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Abstract

In this paper, we consider a class of Finsler metrics called generalized Berwald metrics which contains the class of Berwald metrics as a special case. We prove that every generalized Berwald metrics with non-zero scalar flag curvature or isotropic Berwald curvature is a Randers metric. Then we prove that on generalized Berwald metrics, the notions of generalized Landsberg and Landsberg curvatures are equivalent.

Key Words: Berwald metric, landsberg metric, randers metric

1. Introduction

For a Finsler metric $F = F(x, y)$, its geodesics curves are given by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$. The Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space [6].

On the other hand, various interesting special forms of Cartan, Landsberg and Berwald tensors have been obtained by some Finslerians. The Finsler spaces having such special forms have been called C-reducible, isotropic Berwald curvature and isotropic Landsberg curvature, etc. [4][5][7][9][11][12][13]. In [8], Matsumoto introduced the notion of C-reducible metrics and proved that any Randers metric is C-reducible. Later on, Matsumoto-Hōjō proved that the converse is true, too [10]. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric α perturbated by a one-form β which has important applications both in mathematics and physics [14]. In [4], Shen-Chen by using the structure of Funk metric, introduce the notion of isotropic Berwald metrics. This motivates us to study special forms of Berwald curvature for other important special Finsler metrics.

We call a Finsler metric F to be *generalized Berwald metric* if its Berwald curvature satisfies the relation

$$B^i{}_{jkl} = (\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk})y^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{1}$$

where $\mu_i = \mu_i(x, y)$ and $\lambda = \lambda(x, y)$ are homogeneous functions on TM of degrees -2 and -1 with respect to y, respectively and $h_{ij} := FF_{y^iy^j}$ is the angular metric [18]. It is remarkable that every two-dimensional Finsler metric is a generalized Berwald metric. The study on generalized Berwald metrics will enhance our understanding on the two-dimensional Finsler metric and geometric meaning of non-Riemannian quantities.

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Example 1 *Let (M,F) be a two-dimensional Finsler manifold. We refer to the Berwald's frame (ℓ^i, m^i) where $\ell^i = y^i/F(y)$, m^i is the unit vector with $\ell_i m^i = 0$, $\ell_i = g_{ij}\ell^j$ and g_{ij} is the fundamental tensor of Finsler metric F. Then the Berwald curvature is given by*

$$B^i{}_{jkl} = F^{-1}(-2I_{,1}\ell^i + I_{,2}m^i)m_j m_k m_l,$$

where I is 0-homogeneous function called the main scalar of Finsler metric and $I_2 = I_{,2} + I_{,1|2}$ (see page 689 in [1]). By the above relation, we have

$$B^i{}_{jkl} = \frac{-2I_{,1}}{3F^2}(m_j h_{kl} + m_k h_{jl} + m_l h_{jk})y^i + \frac{I_2}{3F}(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}),$$

where $h_{ij} := m_i m_j$. Therefore, every two-dimensional Finsler metric is a generalized Berwald metric with $\mu_i = \frac{-2}{3}F^{-2}I_{,1}m_i$ and $\lambda = \frac{I_2}{3F}$.

Other than two-dimensional Finsler metrics, there are many generalized Berwald metrics. An example follows.

Example 2 *Consider following Finsler metric on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$,*

$$F(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)} + <x,y>}{1-|x|^2}, \quad y \in T_x\mathbb{B}^n = \mathbb{R}^n,$$

where $|.|$ and $<,>$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. F is called the Funk metric which is a Randers metric on $\mathbb{B}^n$ [15]. Then F is a generalized Berwald metric with $\lambda = \frac{1}{2F}$ and $\mu_i = \frac{I_i}{(n+1)F}$.

For a Finsler manifold (M,F), the flag curvature is a function $\mathbf{K}(P,y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. F is said to be of scalar flag curvature if $\mathbf{K}(P,y) = \mathbf{K}(x,y)$. One of the important problems in Finsler geometry is to characterize Finsler manifolds of scalar flag curvature. In this paper, we study the generalized Berwald metrics of non-zero scalar flag curvature and prove the following.

Theorem 1.1 *Every generalized Berwald metric of non-zero scalar flag curvature with dimension $n \geq 3$ is a Randers metric.*

A Finsler metric F is said to be isotropic Berwald metric if

$$B^i_{jkl} = c\{F_{y^jy^k}\delta^i_l + F_{y^jy^l}\delta^i_k + F_{y^ky^l}\delta^i_j + F_{y^jy^ky^l}y^i\},$$

where $c = c(x)$ is a non-zero scalar function on M [4]. We show that a generalized Berwald metric with isotropic Berwald curvature is a Randers metric.

