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Contact CR-warped product submanifolds in generalized Sasakian space forms

Sibel Sular and Cihan Özgür

Abstract

We consider a contact CR-warped product submanifold $M = M_{\top} \times_f M_{\perp}$ of a trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. We show that M is a contact CR-product under certain conditions.

Key words and phrases: Warped product manifold, contact CR-warped product submanifold, trans-Sasakian manifold, generalized Sasakian space form

1. Introduction

The notion of a CR-warped product manifold was introduced by B. Y. Chen (see [6] and [7]). He established a sharp relationship between the warping function f of a warped product CR-submanifold of a Kaehler manifold and the squared norm of the second fundamental form. Later, I. Hasegawa and I. Mihai found a similar inequality for contact CR-warped product submanifolds of Sasakian manifolds in [8]. Moreover, I. Mihai [11] improved the same inequality for contact CR-warped products in Sasakian space forms and he gave some applications. A classification of contact CR-warped products in spheres, which satisfy the equality case, identically, was also given.

Furthermore, in [2], K. Arslan, R. Ezentaş, I. Mihai and C. Murathan considered contact CR-warped product submanifolds in Kenmotsu space forms and they obtained sharp estimates for the squared norm of the second fundamental form in terms of the warping function for contact CR-warped products isometrically immersed in Kenmotsu space forms.

Recently, in [3], M. Atçeken studied on the contact CR-warped product submanifolds of a cosymplectic space form and obtained a necessary and sufficient condition for a contact CR-product.

Motivated by the studies of the above authors, in the present study, we consider contact CR-warped product submanifolds of a trans-Sasakian generalized Sasakian space forms and obtain a necessary and sufficient condition for a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form to be a contact CR-product.

The paper is organized as follows: In Section 2, we give a brief information about almost contact metric manifolds. Moreover, in this section the definitions of a generalized Sasakian space form and a contact CR-warped product submanifold are given. In Section 3, warped product manifolds are introduced. In the

last section, we establish a sharp relationship between the warping function f and the squared norm of the second fundamental form σ of a contact CR-warped product submanifold of a trans-Sasakian manifold and give characterizations for a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form to be a contact CR-product submanifold.

2. Preliminaries

An odd-dimensional Riemannian manifold \widetilde{M} is called an *almost contact metric manifold* if there exist on \widetilde{M} a $(1, 1)$ -tensor field φ , a vector field ξ (called a *structure vector field*), a 1-form η and the Riemannian metric g on \widetilde{M} such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad (1)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

$$\eta(X) = g(X, \xi), \quad g(\varphi X, Y) = -g(X, \varphi Y), \quad (3)$$

for all vector fields on \widetilde{M} [4].

Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \varphi Y)$ is called the *fundamental 2-form* of \widetilde{M} .

On the other hand, the almost contact metric structure of \widetilde{M} is said to be *normal* if $[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi$ for any X, Y on \widetilde{M} , where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ , given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a *Sasakian manifold* [4]. It is easy to see that an almost contact metric manifold is Sasakian if and only if

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any X, Y on \widetilde{M} .

In [13], A. Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold \widetilde{M} is said to be a *trans-Sasakian manifold* if there exist two functions α and β on \widetilde{M} such that

$$(\widetilde{\nabla}_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \quad (4)$$

for all vector fields on \widetilde{M} . If $\beta = 0$ (resp. $\alpha = 0$), then \widetilde{M} is said to be an α -*Sasakian manifold* (resp. β -*Kenmotsu manifold*). Sasakian manifolds (resp. Kenmotsu manifolds) appear as examples of α -Sasakian manifolds (resp. β -Kenmotsu manifolds), with $\alpha = 1$ (resp. $\beta = 1$).

