap L$, so $p = |PL : T| = |PL : (P \cap T)L| = |P : P \cap T| = |P : P_1|$, thus P_1 is *CISE*-normal in G . If P_1 has a supersoluble supplement M in G , then $G = P_1M$, and $G/L = P_1M/L = TM/L = (T/L)(ML/L)$. Since $ML/L \cong M/M \cap L$ is supersoluble, T/L has a supersoluble supplement ML/L in G/L . Suppose that P_1 is *ISE*-normal in G , then there is a subnormal subgroup B of G such that $G = P_1B$ and $P_1 \cap B$ is π -quasinormally embedded in G . We have $P_1L \cap BL = (P_1L \cap B)L$. Now let $\pi(G) = \{p_1, p_2, \dots, p_n\}$ where $p_1 = p$, and B_{p_i} denotes a Sylow p_i -subgroup of B ($i = 2, \dots, n$). Then B_{p_i} is a Sylow p_i -subgroup of G , hence $B_{p_i} \cap N$ is a Sylow p_i -subgroup of N ($i = 2, \dots, n$). Write $V = \langle L \cap B_{p_2}, \dots, L \cap B_{p_n} \rangle$, then $V \leq B$ and $L = (P_1 \cap L)V$, thus $P_1L \cap BL = (P_1L \cap B)L = (P_1V \cap B)L = (P_1 \cap B)V L = (P_1 \cap B)L$. It follows from Lemma 2.1(b) that $(P_1L/L) \cap (BL/L) = (P_1 \cap B)L/L$ is π -quasinormally embedded in G/L . Therefore T/L is *CISE*-normal in G/L , then the hypothesis is true for $(G/L, N/L)$.

Assume that $|L| < |U|$. If $N = G$, then L is a p -subgroup, then for $L < H$ with $|H| = |U|$, we have H/L is *CISE*-normal in G/L by Lemma 2.2 (2) and $1 < |H/L| < |P/L|$. If $N = P$, then by Lemma 2.2 (2), we have H/L is *CISE*-normal in G/L . Hence the hypothesis is still true for $(G/L, N/L)$.

So let $|L| = |U|$ and $|P : U| > p$. Then by **Step 4** every subgroup H of P with order $|H| = |U|$ not having a supersoluble supplement in G is π -quasinormally embedded in G , and if P is a non-abelian 2-group and $|U| = 2$, then every cyclic subgroup H of P with order 4 not having a supersoluble supplement in G is also π -quasinormally embedded in G . By **Step 3**, L is non-cyclic, hence every subgroup of G containing L is not cyclic. Let $L < K \leq P$, where $|K : L| = p$. Since K is non-cyclic, it has a maximal subgroup $M \neq L$. If M has a supersoluble supplement in G , then K has a supersoluble supplement in G . If M is π -quasinormally embedded in G , so $K = LM$ does by Lemma 2.1(b). Thus if P/L is p -group ($p > 2$) or an abelian 2-group or a non-abelian 2-group with $|U| > 2$, the hypothesis is true for $(G/L, N/L)$ by Lemma 2.2 (2) and **Step 4**. If P/L is a non-abelian 2-group and $|U| = 2$, then P is a non-abelian 2-group and so every cyclic subgroup of P with order 4 not having a supersoluble supplement in G is π -quasinormally embedded in G . In this case, using the same method as above, one can show that every subgroup X of P containing L such that X/L is

a cyclic subgroup of order 4 either has a supersoluble supplement in G or is π -quasinormally embedded in G . Thus again the hypothesis is still true for $(G/L, N/L)$.

Step 7. N is solvable.

