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# On \alpha-skew McCoy modules

JIAN CUI

JIANLONG CHEN

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# On $\alpha$-skew McCoy modules*

*Jian Cui and Jianlong Chen*

**Abstract**

Let $\alpha$ be a ring endomorphism. Extending the notions of McCoy modules and $\alpha$-skew McCoy rings, we introduce the notion of $\alpha$-skew McCoy modules, which can also be regarded as a generalization of $\alpha$-skew Armendariz modules. A number of illustrative examples are given. Various properties of these modules are developed, and equivalent conditions for $\alpha$-skew McCoy modules are established. Furthermore, we study the relationship between a module and its polynomial module.

**Key Words:** $\alpha$-skew McCoy module; $\alpha$-skew Armendariz module; $\alpha$-skew McCoy ring; polynomial module; zip module

## 1. Introduction

Throughout this paper all rings considered are associative with unity and all modules are unitary right modules. $R[x]$ denotes the polynomial ring over a ring $R$ and $M[x]$ denotes the polynomial module over a module $M$. Let $\mathbb{Z}_n$ be the ring of integers modulo $n$. The symbol $I_n$ stands for the $n \times n$ identity matrix. For a set $X \subseteq M$, $r_R(X)$ stands for the right annihilator of $X$ in $R$.

Rege and Chhawchharia [23] and Nielsen [22] independently called a ring $R$ *right McCoy* if whenever $f(x)g(x) = 0$ for $f(x) \in R[x]$ and $g(x) \in R[x]\backslash\{0\}$, there exists a nonzero $r \in R$ with $f(x)r = 0$. Left McCoy rings are defined similarly. A ring is said to be *McCoy* if it is both right and left McCoy. The term "McCoy ring" was coined because McCoy [21] had shown that every commutative ring satisfies the above mentioned condition. The class of McCoy rings properly contains the class of Armendariz rings. (These rings are defined through the condition: whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for every $i$ and $j$. See [23] for basic results on Armendariz rings). Recall that a ring $R$ is *semicommutative* provided $ab = 0$ implies $aRb = 0$ for $a, b \in R$. In [13] it was claimed that all semicommutative rings were McCoy. However, Hirano's claim assumed that $R[x]$ is semicommutative if $R$ is semicommutative, and this was shown to be false in [16]. In 2006, Nielsen [22] gave an example of semicommutative ring which is not right McCoy. Some other properties on McCoy rings have appeared in [5], [11], [18], [20], [23, 24, 25], etc. As a generalization of McCoy rings (resp., Armendariz rings), McCoy

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modules [7] (resp., Armendariz modules [4]) were introduced (Maybe the first result, without a naming, McCoy module, obtained in [2]). A module $M_R$ is said to be *McCoy* (resp., *Armendariz*) if whenever polynomials $m(x)=\sum_{i=0}^{p} m_i x^i \in M[x]$ and $g(x)=\sum_{j=0}^{q} b_j x^j \in R[x]\backslash\{0\}$ satisfy $m(x)g(x)=0$, there exists $r\in R\backslash\{0\}$ such that $m(x)r=0$ (resp., $m_i b_j=0$ for every $i$ and $j$). Armendariz modules are clearly McCoy.

Given an endomorphism $\alpha$ of a ring $R$, the *skew polynomial* ring $R[x;\alpha]$ consists of the polynomials in $x$ with coefficients in $R$ written on the left, subject to the relation $xr=\alpha(r)x$ for all $r\in R$. Recently, Başer, Kwak and Lee [3] called a ring $R$ *$\alpha$-skew McCoy* with respect to an endomorphism $\alpha$ of $R$ if for any nonzero polynomials $f(x)$ and $g(x)\in R[x;\alpha]$, $f(x)g(x)=0$ implies $f(x)r=0$ for some nonzero $r\in R$. This notion generalized both concepts of McCoy rings and $\alpha$-skew Armendariz rings (see [14]).

In this paper, we introduce the notion of $\alpha$-skew McCoy modules as a straightforward extensions to modules. Many examples of $\alpha$-skew McCoy modules are given, and properties of this class of modules are investigated. Various results of $\alpha$-skew McCoy rings are extended to $\alpha$-skew McCoy modules. We also study the relationship between a module and its polynomial module.

## 2. $\alpha$-skew McCoy modules

Let $\alpha$ be an endomorphism of a ring $R$ and $M$ be a right $R$-module. $M[x;\alpha]=\{\sum_{i=0}^{s} m_i x^i; s\geq 0, m_i\in M\}$ is an abelian group under an obvious addition operation. Moreover, $M[x;\alpha]$ becomes a module over $R[x;\alpha]$ under the following scalar product operation: For $m(x)=\sum_{i=0}^{p} m_i x^i\in M[x;\alpha]$ and $f(x)=\sum_{j=0}^{q} a_j x^j\in R[x;\alpha]$, $m(x)f(x)=\sum_k(\sum_{i+j=k} m_i\alpha^i(a_j))x^k$. According to Zhang and Chen [27], $M$ is *$\alpha$-skew Armendariz* if $m(x)f(x)=0$ where $m(x)=\sum_{i=0}^{p} m_i x^i\in M[x;\alpha]$ and $f(x)=\sum_{j=0}^{q} a_j x^j\in R[x;\alpha]$ implies $m_i\alpha^i(a_j)=0$ for all $i$ and $j$.

**Definition 2.1** *Let $\alpha$ be an endomorphism of a ring $R$ and $M$ be an $R$-module. $M$ is called $\alpha$-skew McCoy if whenever $m(x)g(x)=0$ where $m(x)=\sum_{i=0}^{p} m_i x^i\in M[x;\alpha]$ and $g(x)=\sum_{j=0}^{q} b_j x^j\in R[x;\alpha]\backslash\{0\}$, there exists a nonzero element $r\in R$ such that $m(x)r=0$ (i.e., $m_i\alpha^i(r)=0$ for all $i$).*

**Remark 2.2** (1) *$M$ is a McCoy $R$-module if and only if $M$ is $1_R$-skew McCoy, where $1_R$ is the identity endomorphism of $R$.*
(2) *A ring $R$ is $\alpha$-skew McCoy if and only if $R_R$ is an $\alpha$-skew McCoy module.*
(3) *An $R$-module $M$ is $\alpha$-skew McCoy if and only if, for all $m(x)\in M[x;\alpha]$, $r_{R[x;\alpha]}(m(x))\neq 0$ implies that $r_{R[x;\alpha]}(m(x))\bigcap R\neq 0$.*

Any $\alpha$-skew Armendariz module is obviously $\alpha$-skew McCoy, the falsity of the converse can be inferred from [17, Example 3] or [23, Remark 4.3].