Theorem 1.2 *Every non-Berwaldian generalized Berwald metric of isotropic Berwald curvature with dimension $n \geq 3$ is a Randers metric.*

A Finsler metric is called a generalized Landsberg metric if the Riemannian curvature of the Berwald and Chern connections coincide. Landsberg metrics belong to this class of metrics. We prove that on generalized Berwald manifolds, every generalized Landsberg metric reduce to a Landsberg metric.

Theorem 1.3 *Let (M,F) be a generalized Berwald manifold with dimension $n \geq 3$. Then F is a generalized Landsberg metric if and only if it is a Landsberg metric.*

There are many connections in Finsler geometry [16][17]. In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by symbols "$|$" and "," respectively.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F: TM \to [0,\infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y+su+tv)\right]|_{s,t=0}, \quad u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]|_{t=0}, \quad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Deicke's Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$ [15].

Let (M,F) be a Finsler manifold. For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

and $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric F is said to be C-reducible metric if $\mathbf{M}_y = 0$. This quantity is introduced by Matsumoto [7]. Matsumoto proves that every Randers metric satisfies that $\mathbf{M}_y = 0$. Later on, Matsumoto-Hōjō proves that the converse is true too. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha := \sqrt{a^{ij}b_ib_j} < 1$.

Lemma 2.1 ([10]) A positive-definite Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ defined by

$$\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k,$$

where $L_{ijk} := C_{ijk|s}y^s$, $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L=0}$. The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := g^{jk}L_{ijk}$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$.

Define $\bar{\mathbf{M}}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\bar{\mathbf{M}}_y(u, v, w) := \bar{M}_{ijk}(y)u^iv^jw^k$, where

$$\bar{M}_{ijk} := L_{ijk} - \frac{1}{n+1}\{J_ih_{jk} + J_jh_{ik} + J_kh_{ij}\}.$$

A Finsler metric F is said to be P-reducible if $\bar{\mathbf{M}}_y = 0$. The notion of P-reducibility was given by Matsumoto-Shimada [9]. It is obvious that every C-reducible metric is a P-reducible metric.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i(x, y) := \frac{1}{4}g^{il}(y)\Big\{\frac{\partial^2[F^2]}{\partial x^k\partial y^l}(x, y)y^k - \frac{\partial[F^2]}{\partial x^l}(x, y)\Big\}, \quad y \in T_xM$$

are local functions on TM. $\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^jv^k$ where

$$B^i{}_{jkl} := \frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}, \quad E_{jk} := \frac{1}{2}B^m{}_{jkm}.$$

The $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$, respectively [15].

In [4], Shen-Chen by using the structure of Funk metric, introduce the notion of isotropic Berwald metrics. A Finsler metric F is said to be isotropic Berwald metric if

$$B^i{}_{jkl} = c\{F_{y^jy^k}\delta^i_l + F_{y^jy^l}\delta^i_k + F_{y^ky^l}\delta^i_j + F_{y^jy^ky^l}y^i\},$$

where $c = c(x)$ is a scalar function on M [4][19].

The Riemann curvature $\mathbf{R}_y = R^i{}_kdx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2G^i}{\partial x^j\partial y^k} + 2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry was first introduced by L. Berwald [3]. For a flag $P = \text{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{R}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}.$$

When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$, and is the sectional curvature of P. We say that a Finsler metric F is of scalar curvature if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on TM_0.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. To do this, we need the following lemma.

Lemma 3.1 *Every generalized Berwald metric is a P-reducible metric.*

Proof. By assumption, we have

$$B^i{}_{jkl} = (\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk})y^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{2}$$

where $\mu_i = \mu_i(x, y)$ and $\lambda = \lambda(x, y)$ are homogeneous functions of degrees -2 and -1 with respect to y, respectively. Multiplying (2) with y^j and using $y^j B^i{}_{jkl} = 0$ and $y^j h^i_j = y^j(\delta^i_j - F^{-2}y^i y_j) = 0$ implies that $y^i \mu_i = 0$. Contracting (2) with y_i yields

$$y_i B^i{}_{jkl} = F^2(\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk}) + \lambda y_i(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}). \tag{3}$$

Using $y_i B^i{}_{jkl} = -2L_{jkl}$ and $y_i h^i_m = 0$, equation (3) reduces to

$$L_{jkl} = -\frac{1}{2}F^2\{\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk}\}. \tag{4}$$

Contracting (4) with g^{kl} yields

$$J_j = -\frac{1}{2}(n+1)F^2\mu_j. \tag{5}$$

Putting (5) in (4), we get

$$L_{jkl} = \frac{1}{n+1}\{J_j h_{kl} + J_k h_{jl} + J_l h_{jk}\}. \tag{6}$$

It means that F is a P-reducible metric. □

There is a straightforward relation between the Landsberg curvature and Riemannian curvature as follows.