From the above equation, for a trans-Sasakian manifold we also have

$$\widetilde{\nabla}_X \xi = -\alpha\varphi X + \beta[X - \eta(X)\xi]. \quad (5)$$

A plane section in the tangent space $T_x\widetilde{M}$ at $x \in \widetilde{M}$ is called a φ -section if it is spanned by a vector X orthogonal to ξ and φX . The sectional curvature $K(X \wedge \varphi X)$ with respect to a φ -section denoted by a vector X is called a φ -sectional curvature. A Sasakian manifold with constant φ -sectional curvature c is a *Sasakian space form* [4] and its Riemannian curvature tensor is given by

$$\begin{aligned} \widetilde{R}(X, Y)Z &= \frac{1}{4}(c+3)\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \frac{1}{4}(c-1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &+ g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\}. \end{aligned} \tag{6}$$

Given an almost contact metric manifold \widetilde{M} , it is said to be a *generalized Sasakian space form* [1] if there exist three functions f_1, f_2 and f_3 on \widetilde{M} such that

$$\begin{aligned} \widetilde{R}(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_2\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned} \tag{7}$$

for any vector fields X, Y, Z on \widetilde{M} , where \widetilde{R} denotes the curvature tensor of \widetilde{M} . If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then \widetilde{M} is a Sasakian space form [4], if $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then \widetilde{M} is a Kenmotsu space form [9], if $f_1 = f_2 = f_3 = \frac{c}{4}$, then \widetilde{M} is a cosymplectic space form [10].

Let $f : M \rightarrow \widetilde{M}$ be an isometric immersion of an n -dimensional Riemannian manifold M into an $(n+d)$ -dimensional Riemannian manifold \widetilde{M} . We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and \widetilde{M} , respectively. Then we have the Gauss and Weingarten formulas

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \tag{8}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{9}$$

where ∇^\perp denotes the normal connection on $T^\perp M$ of M and A_N is the shape operator of M , for $X, Y \in \chi(M)$ and a normal vector field N on M . We call σ the *second fundamental form* of the submanifold M . If $\sigma = 0$ then the submanifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$g(A_N X, Y) = g(\sigma(X, Y), N),$$

for any vector fields X, Y tangent to M .

The equation of Gauss and Codazzi are defined by

$$(\widetilde{R}(X, Y)Z)^\top = R(X, Y)Z + A_{\sigma(X, Z)}Y - A_{\sigma(Y, Z)}X \tag{10}$$

and

$$(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z), \tag{11}$$

for all vector fields X, Y, Z on \tilde{M} , where $(\tilde{R}(X, Y)Z)^\top$ and $(\tilde{R}(X, Y)Z)^\perp$ denote the tangent and normal components of $\tilde{R}(X, Y)Z$, respectively.

Moreover, the first derivative $\bar{\nabla}\sigma$ of the second fundamental form σ is given by

$$(\bar{\nabla}_X \sigma)(Y, Z) = \nabla_X^\perp \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{12}$$

where $\bar{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M [5].

An m -dimensional Riemannian submanifold M of a trans-Sasakian manifold \tilde{M} , where ξ is tangent to M , is called a *contact CR-submanifold* if it admits an invariant distribution D whose orthogonal complementary distribution D^\perp is anti-invariant, that is

$$TM = D \oplus D^\perp \oplus sp\{\xi\}$$

with $\varphi D_x \subseteq D_x$ and $\varphi D_x^\perp \subseteq T_x^\perp M$ for each $x \in M$, where $sp\{\xi\}$ denotes 1-dimensional distribution which is spanned by ξ .

Let us denote the orthogonal complementary of φD^\perp in $T^\perp M$ by v . Then we have

$$T^\perp M = \varphi D^\perp \oplus v.$$

It is obvious that $\varphi v = v$.

For any vector field X tangent to M , we can write

$$\varphi X = TX + NX,$$

where TX (resp. NX) denotes tangential (resp. normal) component of φX .

Similarly, for any vector field N normal to M , we put

$$\varphi N = BN + CN,$$

where BN (resp. CN) denotes the tangential (resp. normal) component of φN .

3. Warped product manifolds

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and f is a positive differentiable function on B . Consider the product manifold $B \times F$ with its projections $\pi : B \times F \rightarrow B$ and $\sigma : B \times F \rightarrow F$. The *warped product* $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that

$$\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p)) \|\sigma^*(X)\|^2,$$

for any vector field X on M . Thus we have

$$g = g_B + f^2 g_F, \tag{13}$$

holds on M . The function f is called the *warping function* of the warped product [12].