**Example 2.3** (1) *Let $R=\mathbb{Z}_2\oplus\mathbb{Z}_2$, and $\alpha: R\to R$ be defined by $\alpha((a,b))=(b,a)$. Then $R_R$ is McCoy but not $\alpha$-skew McCoy by [3, Example 4] and Remark 2.2(2).*
(2) *For any given ring $S$, let $R=\mathbb{T}_2(S)$ be the ring of all $2\times 2$ upper triangular matrices over $S$. Then $R_R$ is not McCoy by [5, Proposition 10.2]. Define $\alpha: R\to R$ by $\alpha(\left(\begin{smallmatrix} a & b\\ 0 & c\end{smallmatrix}\right))=\left(\begin{smallmatrix} a & 0\\ 0 & 0\end{smallmatrix}\right)$. We conclude that $R_R$ is*

*$\alpha$-skew McCoy. Indeed, suppose that $F(x)G(x)=0$ for $F(x)=\sum_{i=0}^{p}A_ix^i\in R[x;\alpha]$ and $G(x)=\sum_{j=0}^{q}B_jx^j\in R[x;\alpha]\backslash\{0\}$. We may assume that $B_0\neq 0$ and write $B_0=\left(\begin{smallmatrix}b_1 & b_2\\ 0 & b_3\end{smallmatrix}\right)$. If $b_1=0$ then let $C=\left(\begin{smallmatrix}0 & b_2\\ 0 & b_3\end{smallmatrix}\right)$, otherwise, let $C=\left(\begin{smallmatrix}0 & b_1\\ 0 & 0\end{smallmatrix}\right)$. It easily checks that $A_i\alpha^i(C)=0$ for both cases, where $i=0,\ldots,p$.*

An ideal $I$ of a ring $R$ is an *$\alpha$-ideal* if $\alpha(I)\subseteq I$, where $\alpha$ is an endomorphism of $R$.

**Proposition 2.4** (1) *Every submodule of an $\alpha$-skew McCoy module is $\alpha$-skew McCoy. In particular, if $I$ is a right ideal of an $\alpha$-skew McCoy ring $R$, then $I$ is $\alpha$-skew McCoy.*
(2) *$M$ is an $\alpha$-skew McCoy module if and only if every finitely generated submodule of $M$ is $\alpha$-skew McCoy.*
(3) *For any index set $\Gamma$, if $M_i$ is an $\alpha_i$-skew McCoy $R_i$-module for each $i\in\Gamma$, then $\prod_{i\in\Gamma}M_i$ is an $\alpha$-skew McCoy $\prod_{i\in\Gamma}R_i$-module, where $\alpha=(\alpha_i)_{i\in\Gamma}$.*
(4) *Let $I$ be any nonzero $\alpha$-ideal of a ring $R$, then $R/I$ is an $\alpha$-skew McCoy $R$-module.*

**Proof.** (1) - (3) are obvious. (4) For each $\overline{f(x)}\in(R/I)[x;\alpha]$, take any nonzero $r\in I\ (\subseteq R)$. Since $\alpha(I)\subseteq I$, $f(x)r\in I[x;\alpha]$, i.e., $\overline{f(x)}r=0$. □

**Remark 2.5** *The condition "$I$ is an $\alpha$-ideal" in Proposition 2.4(4) is necessary. Take the ring and the ring endomorphism in Example 2.3(1). Let $I=0\oplus\mathbb{Z}_2\subseteq R$. Then $I$ is an ideal but $\alpha(I)\subsetneq I$. Note that $R/I\cong\mathbb{Z}_2\oplus 0$. We show that $R/I$ is not $\alpha$-skew McCoy as a right $R$-module. For $f(x)=(1,0)+(1,0)x\in(\mathbb{Z}_2\oplus 0)[x;\alpha]$ and $g(x)=(0,1)+(1,0)x\in R[x;\alpha]$, $f(x)g(x)=0$. However, $f(x)r=0$ implies $r=0$ for $r\in R$.*

A module $M_R$ is *semicommutative* [4] if for any $m\in M$ and $a\in R$, $ma=0$ implies $mRa=0$. In [27], a module $M_R$ with a ring endomorphism $\alpha$ of $R$ is called *$\alpha$-semicommutative* if whenever $ma=0$ for $m\in M$ and $a\in R$, $mR\alpha(a)=0$; a ring $R$ is *$\alpha$-semicommutative* if $R_R$ is $\alpha$-semicommutative. We can infer that $1_R$-semicommutative modules need not be $1_R$-McCoy from Section 3 of [22].

**Proposition 2.6** *Let $\alpha$ be an endomorphism of a ring $R$. Then a semicommutative module $M_R$ with $m\alpha(a)=0$ whenever $m\alpha(a)a=0$ for $m\in M$ and $a\in R$ is $\alpha$-skew Armendariz.*

**Proof.** Firstly, we show that $M_R$ is $\alpha$-semicommutative. Let $ma=0$ for $m\in M$ and $a\in R$. Then $mRa=0$. Clearly, $m\alpha(a)a=0$. Thus $m\alpha(a)=0$ and $mR\alpha(a)=0$ by the hypotheses.

Let $m(x)=\sum_{i=0}^{p}m_ix^i\in M[x;\alpha]$ and $f(x)=\sum_{j=0}^{q}a_jx^j\in R[x;\alpha]\backslash\{0\}$ with $m(x)f(x)=0$. Then $\sum_{i+j=k}m_i\alpha^i(a_j)=0$ for $k=0,\ldots,p+q$. So $m_0a_0=0$ and $m_0a_1+m_1\alpha(a_0)=0$, and then $m_0a_1a_0+m_1\alpha(a_0)a_0=0$. Since $M_R$ is semicommutative, $m_0a_1a_0=0$. So we have $m_1\alpha(a_0)a_0=0$, and $m_1\alpha(a_0)=0$ by the hypothesis. Hence $m_0a_1=m_1\alpha(a_0)=0$. Assume that $s\geq 1$ and $m_i\alpha^i(a_j)=0$ for all $i,j$ with $i+j\leq s$. Note that

$$m_0a_{s+1}+m_1\alpha(a_s)+\cdots+m_s\alpha^s(a_1)+m_{s+1}\alpha^{s+1}(a_0)=0, \tag{2.1}$$

where $m_i$ and $a_j$ are 0 if $i>p$ and $j>q$. Multiplying (2.1) by $\alpha^s(a_0)$ on the right yields

$$m_0a_{s+1}\alpha^s(a_0)+m_1\alpha(a_s)\alpha^s(a_0)+\cdots+m_s\alpha^s(a_1)\alpha^s(a_0)+m_{s+1}\alpha^{s+1}(a_0)\alpha^s(a_0)=0. \tag{2.2}$$

Since $M_R$ is $\alpha$-semicommutative and $m_i\alpha^i(a_0) = 0$ for $i \leq s$, it follows that $m_iR\alpha^s(a_0) = 0$. Thus (2.2) becomes $m_{s+1}\alpha^{s+1}(a_0)\alpha^s(a_0) = m_{s+1}\alpha(\alpha^s(a_0))\alpha^s(a_0) = 0$, which implies $m_{s+1}\alpha^{s+1}(a_0) = 0$ by the assumption. So (2.1) becomes

$$m_0a_{s+1} + m_1\alpha(a_s) + \cdots + m_{s-1}\alpha^{s-1}(a_2) + m_s\alpha^s(a_1) = 0. \tag{2.3}$$