Lemma 3.2 ([15]) The Landsberg and Riemann Curvatures are related by

$$L_{ijk|m}y^m + C_{ijm}R^m_{\ k} = -\frac{1}{6}\big\{g_{im}(2R^m_{\ k,j} + R^m_{\ j,k}) + g_{jm}(2R^m{}_{k,i} + R^m{}_{i,k})\big\}. \tag{7}$$

Using the relation between the Landsberg curvature and Riemannian curvature mentioned in lemma 3.2, we prove the following lemma.

Lemma 3.3 *Let (M, F) be a P-reducible manifold. Suppose that F is of non-zero scalar flag curvature. Then F is a C-reducible metric.*

Proof. Contracting (7) with g^{ij} gives

$$J_{k|m}y^m + I_m R^m{}_k = -\frac{1}{3}\Big\{2R^m{}_{k,m} + R^m{}_{m,k}\Big\}. \tag{8}$$

Let F is of scalar curvature $\mathbf{K} = \mathbf{K}(x,y)$. This is equivalent to

$$R^i{}_k = \mathbf{K}F^2\, h^i_k, \tag{9}$$

where $h^i_k := g^{ij}h_{jk}$. Differentiating (9) yields

$$R^i{}_{k,l} = \mathbf{K}_{,l}F^2\, h^i_k + \mathbf{K}\Big\{2g_{lp}y^p\delta^i_k - g_{kp}y^p\delta^i_l - g_{kl}y^i\Big\}. \tag{10}$$

By (7), (8) and (10), we obtain

$$L_{ijk|m}y^m = -\frac{1}{3}F^2\Big\{\mathbf{K}_{,i}h_{jk} + \mathbf{K}_{,j}h_{ik} + \mathbf{K}_{,k}h_{ij} + 3\mathbf{K}C_{ijk}\Big\}, \tag{11}$$

$$J_{k|m}y^m = -\frac{1}{3}F^2\Big\{(n+1)\mathbf{K}_{,k} + 3\mathbf{K}I_k\Big\}. \tag{12}$$

Taking a horizontal derivation of P-reducibility yields

$$L_{ijk|m}y^m = \frac{1}{n+1}\Big\{J_{i|m}h_{jk} + J_{j|m}h_{ik} + J_{k|m}h_{ij}\Big\}y^m. \tag{13}$$

By plugging (11) and (12) into (13) we get

$$\mathbf{K}F^2M_{ijk} = 0.$$

Since $\mathbf{K} \neq 0$, thus $M_{ijk} = 0$, and this means that F is a C-reducible metric. □

Proof of Theorem 1.1: By Lemmas 2.1, 3.1 and 3.3, we get the proof. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, we remark the following.

Lemma 4.1 ([18]) Let the Cartan tensor of Finsler metric F satisfies in relation $C_{ijk} = B_i h_{jk} + B_j h_{ik} + B_k h_{ij}$ with $y^iB_i = 0$. Then F is a C-reducible metric.

Theorem 4.2 *A two-dimensional Finsler metric is Berwaldian if and only if it is weakly Landsbergian and weakly Berwaldian.*

Proof. By (5) we have

$$\mu_i = \frac{-2}{(n+1)F^2}J_i. \tag{14}$$

Equation (14) implies that F is a weakly Landsbergian if and only if $\mu_i = 0$. Contracting i and l in the definition of generalized Berwald metric yields

$$2E_{ij} = (n+1)\lambda h_{ij}. \tag{15}$$

The equation (15) implies that F is a weakly Berwald metric if and only if $\lambda = 0$. Plugging (14) and (15) in (1) yields

$$B^i{}_{jkl} = \frac{2}{(n+1)}\Big\{(E_{kl}h^i_j + E_{jl}h^i_k + E_{jk}h^i_l) - (J_j h_{kl} + J_k h_{jl} + J_l h_{jk})F^{-2}y^i\Big\}. \tag{16}$$