We need the following lemma from [12], for later use :

Lemma 3.1 *Let us consider $M = B \times_f F$ and denote by ∇ , ${}^B\nabla$ and ${}^F\nabla$ the Riemannian connections on M , B and F , respectively. If X, Y are vector fields on B and V, W on F , then:*

- (i) $\nabla_X Y$ is the lift of ${}^B\nabla_X Y$,
- (ii) $\nabla_X V = \nabla_V X = (Xf/f)V$,
- (iii) The component of $\nabla_V W$ normal to the fibers is $-(g(V, W)/f)\text{grad}f$,
- (iv) The component of $\nabla_V W$ tangent to the fibers is the lift of ${}^F\nabla_V W$.

Let us choose a local orthonormal frame e_1, \dots, e_n such that e_1, \dots, e_{n_1} are tangent to B and e_{n_1+1}, \dots, e_n are tangent to F . The gradient and Hessian form of f are defined by

$$X(f) = g(\text{grad}f, X) \tag{14}$$

and

$$H^f(X, Y) = X(Y(f)) - (\nabla_X Y)f = g(\nabla_X \text{grad}f, Y), \tag{15}$$

for any vector fields X, Y on M , respectively.

Moreover, the Laplacian of f is given by

$$\Delta f = \sum_{i=1}^n \{(\nabla_{e_i} e_i)f - e_i(e_i(f))\} = -\sum_{i=1}^n g(\nabla_{e_i} \text{grad}f, e_i), \tag{16}$$

(see [12]).

From the Green Theory for compact orientable Riemannian manifolds without boundary, it is well-known that

$$\int_M \Delta f dV = 0, \tag{17}$$

where dV denotes the volume element of M .

4. Contact CR-warped product submanifolds

In this section, we establish a sharp relationship between the warping function f and the squared norm of the second fundamental form σ of a contact CR-warped product submanifold of a trans-Sasakian manifold and give characterizations for a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form to be a contact CR-product submanifold.

Now, let's begin with the following lemma.

Lemma 4.1 *Let $M = M_\top \times_f M_\perp$ be a contact CR-warped product submanifold of a trans-Sasakian manifold \widetilde{M} . Then we have*

$$g(\sigma(\varphi X, Y), \varphi Y) = X(\ln f)g(Y, Y), \tag{18}$$

$$g(\sigma(X, Y), \varphi Y) = -\varphi X(\ln f)g(Y, Y) \tag{19}$$

and

$$g(\sigma(\varphi X, Z), \varphi Y) = 0, \tag{20}$$

for any vector fields X, Z on M_\top and Y on M_\perp .

Proof. Assume that M is a contact CR-warped product submanifold of a trans-Sasakian manifold \widetilde{M} . From the Gauss formula we can write

$$\widetilde{\nabla}_Y \varphi X = \nabla_Y \varphi X + \sigma(\varphi X, Y), \tag{21}$$

for vector fields X on M_\top and Y on M_\perp . Taking the inner product of the above equation with φY we get

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\widetilde{\nabla}_Y \varphi X, \varphi Y). \tag{22}$$

Since \widetilde{M} is a trans-Sasakian manifold, from (4) we have

$$(\widetilde{\nabla}_Y \varphi)X = \alpha[g(X, Y)\xi - \eta(X)Y] + \beta[g(\varphi Y, X)\xi - \eta(X)\varphi Y]. \tag{23}$$

By the use of M is a contact CR-warped product submanifold, the equation (23) reduces to

$$(\widetilde{\nabla}_Y \varphi)X = 0,$$

which implies that

$$\widetilde{\nabla}_Y \varphi X = \varphi \widetilde{\nabla}_Y X. \tag{24}$$

In view of (24) in (22), we obtain

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\varphi \widetilde{\nabla}_Y X, \varphi Y).$$

Using (2), the last equation turns into

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\widetilde{\nabla}_Y X, Y).$$

By making use of the Gauss equation again, we get

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\nabla_Y X, Y).$$

Since $\nabla_X Y - \nabla_Y X = [X, Y] = 0$ for vector fields X on M_\top and Y on M_\perp , from [12], the above equation can be written as

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\nabla_X Y, Y). \tag{25}$$

So by virtue of the Lemma 3.1, (25) gives us (18).

Similarly by the use of the Gauss formula we can write

$$g(\sigma(X, Y), \varphi Y) = g(\widetilde{\nabla}_Y X, \varphi Y).$$

From (3), the last equation shows us

$$g(\sigma(X, Y), \varphi Y) = -g(\varphi \widetilde{\nabla}_Y X, Y).$$

In view of (24), we get

$$g(\sigma(X, Y), \varphi Y) = -g(\widetilde{\nabla}_Y \varphi X, Y).$$

Then, by the use of the Gauss formula and Lemma 3.1 we obtain (19).