Analogously, multiplying (2.3) by $\alpha^{s-1}(a_1)$ on the right, one obtains

$$m_0a_{s+1}\alpha^{s-1}(a_1) + m_1\alpha(a_s)\alpha^{s-1}(a_1) + \cdots + m_{s-1}\alpha^{s-1}(a_2)\alpha^{s-1}(a_1) + m_s\alpha^s(a_1)\alpha^{s-1}(a_1) = 0.$$

The similar argument as the above reveals that $m_s\alpha^s(a_1)\alpha^{s-1}(a_1) = 0$. Thus $m_s\alpha^s(a_1) = 0$. Continuing this process, we have $m_s\alpha^s(a_1) = \cdots = m_1\alpha(a_s) = m_0a_{s+1} = 0$. So we prove that $m_i\alpha^i(a_j) = 0$ for all $i, j$ with $i + j \leq s + 1$. By the induction principle, $m_i\alpha^i(a_j) = 0$ for every $i$ and $j$. □

The converse of Proposition 2.6 is not true. We use the ring given in [14].

**Example 2.7** *Let* $R = \{\left(\begin{smallmatrix} a & \overline{b} \\ 0 & a \end{smallmatrix}\right) | a \in \mathbb{Z}, \overline{b} \in \mathbb{Z}_4\}$. *Clearly,* $R$ *is commutative. Let* $\alpha : R \to R$ *be an endomorphism defined by*

$$\alpha(\left(\begin{smallmatrix} a & \overline{b} \\ 0 & a \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & -\overline{b} \\ 0 & a \end{smallmatrix}\right).$$

*Then* $R_R$ *is* $\alpha$*-skew Armendariz by [14, Example 7]. However,* $I_2\alpha(\left(\begin{smallmatrix} 0 & \overline{b} \\ 0 & 0 \end{smallmatrix}\right))\left(\begin{smallmatrix} 0 & \overline{b} \\ 0 & 0 \end{smallmatrix}\right) = 0$*, but* $I_2\alpha(\left(\begin{smallmatrix} 0 & \overline{b} \\ 0 & 0 \end{smallmatrix}\right)) \neq 0$ *in case* $\overline{b} \neq 0$.

Let $\alpha$ be an endomorphism of a ring $R$ and $M$ be an $R$-module. According to Lee and Zhou [19], $M$ is called $\alpha$*-reduced* if the following conditions hold: For any $m \in M$ and $a \in R$, (1) $ma = 0$ implies $mRa = mR\alpha(a) = 0$; (2) $ma\alpha(a) = 0$ implies $ma = 0$; (3) $ma^2 = 0$ implies $ma = 0$. A ring is *reduced* if $R_R$ is $1_R$-reduced.

**Remark 2.8** *Assume that* $M$ *is an* $\alpha$*-reduced* $R$*-module. For some* $m \in M$ *and* $a \in R$ *with* $m\alpha(a)a = 0$*, by* (1) *we have* $m\alpha(a)\alpha(a) = m[\alpha(a)]^2 = 0$*, and so* $m\alpha(a) = 0$ *by applying condition* (3)*. In view of Proposition 2.6, it is clear that any* $\alpha$*-reduced module is* $\alpha$*-skew Armendariz and is therefore* $\alpha$*-skew McCoy.*

**Proposition 2.9** *Let* $\alpha$ *be an endomorphism of a reduced ring* $R$*. Then every* $\alpha$*-semicommutative module* $M_R$ *is* $\alpha$*-skew McCoy.*

**Proof.** Since $M_R$ is $\alpha$-semicommutative, it is easy to obtain that for $m \in M$ and $a \in R$,

$$ma = 0 \Rightarrow mR\alpha^s(a^t) = 0 \text{ for any } s, t \geq 1. \tag{2.4}$$

Suppose that $m(x) = m_0 + m_1x + \cdots + m_px^p \in M[x; \alpha]$, $f(x) = a_0 + a_1x + \cdots + a_qx^q \in R[x; \alpha]\backslash\{0\}$ satisfy $m(x)f(x) = 0$. We may assume that $m(x) \neq 0$ and $k$ is minimal such that $m_k \neq 0$, and let $l$ be minimal such that $a_l \neq 0$. Since $m(x)f(x) = 0$, we have the following equations:

$$
\begin{array}{ll}
(0) & m_k\alpha^k(a_l)=0,\\
(1) & m_{k+1}\alpha^{k+1}(a_l)+m_k\alpha^k(a_{l+1})=0,\\
 & \dots\\
(p+q-k-l) & m_p\alpha^p(a_q)=0.
\end{array}
$$

If $\alpha^k(a_l)=0$ then $m(x)a_l=0$, and we are done. Next assume that $\alpha^k(a_l)\neq 0$. Multiplying Eq. (1) by $\alpha^{k+1}(a_l)$ from the right, we obtain $m_{k+1}\alpha^{k+1}(a_l)\alpha^{k+1}(a_l)+m_k\alpha^k(a_{l+1})\alpha^{k+1}(a_l)=0$. Combining Eq. (0) with (2.4), one has $m_k\alpha^k(a_{l+1})\alpha^{k+1}(a_l)=0$. Thus $m_{k+1}\alpha^{k+1}(a_l)\alpha^{k+1}(a_l)=m_{k+1}\alpha^{k+1}(a_l^2)=0$. Continuing this procedure, multiplying Eq. $(i)$ on the right by $\alpha^{k+i}(a_l^i)$ yields $m_{k+i}\alpha^{k+i}(a_l^{i+1})=0$, where $i=1,2,\dots,p-k$. Let $r=a_l^p$. Then $r\neq 0$ since $R$ is reduced. So, by (2.4) we get $m_i\alpha^i(r)=0$ for each $i$, proving that $M_R$ is $\alpha$-skew McCoy. □

In view of [7, Example 2.5], the converse of Proposition 2.9 does not hold generally.

Following [12], an endomorphism $\alpha$ of a ring $R$ is called *compatible* if for each $a,\ b\in R,\ ab=0\Leftrightarrow a\alpha(b)=0$. A ring $R$ is said to be $\alpha$*-compatible* if there exists a compatible endomorphism $\alpha$ of $R$. We define the following:

**Definition 2.10** (1) *An endomorphism $\alpha$ of a ring $R$ is called weakly compatible (or W-compatible for short) if whenever $ab=0$ for $a,\ b\in R$, $a\alpha(b)=0$.*
(2) *An endomorphism $\alpha$ of a ring $R$ is called weakly finitely compatible (or WF-compatible for short) if for a finite number of elements $a_i,b_i\in R$, $\sum_i a_ib_i=0$ implies $\sum_i a_i\alpha(b_i)=0$.*

The following examples reveal the relationships among the above endomorphisms (for a given ring).