By (16), we conclude that for every two-dimensional Finsler metric, $\mathbf{B} = 0$ if and only if $\mathbf{E} = 0$ and $\mathbf{J} = 0$. □

Corollary 4.1 *Let (M, F) be a generalized Berwald manifold. Suppose that F is a weakly Berwald metric. Then the following are equivalent:*

1. *F is a Berwald metric,*
2. *F is a Landsberg metric,*
3. *F is a weakly Landsberg metric.*

Proof. By Lemma 3.1, F is a P-reducible metric. Then $\mathbf{J} = 0$ if and only if $\mathbf{L} = 0$. By Theorem 4.2, the proof is complete. □

Proof of Theorem 1.2: Let F be a isotropic Berwald metric

$$B^i{}_{jkl} = cF^{-1}(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk} + 2C_{jkl}y^i). \tag{17}$$

We remark that since F is non-Berwaldian metric then $c \neq 0$. Then we obtain

$$E_{jk} = \frac{1}{2}B^m{}_{jkm} = \frac{1}{2}(n+1)cF^{-1}h_{jk}. \tag{18}$$

Comparing (15) with (18) yields

$$\lambda = cF^{-1}. \tag{19}$$

Putting (19) in (1) implies that

$$B^i{}_{jkl} = (\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk})y^i + cF^{-1}(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}). \tag{20}$$

By (17) and (20) we have

$$\begin{aligned} B^i{}_{jkl} &= cF^{-1}\{h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk} + (\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk})c^{-1}Fy^i\} \\ &= cF^{-1}\{h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk} + 2C_{jkl}y^i\}. \end{aligned} \tag{21}$$

Comparing the above two identities yields

$$C_{jkl} = \frac{1}{2c}(\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk})F. \tag{22}$$

By Lemmas 4.1 and 2.1, F is a Randers metric. □

5. Proof of Theorem 1.3

A Finsler manifold is called a Landsberg manifold if the Berwald connection coincides with the Chern connection. With this definition of the Landsberg manifolds in mind, we may introduce a new class of Finsler manifolds, as follows. The relation between Riemannian curvatures of Berwald and Chern connections is given by

$$H^i{}_{jkl} = R^i{}_{jkl} + [L^i{}_{jl|k} - L^i{}_{jk|l} + L^i{}_{sk}L^s{}_{jl} - L^i{}_{sl}L^s{}_{jk}], \tag{23}$$

where $H^i{}_{jkl}$ and $R^i{}_{jkl}$ denote the Riemannian curvatures of Berwald and Chern connections, respectively. We say that a Finsler metric F is a generalized Landsberg metric if $H^i{}_{jkl} = R^i{}_{jkl}$ [2]. By definition of generalized Landsberg metric we have

$$L^i{}_{jl|k} - L^i{}_{jk|l} + L^i{}_{sk}L^s{}_{jl} - L^i{}_{sl}L^s{}_{jk} = 0. \tag{24}$$

Lemma 5.1 *Let (M, F) be a Finsler manifold. Then F is a generalized Landsberg metric if and only if the following equations hold:*

$$L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk} = 0, \tag{25}$$

$$L_{ijl|k} - L_{ijk|l} = 0. \tag{26}$$

Proof. Fix k and l and put

$$Q_{ij} := L_{ijl|k} - L_{ijk|l} + L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk}.$$

One can write

$$Q_{ij} := Q^s_{ij} + Q^a_{ij},$$

where

$$Q^s_{ij} := \frac{1}{2}(Q_{ij} + Q_{ji}),$$
$$Q^a_{ij} := \frac{1}{2}(Q_{ij} - Q_{ji}).$$

It is easy to see that $Q_{ij} = 0$ if and only if $Q^s_{ij} = 0$ and $Q^a_{ij} = 0$. On the other hand, we have

$$Q_{ji} = L_{jil|k} - L_{jik|l} + L_{jsk}L^s{}_{il} - L_{jsl}L^s{}_{ik}$$
$$= L_{ijl|k} - L_{ijk|l} + L^s{}_{jk}L_{sil} - L^s{}_{jl}L_{sik}.$$

Hence

$$Q^s_{ij} = L_{jil|k} - L_{jik|l},$$

and consequently

$$Q^a_{ij} = L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk}.$$

This proves the Lemma. □

Lemma 5.2 *Let (M, F) be a P-reducible manifold. Then F is a generalized Landsberg metric if and only if F is a Landsberg metric.*