Similar to the proof of (18) and (19) we can easily show that

$$g(\sigma(\varphi X, Z), \varphi Y) = g(\nabla_Z X, Y),$$

for any vector fields X, Z on M_\top and Y on M_\perp . Since M_\top is totally geodesic in M , the above equation gives us (20). Hence, we finish the proof of the lemma. \square

Lemma 4.2 *Let $M = M_\top \times_f M_\perp$ be a contact CR-warped product submanifold of a trans-Sasakian manifold \widetilde{M} . Then we have*

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = \|\sigma(X, Y)\|^2 - [\varphi X(\ln f)]^2 \|Y\|^2, \quad (26)$$

for any vector fields X on M_\top and Y on M_\perp .

Proof. Taking the inner product of (21) with $\varphi\sigma(X, Y)$ we get

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = g(\widetilde{\nabla}_Y \varphi X - \nabla_Y \varphi X, \varphi\sigma(X, Y)),$$

for any vector fields X on M_\top and Y on M_\perp .

Since the ambient space \widetilde{M} is trans-Sasakian, by the use of (24) and Lemma 3.1 we find

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = g(\varphi \widetilde{\nabla}_Y X, \varphi\sigma(X, Y)) - g(\varphi X(\ln f)Y, \varphi\sigma(X, Y)). \quad (27)$$

In view of (2) and (3), the equation (27) reduces to

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = g(\widetilde{\nabla}_Y X, \sigma(X, Y)) + \varphi X(\ln f)g(\varphi Y, \sigma(X, Y)).$$

Then, from the Gauss formula and the equation (19) we obtain

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = g(\sigma(X, Y), \sigma(X, Y)) - [\varphi X(\ln f)]^2 g(Y, Y),$$

which gives us (26). Thus, the proof of the lemma is completed. \square

Lemma 4.3 *Let $M = M_\top \times_f M_\perp$ be a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. Then we have*

$$\begin{aligned} 2 \|\sigma(X, Y)\|^2 &= \{H^{\ln f}(X, X) + H^{\ln f}(\varphi X, \varphi X) \\ &\quad + 2[\varphi X(\ln f)]^2 + 2f_2 \|X\|^2\} \|Y\|^2, \end{aligned} \quad (28)$$

for any vector fields X on M_\top and Y on M_\perp .

Proof. In view of the equation (11), we can write

$$g(\widetilde{R}(X, \varphi X)Y, \varphi Y) = g((\widetilde{\nabla}_X \sigma)(\varphi X, Y) - (\widetilde{\nabla}_{\varphi X} \sigma)(X, Y), \varphi Y), \quad (29)$$

for any vector fields X on M_{\top} and Y on M_{\perp} . Then, by the use of (12) the equation (29) reduces to

$$g(\tilde{R}(X, \varphi X)Y, \varphi Y) = g(\nabla_X^\perp \sigma(\varphi X, Y) - \sigma(\nabla_X \varphi X, Y) - \sigma(\nabla_X Y, \varphi X), \varphi Y) - g(\nabla_{\varphi X}^\perp \sigma(X, Y) + \sigma(\nabla_{\varphi X} X, Y) + \sigma(\nabla_{\varphi X} Y, X), \varphi Y).$$

By making use of the Weingarten formula in the above equation, we get

$$g(\tilde{R}(X, \varphi X)Y, \varphi Y) = g(\tilde{\nabla}_X \sigma(\varphi X, Y), \varphi Y) - g(\sigma(\nabla_X \varphi X, Y), \varphi Y) - g(\sigma(\nabla_X Y, \varphi X), \varphi Y) - g(\tilde{\nabla}_{\varphi X} \sigma(X, Y), \varphi Y) + g(\sigma(\nabla_{\varphi X} X, Y), \varphi Y) + g(\sigma(\nabla_{\varphi X} Y, X), \varphi Y).$$