By **Step 1** and the choice of G we only need consider the case $N = G$. Let $2 = \min \pi(N)$. Then the Sylow 2-subgroup P of N are not cyclic by **Step 2**. Assume that $|P : U| = 2$. Let G be a counterexample of minimal order. By **Step 6** and Lemma 2.7, we get G has the unique minimal normal subgroup L of G such that G/L is solvable and $L \neq 1$. Suppose that $L \cap P \leq \Phi(P)$, then L is 2-nilpotent by J. Tate Theorem ([10, Theorem 4.4.7]), so G is solvable, a contradiction. Then $L \cap P \not\leq \Phi(P)$, so there exists a maximal subgroup P_1 of P such that $(L \cap P)P_1 = P$. By the hypothesis, P_1 is *CISE*-normal in G . If P_1 is *ISE*-normal in G , then there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T$ is π -quasinormally embedded in G . So there exists a π -quasinormal subgroup K of G such that $P_1 \cap T \in \text{Syl}_2(K)$. Assume that $K_G \neq 1$, then $L \leq K_G \leq K$, so $P_1 \cap T \cap L \in \text{Syl}_2(L)$ and $P_1 \cap T \cap L \leq P_1 \cap L \leq P \cap L$. Thus $P_1 \cap T \cap L = P \cap L = P_1 \cap L$, hence $P = (P \cap L)P_1 = P_1$, a contradiction. So $K_G = 1$. By Lemma 2.3, $P_1 \cap T$ is π -quasinormal in G . If $P_1 \cap T \neq 1$, then $P_1 \cap T \leq O_2(G)$, $O_2(G) \neq 1$, so $L \leq O_2(G)$, thus G is solvable, a contradiction. If $P_1 \cap T = 1$, then $2 \mid |T|$, but $4 \nmid |T|$, so T is solvable, thus G is solvable by the subnormality of T , a contradiction. Thus every maximal of P has a supersoluble supplement in G , then G is q -closed by [21, Lemma 2.2] and **Step 2**, where $q = \max \pi(G)$, so G is solvable, a contradiction.

Assume that $|P : U| > 2$. By **Step 4**, every subgroup H of P of order $|U|$ not having a supersoluble supplement in G is π -quasinormally embedded in G . If P is a non-abelian 2-group and $|U| = 2$, then every subgroup H of P of order 4 not having a supersoluble supplement in G is also π -quasinormally embedded in G . By **Step 1** and **Step 6**, we may assume that $O_{2'}(G) = 1$ and $O_2(G) = 1$. Suppose that H is π -quasinormally embedded in G , then there exists a π -quasinormal subgroup K such that $H \in \text{Syl}_2(K)$. If $K_G = 1$, then H is π -quasinormal subgroup of G by Lemma 2.3, so $H \leq O_2(G)$, thus $O_2(G) \neq 1$, a contradiction. If $K_G \neq 1$, we choose $H \triangleleft H_1 \leq P$, then H_1K_G satisfies the hypothesis. By the first paragraph discussion, we have H_1K_G is solvable, so is K_G . Thus $O_2(K_G) \neq 1$ or $O_{2'}(K_G) \neq 1$, hence $O_2(G) \neq 1$ or $O_{2'}(G) \neq 1$, a contradiction. Therefore, every subgroup H of P of order $|U|$ has a supersoluble supplement in G , that is, every maximal subgroup of P has a supersoluble supplement in G , then G is q -closed by [21, Lemma 2.2] and **Step 2**, where $q = \max \pi(G)$, so G is solvable, the final contradiction. This contradiction implies that G is solvable.

Step 8. Let $q = \max \pi(N)$, then N is q -closed.

Assume that N_q is not normal in N and let N be a counterexample with $|N| + |G|$ of minimal order for q -closed.

By **Step 7**, we can assume that $\{N_r \mid r \in \pi(N)\}$ is a Sylow system of N . Let $K = N_q N_r$ for any $r \in \pi(N)$ with $r \neq q$. By Step 1, the hypothesis is still true for (K, K) . If $|\pi(N)| > 3$ or $G \neq N$, then $N_q \triangleleft K$, which implies that $N_q \triangleleft N$, a contradiction. Thus we may assume that $G = N$ and $|G| = p^a q^b$.