**Example 2.11** (1) *Both compatible and WF-compatible endomorphisms of given rings are W-compatible, but the converse is not true. Let $R=\mathbb{Z}_2[x]$, and $\alpha: R\to R$ be defined by $\alpha(f(x))=f(0)$ for $f(x)\in R$. Since $R$ is an integral domain, $g(x)h(x)=0$ implies that either $g(x)=0$ or $h(x)=0$, so $g(x)\alpha(h(x))=0$. Hence $\alpha$ is W-compatible. But $\alpha$ is neither compatible nor WF-compatible. Indeed, let $f_1(x)=x,\ g_1(x)=1,\ f_2(x)=1$ and $g_2(x)=x$. Then $f_2(x)\alpha(g_2(x))=0$ and $f_1(x)g_1(x)+f_2(x)g_2(x)=0$, but both $f_2(x)g_2(x)$ and $f_1(x)\alpha(g_1(x))+f_2(x)\alpha(g_2(x))$ do not equal* 0.
(2) *WF-compatible endomorphisms need not be compatible. Given the ring and the ring endomorphism in Example 2.3(2), it is easy to check that $\alpha$ is WF-compatible. However, since $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\alpha(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right))=0$ and $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\neq 0$, $\alpha$ is not compatible.*
(3) *Let $R=\mathbb{Z}_2[x_1,x_2,\dots]$ be a ring of polynomials in infinitely countably many indeterminates. Define $\alpha: R\to R$ by $x_i\mapsto x_{i+1}$ for $i=1,2,\dots$. Notice that $R$ is an integral domain and $\alpha$ is monic. So $\alpha$ is compatible. Nevertheless, since $x_2x_3+x_3x_2=0$ and $x_2\alpha(x_3)+x_3\alpha(x_2)=x_2x_4+(x_3)^2\neq 0$, $\alpha$ is not WF-compatible.*

For an endomorphism $\alpha$ of a ring $R$, write $\alpha(h(x))=\sum_{i=0}^n\alpha(c_i)x^i$, where $h(x)=\sum_{i=0}^n c_ix^i\in R[x;\alpha]$.

**Lemma 2.12** *Let $\alpha$ be a WF-compatible endomorphism of a ring $R$. If $f(x)g(x)=0$ in $R[x;\alpha]$, then $f(x)\alpha(g(x))=0$.*

**Proof.** Suppose that $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ are elements of $R[x;\alpha]$ with $f(x)g(x) = 0$. Then looking at the degree $k$ part of the equation $f(x)g(x) = 0$ we have $\sum_{i+j=k} a_i\alpha^i(b_j) = 0$. Because $\alpha$ is WF-compatible, $0 = \sum_{i+j=k} a_i\alpha^{i+1}(b_j) = \sum_{i+j=k} a_i\alpha^i(\alpha(b_j))$ for each $k$. Thus $f(x)\alpha(g(x)) = 0$. □

**Lemma 2.13** *Let $R$ be a semicommutative ring and $\alpha$ be a W-compatible endomorphism of $R$. Suppose that $f(x)g(x) = 0$ for nonzero $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ of $R[x;\alpha]$. If $a_l g(x) \neq 0$ for some minimal index $l$ then $a_l^{n+1}\alpha^l(g(x)) = 0$.*

**Proof.** A direct check shows that $R$ is $\alpha$-semicommutative. By hypothesis, $a_k g(x) = 0$ for every $k < l$. So $a_k b_j = 0$ for $j = 0, \dots, n$. Since $\alpha$ is W-compatible, we have $a_k\alpha^k(b_j) = 0$. It follows that $0 = f(x)g(x) = (\sum_{i=l}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j)$. One easily obtains the following system of equations:

$$
\begin{array}{ll}
(l) & a_l\alpha^l(b_0) = 0, \\
(l+1) & a_l\alpha^l(b_1) + a_{l+1}\alpha^{l+1}(b_0) = 0, \\
(l+2) & a_l\alpha^l(b_2) + a_{l+1}\alpha^{l+1}(b_1) + a_{l+2}\alpha^{l+2}(b_0) = 0, \\
 & \cdots \\
(m+n) & a_m\alpha^m(b_n) = 0.
\end{array}
$$

Since $R$ is $\alpha$-semicommutative, by Eq. $(l)$ one has $a_l a_{l+1}\alpha^{l+1}(b_0) = 0$. Now multiplying Eq. $(l+1)$ by $a_l$ on the left yields $a_l^2\alpha^l(b_1) = 0$. Similarly, multiplying Eq. $(l+2)$ by $a_l^2$ from the left, we have $a_l^3\alpha^l(b_2) = 0$ by using the $\alpha$-semicommutativity of $R$. Repeating this process finite times, we obtain $a_l^{j+1}\alpha^l(b_j) = 0$ for $j = 0, \dots, n$. Thus $a_l^{n+1}\alpha^l(b_j) = 0$, and so $a_l^{n+1}\alpha^l(g(x)) = 0$. □

A ring is said to be *right duo* (resp., *left duo*) if all its right (resp., left) ideals are two-sided ideals. It is not difficult to show that one-sided duo rings are semicommutative.

**Lemma 2.14** *Suppose that $R$ is a right duo ring and $\alpha$ is a WF-compatible automorphism of $R$. If $f(x)g(x) = 0$ for nonzero $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ of $R[x;\alpha]$ and $a_k g(x) \neq 0$ for minimal $k \geq 0$, then there exists $h(x) \in R[x;\alpha]\backslash\{0\}$ such that $f(x)h(x) = 0$ and $a_i h(x) = 0$ for all $i \leq k$.*

**Proof.** Since $a_k g(x) \neq 0$, there exists a minimal index $l$ such that $a_k b_l \neq 0$. If $a_k\alpha^k(b_l) = 0$ then let $h_1(x) = \alpha^k(g(x))$. As $\alpha$ is monic, $h_1(x) \neq 0$. Next assume that $a_k\alpha^k(b_l) \neq 0$. Note that $k$ is minimal such that $a_k g(x) \neq 0$. By Lemma 2.13, there exists an integer $p \geq 1$ such that $a_k^{p+1}\alpha^k(b_l) = 0 \neq a_k^p\alpha^k(b_l)$. Since $R$ is right duo, there exists $s \in R$ with $a_k^p\alpha^k(b_l) = \alpha^k(b_l)s$. As $\alpha$ is an automorphism, we may let $s = \alpha^l(r)$ for some $r \in R$. Write $h_1(x) = \alpha^k(g(x))r$. Then $h_1(x) \neq 0$ since $\alpha^k(b_l)\alpha^l(r) = \alpha^k(b_l)s \neq 0$. By Lemma 2.12, $f(x)h_1(x) = 0$ for both cases. In addition, since $a_i b_j = 0$ for all $j$ and $i < k$, it follows that $a_0 h_1(x) = a_1 h_1(x) = a_{k-1}h_1(x) = 0$ by using the W-compatibility of $\alpha$, and $a_k$ annihilates the first $l$ coefficients of $h_1(x)$.