By virtue of the properties of the Levi-Civita connection $\tilde{\nabla}$, the above equation can be written as follows

$$g(\tilde{R}(X, \varphi X)Y, \varphi Y) = X[g(\sigma(\varphi X, Y), \varphi Y)] - g(\sigma(\varphi X, Y), \tilde{\nabla}_X \varphi Y) - g(\sigma(\nabla_X \varphi X, Y), \varphi Y) - g(\sigma(\nabla_X Y, \varphi X), \varphi Y) - \varphi X[g(\sigma(X, Y), \varphi Y)] + g(\sigma(X, Y), \tilde{\nabla}_{\varphi X} \varphi Y) + g(\sigma(\nabla_{\varphi X} X, Y), \varphi Y) + g(\sigma(\nabla_{\varphi X} Y, X), \varphi Y).$$

Then, in view of Lemma 3.1, Lemma 4.1 and (24), the last equation turns into

$$g(\tilde{R}(X, \varphi X)Y, \varphi Y) = X[X(\ln f)g(Y, Y)] - g(\sigma(\varphi X, Y), \varphi \tilde{\nabla}_X Y) + \varphi \nabla_X \varphi X(\ln f)g(Y, Y) - X(\ln f)g(\sigma(\varphi X, Y), \varphi Y) + \varphi X[\varphi X(\ln f)g(Y, Y)] + g(\sigma(X, Y), \varphi \tilde{\nabla}_{\varphi X} Y) - \varphi \nabla_{\varphi X} X(\ln f)g(Y, Y) + \varphi X(\ln f)g(\sigma(X, Y), \varphi Y). \tag{30}$$

Taking into account of the covariant derivative and the Gauss formula in (30) we obtain

$$g(\tilde{R}(X, \varphi X)Y, \varphi Y) = X(X(\ln f))g(Y, Y) + 2X(\ln f)g(\nabla_X Y, Y) - g(\sigma(\varphi X, Y), \varphi \nabla_X Y) - g(\sigma(\varphi X, Y), \varphi \sigma(X, Y)) + \varphi \nabla_X \varphi X(\ln f)g(Y, Y) - X(\ln f)g(\sigma(\varphi X, Y), \varphi Y) + \varphi X(\varphi X(\ln f))g(Y, Y) + 2\varphi X(\ln f)g(\nabla_{\varphi X} Y, Y) + g(\sigma(X, Y), \varphi \nabla_{\varphi X} Y) + g(\sigma(X, Y), \varphi \sigma(X, Y)) - \varphi \nabla_{\varphi X} X(\ln f)g(Y, Y) + \varphi X(\ln f)g(\sigma(X, Y), \varphi Y).$$

By the use of Lemma 3.1, Lemma 4.1 and Lemma 4.2 in the above equation we get

$$g(\tilde{R}(X, \varphi X)Y, \varphi Y) = \{X(X(\ln f)) + \varphi \nabla_X \varphi X(\ln f) - \varphi \nabla_{\varphi X} X(\ln f) + \varphi X(\varphi X(\ln f)) + 2[\varphi X(\ln f)]^2\}g(Y, Y) - 2\|\sigma(X, Y)\|^2. \tag{31}$$

Since M_T is totally geodesic in M and it is an invariant submanifold of a trans-Sasakian manifold \widetilde{M} , from (4) we have

$$\varphi \nabla_X \varphi X = -\nabla_X X \tag{32}$$

and

$$\varphi \nabla_{\varphi X} X = \nabla_{\varphi X} \varphi X + \beta g(X, X)\xi. \tag{33}$$

By making use of (32) and (33) in (31), we obtain

$$\begin{aligned} g(\widetilde{R}(X, \varphi X)Y, \varphi Y) &= \{X(X(\ln f)) - \nabla_X X(\ln f) - \nabla_{\varphi X} \varphi X(\ln f) \\ &\quad - \beta g(X, X)\xi(\ln f) + \varphi X(\varphi X(\ln f)) \\ &\quad + 2[\varphi X(\ln f)]^2\}g(Y, Y) - 2\|\sigma(X, Y)\|^2. \end{aligned}$$