Let L be a minimal normal subgroup of G , then G/L is q -closed by **Step 6**. Since q -closed groups are a saturated formation, we may assume that $L \not\leq \Phi(G)$ and L is the only minimal normal subgroup of G . If L is a q -group, then $G_q \triangleleft G$, where G_q denotes a Sylow q -subgroup of G , a contradiction. Thus $L \leq P$ and so $L \leq O_p(G)$. Now we show that $L = O_p(G)$. Let W be a maximal subgroup of G such that $L \not\leq W$, then $G = LW$ and $L \cap W = 1$. Since $W \cong G/L$, W is q -closed. By $L \leq O_p(G)$, $G = LW = O_p(G)W$. From $O_p(G) \leq F(G) \leq C_G(L)$, it is easy to see that L and W normalize $O_p(G) \cap W$, thus $O_p(G) \cap W \triangleleft G$. So

$O_p(G) \cap W = 1$ or $L \leq O_p(G) \cap W$. If the later case happened, then $L \leq W$, that is, $G = L \rtimes W = W$, a contradiction. So $O_p(G) \cap W = 1$, thus $|O_p(G)| = |G : W| = |L|$, hence $L = O_p(G)$.

Assume that $|P : U| = p$. For every maximal subgroup A of P containing L we have $G = AW$, hence A has a supplement W in G such that W is q -closed. If every maximal subgroup of P not containing L has a supersolvable supplement M in G , then M is q -closed. Thus every maximal subgroup of P has a supplement M such that M is q -closed. By [21, Lemma 2.2], G is q -closed, a contradiction. Thus there exists one maximal subgroup S of P neither containing L nor having a supersolvable supplement in G . By the hypothesis, S is *ISE*-normal in G . It follows that there exists a subnormal subgroup K of G such that $G = SK$ and $S \cap K$ is π -quasinormally embedded in G . It is easy to prove that all Sylow q -subgroups of G are in every subnormal subgroup of G containing a Sylow q -subgroup of G . Since for any $g \in G$, K^g is subnormal in G , all Sylow q -subgroups of G are in K^g , so $K_G \neq 1$ and $G_q \leq K_G$. By the uniqueness of L , $L \leq K_G \leq K$. If $S \cap K = 1$, then $|K| = pq^b$. Since $q > p$, $K_q \triangleleft K$, which implies that G is q -closed, a contradiction. If $S \cap K \neq 1$, then $S \cap K$ is a Sylow p -subgroup of some π -quasinormal subgroup K_1 of G . Now we claim that $(K_1)_G = 1$. If $(K_1)_G \neq 1$, by the uniqueness of L , we have $L \leq (K_1)_G$, $O_p((K_1)_G) \neq 1$. Thus $L \leq O_p((K_1)_G) \leq (K_1)_p = S \cap K \leq S$, where $(K_1)_p \in \text{Syl}_p(K_1)$, which contradicts the choice of S . Hence $S \cap K$ is π -quasinormal in G by Lemma 2.3, so $S \cap K$ is a subnormal subgroup of G , thus $S \cap K \leq O_p(G) = L$, then $S \cap K \leq S \cap L \leq S \cap K$, so $S \cap K = S \cap L$. It is clear that $S \cap K = S \cap L$ is normalized by P . Since $S \cap K$ is also a subnormal Sylow subgroup of $(S \cap K)G_q$, $S \cap K$ is normalized by G_q . By $G = PG_q$, $S \cap K \triangleleft G$, then $L \leq S \cap K \leq S$, which contradicts the choice of S .