If $a_k$ annihilates all coefficients of $h_1(x)$ then we are done by letting $h(x) = h_1(x)$. Otherwise, repeating the above procedure, and after finite times we can construct $h(x) \in R[x;\alpha]\backslash\{0\}$ satisfying $f(x)h(x) = 0$ and for each $i \leq k$, $a_i h(x) = 0$. □

**Theorem 2.15** *Let $R$ be a right duo ring and $\alpha$ be a WF-compatible automorphism of $R$. Then $R$ is an $\alpha$-skew McCoy ring.*

**Proof.** Let $f(x)=\sum_{i=0}^{m} a_i x^i,\ g(x)=\sum_{j=0}^{n} b_j x^j \in R[x;\alpha]\backslash\{0\}$ satisfy $f(x)g(x)=0$. It suffices to show the following.

There exists $g'(x)\in R[x;\alpha]\backslash\{0\}$ such that $f(x)g'(x)=0$ and for all $i,\ a_i g'(x)=0$.

Note that $\alpha$ is WF-compatible. If $a_i g(x)=0$ for all $i$ then let $g'(x)=g(x)$. So $a_i\alpha^i(b_j)=0$, which implies that $f(x)b_j=0$ for each $j$, and we are done. Next we assume that $a_i g(x)\neq 0$ for some $i$. Let $k$ be minimal such that $a_k g(x)\neq 0$. Then by Lemma 2.14, there exists a nonzero $h(x)\in R[x;\alpha]$ such that $f(x)h(x)=0$ and $a_0h(x)=a_1h(x)=\cdots=a_kh(x)=0$. Now, if $a_ih(x)=0$ for all $i$, then the proof is finished by letting $g'(x)=h(x)$. If not, there must exist an integer $i_0$ ($>k$) satisfying $a_{i_0}h(x)\neq 0$, and apply Lemma 2.14 again. So after finite times check, we can produce a nonzero polynomial $g'(x)\in R[x;\alpha]$ such that $f(x)g'(x)=0$ and $a_ig'(x)=0$ for all $i$. The proof is complete. □

**Remark 2.16** *Notice that in Example 2.3(1), the ring $R=\mathbb{Z}_2\oplus\mathbb{Z}_2$ is commutative, and thus duo. But $R$ is not $\alpha$-skew McCoy. So we conclude that the condition "$\alpha$ is a WF-compatible automorphism" in Theorem 2.15 is not superfluous.*

**Corollary 2.17** *[5, Theorem 8.2] Right duo rings are necessarily right McCoy.*

Let $\alpha$ be an endomorphism of a ring $R$ and $M$ be an $R$-module. $M$ is said to be *$\alpha$-compatible* if for any $m\in M$ and $r\in R$, $mr=0 \Leftrightarrow m\alpha(r)=0$ (see [1]). Based on this, we call $M$ *weakly $\alpha$-compatible* (or *W-$\alpha$-compatible* for short) if $m\alpha(r)=0$ whenever $mr=0$; and call $M$ *weakly finitely $\alpha$-compatible* (or *WF-$\alpha$-compatible* for short) if for a finite number of elements $m_i\in M$ and $r_i\in R$, $\sum_i m_ir_i=0$ implies $\sum_i m_i\alpha(r_i)=0$.

**Proposition 2.18** *Let $R$ be a right duo ring and $\alpha$ be an automorphism of $R$. Then every WF-$\alpha$-compatible cyclic $R$-module is $\alpha$-skew McCoy.*

**Proof.** In view of Theorem 2.15, $R_R$ is $\alpha$-skew McCoy. Let $N$ be a cyclic $R$-module. Then $N\cong R/I$ with $I=r_R(n)$ for some $n\in N$. By hypothesis, $N$ is WF-$\alpha$-compatible. Then for any $s\in I$, we have $ns=0$, implying $n\alpha(s)=0$. Thus $\alpha(I)\subseteq I$. Therefore, the result follows from Proposition 2.4(4). □

Recall that a module is called a *Bezout module* if each of its finitely generated submodules is cyclic.

**Corollary 2.19** *Let $R$ be a right duo ring with an automorphism $\alpha$. Then WF-$\alpha$-compatible Bezout $R$-modules are $\alpha$-skew McCoy.*

**Proof.** By Proposition 2.18, every WF-$\alpha$-compatible cyclic $R$-module is $\alpha$-skew McCoy. Hence Bezout $R$-modules are $\alpha$-skew McCoy by Proposition 2.4(2). □

In what follows $R_n$ denotes (for a positive integer $n$) the following subring of the upper triangular matrix ring $T_n(R)$ over a ring $R$:

$$R_n=\{(a_{ij})\in T_n(R): a_{ij}\in R, a_{11}=a_{22}=\cdots=a_{nn}\};$$

we also consider the following subgroup of the additive group of all formal upper triangular matrices over $M$, namely,

$$M_n = \{(m_{ij}) \in T_n(M) : m_{ij} \in M, m_{11} = m_{22} = \cdots = m_{nn}\}.$$

Then $M_n$ is an $R_n$-module under the usual matrix addition operation and the following scalar product operation. For $W = (w_{ij}) \in M_n$ and $A = (a_{ij}) \in R_n$, $WA = (m_{ij})$ with $m_{ij} = \sum_{k=1}^{n} w_{ik}a_{kj}$, for $i, j = 1, 2, \ldots, n$. An endomorphism $\alpha$ of $R$ can be extended to an endomorphism $\overline{\alpha}$ of $R_n$ defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$.

**Proposition 2.20** *A module $M_R$ is $\alpha$-skew McCoy if and only if $M_n$ is $\overline{\alpha}$-skew McCoy as an $R_n$-module.*

**Proof.** The result for modules can be proved in exactly the same manner as that results for rings in [3, Theorem 14]. □

For a commutative domain $R$ and a module $M_R$, the *torsion submodule* of $M$ is defined by $T(M) = \{x \in M | r_R(x) \neq 0\}$; $M$ is called *torsion free* if $T(M) = 0$.