Since $\xi(\ln f) = 0$, the above equation reduces to

$$\begin{aligned} g(\widetilde{R}(X, \varphi X)Y, \varphi Y) &= \{X(X(\ln f)) - \nabla_X X(\ln f) \\ &\quad + \varphi X(\varphi X(\ln f)) - \nabla_{\varphi X} \varphi X(\ln f) \\ &\quad + 2[\varphi X(\ln f)]^2\}g(Y, Y) - 2\|\sigma(X, Y)\|^2, \end{aligned}$$

which gives us

$$\begin{aligned} g(\widetilde{R}(X, \varphi X)Y, \varphi Y) &= \{H^{\ln f}(X, X) + H^{\ln f}(\varphi X, \varphi X) \\ &\quad + 2[\varphi X(\ln f)]^2\}g(Y, Y) - 2\|\sigma(X, Y)\|^2. \end{aligned} \tag{34}$$

On the other hand, since \widetilde{M} is a generalized Sasakian space form, in view of (7) we get

$$g(\widetilde{R}(X, \varphi X)Y, \varphi Y) = -2f_2g(X, X)g(Y, Y). \tag{35}$$

Hence, comparing the right hand sides of the equations (34) and (35) we can write

$$\begin{aligned} 2\|\sigma(X, Y)\|^2 &= \{H^{\ln f}(X, X) + H^{\ln f}(\varphi X, \varphi X) \\ &\quad + 2[\varphi X(\ln f)]^2 + 2f_2g(X, X)\}g(Y, Y). \end{aligned}$$

Thus, the proof of the lemma is completed. □

Theorem 4.4 *Let $M = M_T \times_f M_\perp$ be a compact contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. Then M is a contact CR-product if*

$$\sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 \geq f_2 \cdot p \cdot q,$$

where σ_v denotes the component of σ in v , $(2p + 1)$ - $\dim(TM_T)$ and q - $\dim(TM_\perp)$.

Proof. Let $\{e_0 = f, e_1, e_2, \dots, e_p, \varphi e_1, \varphi e_2, \dots, \varphi e_p, e^1, e^2, \dots, e^q\}$ be an orthonormal basis of $\chi(M)$ such that $e_0, e_1, e_2, \dots, e_p, \varphi e_1, \varphi e_2, \dots, \varphi e_p$ are tangent to M_\top and e^1, e^2, \dots, e^q are tangent to M_\perp . Similarly, let $\{\varphi e^1, \varphi e^2, \dots, \varphi e^q, N_1, N_2, \dots, N_{2r}\}$ be an orthonormal basis of $\chi^\perp(M)$ such that $\varphi e^1, \varphi e^2, \dots, \varphi e^q$ are tangent to $\varphi(T(M_\perp))$ and N_1, N_2, \dots, N_{2r} are tangent to $\chi(v)$.

In view of (16), we can write

$$\begin{aligned} \Delta \ln f &= -\sum_{i=1}^p g(\nabla_{e_i} \text{grad} \ln f, e_i) - \sum_{i=1}^p g(\nabla_{\varphi e_i} \text{grad} \ln f, \varphi e_i) \\ &\quad - \sum_{j=1}^q g(\nabla_{e^j} \text{grad} \ln f, e^j) - g(\nabla_\xi \text{grad} \ln f, \xi). \end{aligned}$$

Since \widetilde{M} is trans-Sasakian, the induced connection is Levi-Civita and $\text{grad} f \in \chi(M_\top)$ we have $g(\nabla_\xi \text{grad} \ln f, \xi) = 0$. Hence, by the use of (15), the above equation can be written as

$$\Delta \ln f = -\sum_{i=1}^p \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i)\} - \sum_{j=1}^q g(\nabla_{e^j} \text{grad} \ln f, e^j).$$

Then, similar to the proof of the Theorem 3.4 in [3] we get

$$\begin{aligned} \Delta \ln f &= -\sum_{i=1}^p \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i)\} \\ &\quad - \sum_{j=1}^q \left\{ e^j \left(\frac{g(\text{grad} f, e^j)}{f} \right) - \frac{1}{f} g(\nabla_{e^j} e^j, \text{grad} f) \right\}. \end{aligned}$$

By the use of Lemma 3.1, since $\text{grad} f \in \chi(M_\top)$, we obtain

$$\Delta \ln f = -\sum_{i=1}^p \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i)\} - q \|\text{grad} \ln f\|^2. \tag{36}$$

On the other hand, taking $X = e_i$ and $Y = e^j$ in (28), where $1 \leq i \leq p$ and $1 \leq j \leq q$, we can write

$$\begin{aligned} 2 \sum_{i=1}^p \sum_{j=1}^q \|\sigma(e_i, e^j)\|^2 &= q \sum_{i=1}^p \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i)\} \\ &\quad + 2 \sum_{i=1}^p [\varphi e_i(\ln f)]^2 + 2f_2 \cdot p. \end{aligned} \tag{37}$$