Therefore we may assume that $|P : U| > p$, then by **Step 4**, every subgroup H of P satisfying $|H| = |U|$ and not having a supersolvable supplement in G is a π -quasinormally embedded subgroup. If H has not supersolvable supplement in G , then H is a π -quasinormally embedded subgroup of G , so there exists a π -quasinormal subgroup K of G such that $K_p = H$, where $K_p \in \text{Syl}_p(K)$. If $K_G = 1$, then H is a π -quasinormal subgroup of G by Lemma 2.3, so H is subnormal in G . Since every subnormal p -subgroup is contained in $O_p(G)$ and $O_p(G) = L$ by the previous argument, $H \leq L$. On the other hand, by **Step 5**, $|H| \geq |L|$, so $L = H$. If $K_G \neq 1$, then $L \leq K_G$ and $L \leq K_p = H$. Summing up, we have $L \leq H$, so $G = WL = WH$ and $W \cap L = 1$, H has a q -closed supplement W in G . If H has a supersolvable supplement M in G , then M is also q -closed. We have obtained that every subgroup H of order $|U|$ in P has a q -closed supplement in G . Since every maximal subgroup of P contains at least one subgroup H such that $|H| = |U|$, we get that every maximal subgroup of P has a q -closed supplement in G . By [21, Lemma 2.2], G is q -closed, a final contradiction.

Step 9. Final contradiction.

Let $q = \max \pi(N)$ and Q be a Sylow q -subgroup of N . Then by **Step 8**, Q is normal in N and so we may assume that $Q = N = P$ by **Step 1**. Let L be a minimal normal subgroup of G contained in P . Then by **Step 6**, L is the only minimal normal subgroup of G contained in P and so $L = O_p(G) = P$. But by Lemma 2.4, L is not a minimal normal subgroup of G , a contradiction. This contradiction completes the proof of this theorem. \square

Proposition 3.3 (a) *Let H be a p -subgroup of $F(G)$. If H is an *ISE*-normal subgroup of G , then it is also a weakly s -permutable subgroup of G .*

(b) If every *ISE*-normal subgroup of G is also weakly s -permutable in G , then $G/O_p(G)$ is p -nilpotent for arbitrary $p \in \pi(G)$.

Proof. (a) Let H be an *ISE*-normal subgroup of G , then there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K$ is π -quasinormally embedded in G . Since $H \leq F(G)$, we have $H \leq O_p(G)$, so $H \cap K \leq O_p(G)$, thus $H \cap K$ is π -quasinormal in G by Lemma 2.3, so $H \cap K \leq H_{sG}$, consequently, H is weakly s -permutable in G , as desired.

(b) Because every Sylow subgroup of G is always normally embedded in G , it is of course, *ISE*-normal, it follows that G_p is weakly s -permutable in G . By definition in [21], there exists a subnormal subgroup T such that $G = G_p T$ and $G_p \cap T \leq (G_p)_{sG}$. Since $(G_p)_{sG} \leq O_p(G)$, we have $G_p \cap T = T \cap O_p(G)$, so

$$|(G/O_p(G)) : (TO_p(G)/O_p(G))| = |G : TO_p(G)| = |G_p T : TO_p(G)| = |G_p : O_p(G)|.$$

Hence $TO_p(G)/O_p(G)$ is a Hall p' -subgroup of $G/O_p(G)$. Since $TO_p(G)$ is subnormal in G , we get $TO_p(G)/O_p(G)$ is a normal p -complement of $G/O_p(G)$, so $G/O_p(G)$ is p -nilpotent. \square

Theorem 3.4 Let \mathcal{F} be a saturated formation containing \mathcal{U} and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every non-cyclic Sylow subgroup P of $F^*(N)$ has a subgroup U with $1 < |U| < |P|$ such that every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is *CISE*-normal in G , then $G \in \mathcal{F}$.