**Proposition 2.21** *Let $\alpha$ be a monomorphism of a commutative domain $D$ and $M$ be a $D$-module. Then $M$ is $\alpha$-skew McCoy if and only if its torsion submodule $T(M)$ is $\alpha$-skew McCoy.*

**Proof.** Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\alpha]$ and $d(x) = \sum_{j=0}^{q} d_j x^j \in D[x;\alpha]\backslash\{0\}$ satisfy $m(x)d(x) = 0$. We have

(0) $m_0 d_0 = 0,$
(1) $m_0 d_1 + m_1\alpha(d_0) = 0,$
(2) $m_0 d_2 + m_1\alpha(d_1) + m_2\alpha^2(d_0) = 0,$
$\cdots$
$(p+q)$ $m_p\alpha^p(d_q) = 0.$

We may assume that $d_0 \neq 0$. Then by Eq. (0), $m_0 \in T(M)$. Multiplying Eq. (1) by $d_0$ on the right, one obtains $m_1\alpha(d_0)d_0 = 0$. Since $\alpha$ is monic and $D$ is a domain, $m_1 \in T(M)$. Multiplying Eq. (2) by $\alpha(d_0)d_0$ from the right yields $m_2\alpha^2(d_0)\alpha(d_0)d_0 = 0$, so $m_2 \in T(M)$. Repeating this process, we have $m(x) \in T(M)[x]$. Since $T(M)$ is $\alpha$-skew McCoy, there exists $r \in R\backslash\{0\}$ satisfying $m_i\alpha^i(r) = 0$. This proves that $M$ is an $\alpha$-skew McCoy module. The other implication is trivial. □

By a similar proof as above, we have the following result.

**Proposition 2.22** *Let $\alpha$ be an endomorphism of a commutative domain $D$ and $M$ be a torsion free $D$-module. Then $M$ is an $\alpha$-skew McCoy module.*

A module is *uniform* [8] if any two nonzero submodules have a nonzero intersection.

**Lemma 2.23** *Let $\{M_i\}_{i\in\Lambda}$ be a family of $\alpha$-skew McCoy $R$-modules with $\Lambda$ an index set. If $R_R$ is uniform, then a direct sum $M = \coprod_{i\in\Lambda} M_i$ is $\alpha$-skew McCoy.*

**Proof.** Let $m(x) = \sum_{k=0}^{p}(m_{ik})_{i\in\Lambda}x^k \in M[x;\alpha]$, $g(x) \in R[x;\alpha]\backslash\{0\}$ satisfy $m(x)g(x) = 0$. Let $m_i(x) = \sum_{k=0}^{p} m_{ik}x^k \in M_i[x]$. Since $m_i(x)g(x) = 0$ and $M_i$ is $\alpha$-skew McCoy, there exists $r_i \in R\backslash\{0\}$ such that $m_i(x)r_i = 0$. Note that the set $\Lambda' = \{i \in \Lambda \mid m_i(x) \neq 0\}$ is finite. Put $U = \bigcap_{i\in\Lambda'} r_i R$. Then $U \neq 0$ since $R_R$

is uniform. Take any $r \in U\backslash\{0\}$. Then $m_i(x)r = 0$ for each $i$, whence $m(x)r = 0$. Thus, $M = \coprod_{i\in\Lambda} M_i$ is $\alpha$-skew McCoy. □

**Theorem 2.24** *Let $\alpha$ be an endomorphism of a ring $R$ and $R_R$ be uniform. Then $R$ is $\alpha$-skew McCoy if and only if every flat $R$-module is $\alpha$-skew McCoy.*

**Proof.** Let $M$ be a flat module. Let $0 \to K \to F \to M \to 0$ be an exact sequence with $F$ free. (In what follows, for any $y \in F$, we denote $\overline{y} = y + K$ in $M$). Let $m(x) = \sum_{i=0}^{p} \overline{y}_i x^i \in M[x;\alpha]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\alpha]\backslash\{0\}$ satisfy $m(x)g(x) = 0$, then we have

$$\sum_{i+j=k} \overline{y_i}\alpha^i(b_j) = 0 \text{ for } k = 0, \ldots, p+q.$$

Therefore $y_0b_0, y_0b_1 + y_1\alpha(b_0), \ldots, y_p\alpha^p(b_q)$ all belong to $K$. Since $M$ is a flat $R$-module, there exists an $R$-homomorphism $\nu : F \to K$ such that $\nu(y_0b_0) = y_0b_0, \nu(y_0b_1 + y_1\alpha(b_0)) = y_0b_1 + y_1\alpha(b_0), \ldots, \nu(y_p\alpha^p(b_q)) = y_p\alpha^p(b_q)$. Write $w_i := \nu(y_i) - y_i$ for $i = 0, \ldots, p$. Each $w_i$ is an element of $F$ and therefore the polynomial $n(x) = \sum_{i=0}^{p} w_i x^i \in F[x;\alpha]$ and $n(x)g(x) = 0$. Since $R$ is $\alpha$-skew McCoy and $F_R$ is free, by Lemma 2.23 $F$ is $\alpha$-skew McCoy. Thus, there exists a nonzero $r \in R$ such that $w_i\alpha^i(r) = 0$ for all $i$. It follows that $y_i\alpha^i(r) \in K$, and so $\overline{y}_i\alpha^i(r) = 0$ in $M$, proving that $M$ is $\alpha$-skew McCoy. The other implication is obvious. □

**Question:** Can the words "$R_R$ is uniform" be removed in Theorem 2.24?

Recall that if $\alpha$ is an endomorphism of a ring $R$, then the map $R[x] \to R[x]$ defined by $\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$. We also denote the extended map by $\alpha$. In [27, Theorem 3.3], Zhang and Chen proved that, if the endomorphism $\alpha$ of a ring $R$ satisfies $\alpha^l = 1_R$ for some integer $l \geq 1$, then a module $M_R$ is $\alpha$-skew Armendariz iff $M[x]$ is $\alpha$-skew Armendariz over $R[x]$. We have a similar result.

**Theorem 2.25** *Let $\alpha$ be an endomorphism of a ring $R$ and $\alpha^l = 1_R$ for some integer $l \geq 1$. Then a module $M_R$ is $\alpha$-skew McCoy if and only if $M[x]$ is $\alpha$-skew McCoy over $R[x]$.*

**Proof.** Assume that $M$ is $\alpha$-skew McCoy. Let $n(y) = \sum_{i=0}^{p} n_i(x)y^i \in M[x][y;\alpha]$ and $g(y) = \sum_{j=0}^{q} g_j(x)y^j \in R[x][y;\alpha]$ with $n(y)g(y) = 0$, where $n_i(x) = \sum_{k=0}^{p_i} n_{ik}x^k \in M[x]$ and $g_j(x) = \sum_{l=0}^{q_j} b_{jl}x^l \in R[x]$. Take an integer $u$ such that $u \geq deg(n_0(x)) + deg(n_1(x)) + \cdots + deg(n_p(x)) + deg(g_0(x)) + deg(g_1(x)) + \cdots + deg(g_q(x))$, where the degree of $n_i(x)$ is as polynomial in $M[x]$, the degree of $g_j(x)$ is as polynomial in $R[x]$ and the degree of the zero polynomial is taken to be 0. Put

$$m(x) = n_0(x^l) + n_1(x^l)x^{lu+1} + n_2(x^l)x^{2lu+2} + \cdots + n_p(x^l)x^{plu+p} \in M[x;\alpha],$$
$$h(x) = g_0(x^l) + g_1(x^l)x^{lu+1} + g_2(x^l)x^{2lu+2} + \cdots + g_q(x^l)x^{qlu+q} \in R[x;\alpha].$$

Then $h(x) \neq 0$, and the set of coefficients of $n_i(x)$'s (resp., $g_j(x)$'s) equals the set of coefficients of $m(x)$ (resp., $h(x)$). Since $\alpha^l = 1_R$, $x^l$ commutes with elements of $R$ in $R[x;\alpha]$. By $n(y)g(y) = 0$, we have

$m(x)h(x) = 0 \in M[x;\alpha]$. Since $M$ is $\alpha$-skew McCoy, there exists $r \in R\backslash\{0\}$ such that $m(x)r = 0 \in M[x;\alpha]$. That is, $n_i(x^l)x^{ilu+i}r = 0$ for $i = 0, 1, \ldots, p$. Again, since $\alpha^l = 1_R$, we have $n_{ik}\alpha^i(r) = 0$ for all $i$ and $k$. Hence $n(y)r = 0$ in $M[x][y;\alpha]$. Thus, $M[x]$ is $\alpha$-skew McCoy over $R[x]$.