Comparing the equations (36) and (37), it can be easily seen that

$$-\Delta \ln f = \frac{2}{q} \sum_{i=1}^p \sum_{j=1}^q \|\sigma(e_i, e^j)\|^2 - 2 \sum_{i=1}^p [\varphi e_i(\ln f)]^2 + q \|\text{grad} \ln f\|^2 - 2f_2 \cdot p. \tag{38}$$

Furthermore, we can write the second fundamental form σ as follows

$$\sigma(e_i, e^j) = \sum_{k=1}^q g(\sigma(e_i, e^j), \varphi e^k) \varphi e^k + \sum_{l=1}^{2r} g(\sigma(e_i, e^j), N_l) N_l,$$

for each $1 \leq i \leq p$ and $1 \leq j \leq q$. Taking the inner product of the above equation with $\sigma(e_i, e^j)$ we get

$$\sum_{i=1}^p \sum_{j=1}^q g(\sigma(e_i, e^j), \sigma(e_i, e^j)) = \sum_{i=1}^p \sum_{j,k=1}^q g(\sigma(e_i, e^j), \varphi e^k)^2 + \sum_{i=1}^p \sum_{j=1}^q \sum_{l=1}^{2r} g(\sigma(e_i, e^j), N_l)^2.$$

Then by making use of Lemma 4.1, the last equation turns into

$$\sum_{i=1}^p \sum_{j=1}^q \|\sigma(e_i, e^j)\|^2 = q \sum_{i=1}^p [\varphi e_i(\ln f)]^2 + \sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2. \tag{39}$$

So, comparing the equations (38) and (39) we obtain

$$-\Delta \ln f = \frac{2}{q} \sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 + q \|\text{grad} \ln f\|^2 - 2f_2 \cdot p.$$

Since M is a compact submanifold, by virtue of (17) we can write

$$\int_M \left\{ \sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 + \frac{q^2}{2} \|\text{grad} \ln f\|^2 - f_2 \cdot p \cdot q \right\} dV = 0. \tag{40}$$

If

$$\sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 \geq f_2 \cdot p \cdot q,$$

then (40) gives us $\text{grad} f = 0$, which means that f is a constant on M . So, M is a contact CR-product. Hence, we finish the proof of the theorem. \square

Proposition 4.5 *Let $M = M_{\top} \times_f M_{\perp}$ be a compact contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 = f_2 \cdot p \cdot q. \tag{41}$$

Proof. Assume that M is a compact contact CR-warped product submanifold of trans-Sasakian generalized Sasakian space form \widetilde{M} satisfying

$$\sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 = f_2 \cdot p \cdot q.$$

Then, from (40) it is easy to see that f is a constant on M , which implies that M is a contact CR-product.

Conversely, if M is a contact CR-product, then f is a constant on M . So we get

$$g(\sigma(X, Y), \varphi Y) = -\varphi X(\ln f)g(Y, Y) = 0,$$

for any vector fields X on M_{\top} and Y on M_{\perp} . So, the last equation can be written as

$$g(\varphi\sigma(X, Y), Y) = 0,$$

which gives us $B\sigma(X, Y) = 0$, i. e. $\sigma(X, Y) \in \chi(v)$. Hence, we obtain (41). \square

As a consequence of the above proposition, we can give the following corollaries.

Corollary 4.6 [8] *Let $M = M_{\top} \times_f M_{\perp}$ be a compact contact CR-warped product submanifold of a Sasakian space form $\widetilde{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 = \frac{(c-1)}{4} p.q.$$

Corollary 4.7 [2] *Let $M = M_{\top} \times_f M_{\perp}$ be a compact contact CR-warped product submanifold of a Kenmotsu space form $\widetilde{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 = \frac{(c+1)}{4} p.q.$$

Corollary 4.8 [3] *Let $M = M_{\top} \times_f M_{\perp}$ be a compact contact CR-warped product submanifold of a cosymplectic space form $\widetilde{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^p \sum_{j=1}^q \|\sigma_v(e_i, e^j)\|^2 = \frac{c}{4} p.q.$$

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