Proof. Assume that the result is false and consider a counterexample (G, N) with minimal $|G| + |N|$. If all Sylow subgroups of $F^*(N)$ are cyclic, then by Theorem 3.1, $G \in \mathcal{F}$. Next, we assume always that $F^*(N)$ has a non-cyclic Sylow subgroup. We claim that $F^*(N) = F(N) \neq 1$. In fact, $F^*(N)$ is supersolvable by Theorem 3.2. So $F^*(N) = F(N) \neq 1$ by Lemma 2.8 (2), (3). By the hypothesis, every non-cyclic Sylow subgroup P of $F^*(N) = F(N)$ has a subgroup U with $1 < |U| < |P|$ such that every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is *CISE*-normal in G . If H has supersolvable supplement M in G , then $G = HM = PM$, so $G/P \cong M/P \cap M \in \mathcal{U} \subseteq \mathcal{F}$, thus (G, P) satisfy the condition of Theorem 3.2, hence $G \in \mathcal{F}$, a contradiction. Thus every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is *ISE*-normal in G , by Proposition 3.3 (a), every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is weakly s -permutable in G . Applying [20, Corollary 5.4], $G \in \mathcal{F}$, a contradiction. This contradiction completes the proof of this theorem. \square

It is well known that if a subgroup H of G is c -normal, c^* -normal, S -permutable, S -quasinormally embedded in G respectively and has a supersolvable supplement in G , then H is *CISE*-normal in G . Hence [21, Corollary 5.1~2.24] are corollaries of our Theorem 3.2 and Theorem 3.4. Moreover, we have the following corollaries.

Corollary 3.5 (See [2, Theorem 3.3]) Let \mathcal{F} be a saturated formation containing \mathcal{U} , and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of H are S -quasinormally embedded in G .

Corollary 3.6 (See [2, Corollary 3.4]) *Let \mathcal{F} be a saturated formation containing \mathcal{U} and G a solvable group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of the Sylow subgroups of $F(H)$ are S -quasinormally embedded in G .*

Corollary 3.7 (See [15, Theorem 3.2]) *Let \mathcal{F} be a saturated formation containing \mathcal{U} and G be a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ are π -quasinormally embedded in G , then $G \in \mathcal{F}$.*

Corollary 3.8 (See [15, Theorem 1.1]) *Let \mathcal{F} be a saturated formation containing \mathcal{U} , and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(H)$ are π -quasinormally embedded in G .*

Corollary 3.9 (See [15, Theorem 1.2]) *Let \mathcal{F} be a saturated formation containing \mathcal{U} , and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and the cyclic subgroups of prime order or order 4 of $F^*(H)$ are π -quasinormally embedded in G .*

Corollary 3.10 (See [25, Theorem 4.1]) *Let \mathcal{F} be a saturated formation containing \mathcal{U} , and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(H)$ are c^* -normal in G .*

Corollary 3.11 (See [13, Theorem 3.5]) *Let G be a group and \mathcal{F} be a saturated formation containing \mathcal{U} . Then $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup H such that $G/H \in \mathcal{F}$ and every maximal subgroup of all Sylow subgroups of $F(H)$, the Fitting subgroup of H , is either c -normal or S -quasinormally embedded in G .*

Corollary 3.12 (See [13, Theorem 3.2]) *Let \mathcal{F} be a saturated formation containing \mathcal{U} and let G be a group. Then G is in \mathcal{F} if and only if there is a normal subgroup H such that $G/H \in \mathcal{F}$ and every maximal subgroup of all Sylow subgroups of H is either c -normal or S -quasinormally embedded in G .*

In [14], the following concept was introduced: Let G be a group. A subgroup H of G is said to be an SS -quasinormal subgroup (Supplement-Sylow-quasinormal subgroup) of G if there is a supplement B to H in G such that H permutes with every Sylow subgroup of B . Let H be a nilpotent subgroup of G and $H \leq F(G)$. We know that H is SS -quasinormal in G if and only if H is S -quasinormally embedded in G . our final corollary:

Corollary 3.13 *Let \mathcal{F} be a saturated formation containing \mathcal{U} and G a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of the Sylow subgroups of $F(H)$ (all minimal subgroups and all cyclic subgroups with order 4 of $F(E)$) are SS -quasinormal in G , then $G \in \mathcal{F}$.*

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