Conversely, assume that $M[x]$ is $\alpha$-skew McCoy. Let $m(x)g(x) = 0$ with $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\alpha]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\alpha]\backslash\{0\}$. Set $n(y) = \sum_{i=0}^{p} m_i y^i$ and $h(y) = \sum_{j=0}^{q} b_j y^j$. Then $h(y) \neq 0$ and $n(y)h(y) = 0 \in M[x][y;\alpha]$. By hypothesis, there exists a nonzero element $c(x) = \sum_{i=0}^{m} c_i x^i \in R[x]$ satisfying $n(y)c(x) = 0$. It follows that $m_i\alpha^i(c(x)) = 0$, and so $m_i\alpha^i(c_j) = 0$, implying $m(x)c_j = 0$ in $M[x;\alpha]$, where $0 \leq i \leq p$ and $0 \leq j \leq m$. Thus $M_R$ is $\alpha$-skew McCoy. □

**Corollary 2.26** *[3, Theorem 20] Let $\alpha$ be an endomorphism of a ring $R$ and $\alpha^l = 1_R$ for some positive integer $l$. Then $R$ is $\alpha$-skew McCoy if and only if $R[x]$ is $\alpha$-skew McCoy.*

We write $M_n(R)$ for the $n \times n$ matrix ring over $R$. For a module $M_R$ and $A = (a_{ij}) \in M_n(R)$, let $MA = \{(ma_{ij}) : m \in M\}$. For $n \geq 2$, let $V = \sum_{i=1}^{n-1} E_{i(i+1)}$ where $\{E_{ij} : 1 \leq i, j \leq n\}$ are the matrix units, and set $V_n(R) = RI_n + RV + \cdots + RV^{n-1}$ and $V_n(M) = MI_n + MV + \cdots + MV^{n-1}$. Then $V_n(R)$ is a ring and $V_n(M)$ becomes a right module over $V_n(R)$ under usual addition and multiplication of matrices. There is a ring isomorphism $\theta : V_n(R) \to R[x]/(x^n)$ given by $\theta(r_0 I_n + r_1 V + \cdots + r_{n-1}V^{n-1}) = r_0 + r_1 x + \cdots + r_{n-1}x^{n-1} + (x^n)$, and an abelian group isomorphism $\phi : V_n(M) \to M[x]/(M[x](x^n))$ given by $\phi(m_0 I_n + m_1 V + \cdots + m_{n-1}V^{n-1}) = m_0 + m_1 x + \cdots + m_{n-1}x^{n-1} + M[x](x^n)$ such that $\phi(WA) = \phi(W)\theta(A)$ for all $W \in V_n(M)$ and $A \in V_n(R)$.

Let $\alpha$ be an endomorphism of a ring $R$, the map $V_n(R) \to V_n(R)$ defined by $a_0 I_n + a_1 V + \cdots + a_{n-1}V^{n-1} \mapsto \alpha(a_0)I_n + \alpha(a_1)V + \cdots + \alpha(a_{n-1})V^{n-1}$ is an endomorphism of $V_n(R)$. Similarly the map $R[x]/(x^n) \to R[x]/(x^n)$ defined by $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + (x^n) \mapsto \alpha(a_0) + \alpha(a_1)x + \cdots + \alpha(a_{n-1})x^{n-1} + (x^n)$ is an endomorphism of $R[x]/(x^n)$. We shall denote the two maps above by $\overline{\alpha}$.

**Proposition 2.27** *Let $\alpha$ be an endomorphism of a ring $R$. Then a module $M_R$ is $\alpha$-skew McCoy if and only if $M[x]/M[x](x^n)$ is $\overline{\alpha}$-skew McCoy over $R[x]/R[x](x^n)$ for any $n \geq 2$.*

**Proof.** By the remark above, it suffices to show that $M_R$ is $\alpha$-skew McCoy iff $V_n(M)_{V_n(R)}$ is $\overline{\alpha}$-skew McCoy.

"⇒". Suppose that $W(x)A(x) = 0$ where $W(x) = \sum_{i=0}^{p} W_i x^i \in V_n(M)[x;\overline{\alpha}]$ and $A(x) = \sum_{j=0}^{q} A_j x^j \in V_n(R)[x;\overline{\alpha}]\backslash\{0\}$. Let $W_i = m_{i0}I_n + m_{i1}V + \cdots + m_{i(n-1)}V^{n-1}$ and $A_j = a_{j0}I_n + a_{j1}V + \cdots + a_{j(n-1)}V^{n-1}$ for $0 \leq i \leq p$ and $0 \leq j \leq q$. It follows that $[m_0(x)I_n + m_1(x)V + \cdots + m_{n-1}(x)V^{n-1}][a_0(x)I_n + a_1(x)V + \cdots + a_{n-1}(x)V^{n-1}] = 0$ in $V_n(M)[x;\overline{\alpha}]$, where $m_k(x) = m_{0k} + m_{1k}x + \cdots + m_{pk}x^p \in M[x;\alpha]$ and $a_l(x) = a_{0l} + a_{1l}x + \cdots + a_{ql}x^q \in R[x;\alpha]$ for $0 \leq k, l \leq n-1$, and hence $\sum_{k+l=t} m_k(x)a_l(x) = 0$ in $M[x;\alpha]$ for $t = 0, 1, \ldots, n-1$. In particular, we have

$$m_0(x)a_{l_0}(x) = 0$$

with a minimal index $l_0$ ($l_0$ exists since $A(x) \neq 0$) such that $a_{l_0}(x) \neq 0$. Since $M_R$ is $\alpha$-skew McCoy, there exists a nonzero $r \in R$ such that $m_0(x)r = 0$. Let $A = rE_{1n}$. Then $A \in V_n(R)\backslash\{0\}$ and $W(x)A = 0$. So $V_n(M)_{V_n(R)}$ is $\overline{\alpha}$-skew McCoy.

"$\Leftarrow$". Assume that $m(x)g(x)=0$, where $m(x)\in M[x;\alpha]$ and $g(x)\in R[x;\alpha]\backslash\{0\}$. Let $\alpha(x)=m(x)I_n$ and $\beta(x)=g(x)I_n$. Then $\alpha(x)\in V_n(M)[x;\overline{\alpha}]$, $\beta(x)\in V_n(R)[x;\overline{\alpha}]\backslash\{0\}$ and $\alpha(x)\beta(x)=0$. As $V_n(M)$ is an $\overline{\alpha}$-skew McCoy $V_n(R)$-module, there exists a nonzero $A\in R_n$ such that $\alpha(x)A=0$. Obviously, there is an element $r\in R\backslash\{0\}$ such that $m(x)r=0$. Therefore, $M_R$ is $\alpha$-skew McCoy. □

The following definition is due to Zhang and Chen [28]. A module $M_R$ is a *zip module* if for any subset $X$ of $M$, $r_R(X)=0$ implies $r_R(Y)=0$ for some finite subset $Y$ of $X$. By [6, Proposition 1] and [15, Example 10], (in general) the class of $\alpha$-skew McCoy modules neither contains nor is contained in the class of zip modules. According to [6, Example 2], $R_R$ is a zip module does not imply that $R[x;1_R]_{R[x;1_R]}$ is zip (Some notable results on zip rings have appeared in [9], [10], [26], etc).

**Theorem 2.28** *Let $\alpha$ be an endomorphism of a ring $R$ with $\alpha^l=1_R$ for some positive integer $l$ and $M_R$ be a W-$\alpha$-compatible $\alpha$-skew McCoy module. Then $M$ is a zip $R$-module if and only if $M[x;\alpha]$ is a zip $R[x;\alpha]$-module.*

**Proof.** Suppose that $M[x;\alpha]_{R[x;\alpha]}$ is zip. Let $Y\subseteq M$ with $r_R(Y)=0$. If $f(x)=a_0+a_1x+\cdots+a_nx^n\in r_{R[x;\alpha]}(Y)$, then $mf(x)=0$ for each $m\in Y$. Thus $ma_i=0$, and so $a_i\in r_R(Y)=0$ for $i=1,2,\ldots,n$. Therefore $f(x)=0$, i.e., $r_{R[x;\alpha]}(Y)=0$. Since $M[x;\alpha]$ is zip, there exists a finite subset $Y_0\subseteq Y$ such that $r_{R[x;\alpha]}(Y_0)=0$. Hence, $r_R(Y_0)=r_{R[x;\alpha]}(Y_0)\bigcap R=0$.

Conversely, assume that $M$ is zip. Let $X\subseteq M[x;\alpha]$ with $r_{R[x;\alpha]}(X)=0$. Now let $Y$ be the set of all coefficients of elements in $X$. Then $Y\subseteq M$. If $a\in r_R(Y)$, then $wa=0$ for each $w\in Y$. Since $M_R$ is W-$\alpha$-compatible, $w\alpha^i(a)=0$ for all $i\geq 0$. Thus we have $m(x)a=0$ for every $m(x)\in X$, and so $a\in r_{R[x;\alpha]}(X)=0$. That is $r_R(Y)=0$. Since $M$ is zip, there exists a finite subset $Y_0=\{w_1,w_2,\ldots,w_t\}\subseteq Y$ such that $r_R(Y_0)=0$. For each $w_i\in Y_0$ and $i=1,2,\ldots,t$, let $m_{w_i}(x)\in X$ be such that some coefficient of $m_{w_i}(x)$ is $w_i$. Let $X_0=\{m_{w_1}(x),m_{w_2}(x),\ldots,m_{w_t}(x)\}\subseteq X$ and $Y_1$ be the set of all coefficients of elements in $X_0$, where $m_{w_i}(x)=\sum_{q=0}^{p_{w_i}}a_{w_iq}x^q$. Then $Y_0\subseteq Y_1$ and so $r_R(Y_1)\subseteq r_R(Y_0)=0$. If $f(x)=\sum_{j=0}^{n}b_jx^j\in r_{R[x;\alpha]}(X_0)\backslash\{0\}$, then $m_{w_i}(x)f(x)=0$ for $i=1,2,\ldots,t$. Write $u=\sum_{k=1}^{t}p_{w_k}+n$. Let $n(x)=m_{w_1}(x)+m_{w_2}(x)x^{lu}+\cdots+m_{w_t}(x)x^{lu(t-1)}\in M[x;\alpha]$, by $\alpha^l=1_R$ we have $n(x)f(x)=0$. Since $M_R$ is $\alpha$-skew McCoy, there exists $r\in R\backslash\{0\}$ such that $n(x)r=0$. So $m_{w_i}(x)r=0$ in $M[x;\alpha]$ for each $i$, i.e., $a_{w_iq}\alpha^q(r)=0$. The condition $M_R$ is W-$\alpha$-compatible implies that there exists an integer $z$ such that $a_{w_iq}\alpha^z(r)=0$ for all $w_i$ and $q$. Then $\alpha^z(r)\in r_R(Y_1)=0$, and so $r=0$, a contradiction. Therefore $f(x)=0$, that is, $r_{R[x;\alpha]}(X_0)=0$. □

**Corollary 2.29** *[7, Theorem 3.6] Let $M$ be a McCoy $R$-module. Then $M$ is a zip $R$-module if and only if $M[x]$ is a zip $R[x]$-module.*

**Corollary 2.30** *Let $R$ be a right McCoy ring. Then $R$ is right zip if and only if $R[x]$ is right zip.*

**Remark 2.31** *Notice that all $R$-modules are W-$1_R$-compatible. We conclude that there exists an $\alpha$-skew McCoy module which is not W-$\alpha$-compatible. Consider the ring $R=\{\left(\begin{smallmatrix} a & b\\ 0 & a\end{smallmatrix}\right)\,|\,a,b\in\mathbb{Z}_4\}$. Let $\alpha:R\to R$ be* *an endomorphism defined by* $\alpha(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & -b \\ 0 & a \end{smallmatrix}\right)$. *By [3, Example 7],* $R_R$ *is* $\alpha$*-skew McCoy. Let* $R_2$ *be a ring and the endomorphism* $\overline{\alpha} : R_2 \to R_2$ *both as defined in Proposition 2.20. Write* $M = R_2$. *Then* $M$ *is* $\overline{\alpha}$*-skew McCoy as an* $R_2$*-module also by Proposition 2.20. Nevertheless,* $M$ *is not W-*$\overline{\alpha}$*-compatible. Indeed, for* $A = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} \in M$, $B = \begin{pmatrix} \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} \in R_2$, $AB = 0$ *but* $A\overline{\alpha}(B) \neq 0$.


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Jian CUI, Jianlong CHEN
Department of Mathematics,
Southeast University,
Nanjing 210096, P. R. CHINA
e-mails: seujcui@126.com, jlchen@seu.edu.cn

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