Proof. It is sufficient to prove that every P-reducible generalized Landsberg metric is a Landsberg metric. Let F be a generalized Landsberg metric. Then by Lemma 5.1, we have

$$L^i{}_{sk}L^s{}_{jl} - L^i{}_{sl}L^s{}_{jk} = 0, \tag{27}$$

$$L^i{}_{jl|k} - L^i{}_{jk|l} = 0. \tag{28}$$

On the other hand, we have:

$$h_{ij} = h_{ir}h_{js}g^{rs}, \tag{29}$$

$$J_i = g^{sr}h_{ri}J_s. \tag{30}$$

F is a P-reducible metric

$$L_{ijk} = \frac{1}{1+n}\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}. \tag{31}$$

By using (29), (30) and (31) in (27), we get

$$\{h_{ij}h_{lk} + h_{ik}h_{lj}\}J^sJ_s + \{h_{lk}J_j + h_{lj}J_k\}J_i + \{h_{ij}J_k + h_{ik}J_j\}J_l = 0. \tag{32}$$

Contracting (32) with $g^{ij}g^{tk}$ and using (29), (30) and $g^{ij}h_{ij} = n-1$, we obtain

$$(n+1)(n-2)J^sJ_s = 0. \tag{33}$$

Since F is a positive definite metric and $n > 2$, then we have $J_s = 0$. By considering (31), we conclude that F is a Landsberg metric. □

Proof of Theorem 1.3: By Lemma 3.1 and Lemma 5.2, we get the proof. □

References

[1] Antonelli, P. L.: Handbook of Finsler Geometry, Kluwer Academic Publishers, 2005.

[2] Bejancu, A., Farran, H.: Generalized Landsberg Manifolds of scalar curvature, Bull. Korean Math. Soc. 37(3), 543–550 (2000).

[3] Berwald, L.: Über Parallelübertragung in Räumen mit allgemeiner Massbestimmung, Jber. Deutsch. Math.-Verein. 34, 213–220 (1926).

[4] Chen, X., Shen, Z.: On Douglas Metrics, Publ. Math. Debrecen. 66, 503–512 (2005).

[5] Chen, X., Shen, Z.: Randers metrics with special curvature properties, Osaka J. of Math. 40, 87–101 (2003).

[6] Ichijyō, Y.: Finsler manifolds modelled on a Minkowski space, J. Math. Kyoto Univ. 16, 639–652 (1976).

[7] Matsumoto, M.: On C-reducible Finsler spaces, Tensor, N. S. 24, 29–37 (1972).

[8] Matsumoto, M.: On Finsler spaces with Randers metric and special forms of important tensors, J. Math. Kyoto Univ. 14, 477–498 (1974).

[9] Matsumoto, M., Shimada, H.: On Finsler spaces with the curvature tensors P_{hijk} and S_{hijk} satisfying special conditions, Rep. Math. Phys. 12, 77–87 (1977).

[10] Matsumoto, M., Hōjō, S.: A conclusive theorem on C-reducible Finsler spaces, Tensor. N. S. 32, 225–230 (1978).

[11] Najafi, B., Shen, Z., Tayebi, A.: Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, Geom. Dedicata. 131, 87–97 (2008).

[12] Najafi, B., Tayebi, A., Rezaei, M. M.: General relatively isotropic L-curvature Finsler manifolds, Iranian Journal of Science and Technology, Transaction A. 29, 357–366(2005).

[13] Najafi, B., Tayebi, A., Rezaei, M. M.: General relatively isotropic mean Landsberg Finsler manifolds, Iranian Journal of Science and Technology, Transaction A. 29, 497–505 (2005).

[14] Randers, G.: On an asymmetric metric in the four-space of general relativity, Phys. Rev. 59, 195–199 (1941).

[15] Shen, Z.: Differential Geometry of Spray and Finsler Spaces, Kluwer Academic Publishers, 2001.

[16] Tayebi, A., Azizpour, E., Esrafilian, E.: On a family of connections in Finsler geometry, Publ. Math. Debrecen. 72, 1–15 (2008).

[17] Tayebi, A., Najafi, B.: Shen's processes on Finslerian connection, Bull. Iran. Math. Soc. 36, No. 2, 57–73 (2010).

[18] Tayebi, A., Peyghan, E.: On special Berwald metrics, Symmetry, Integrability and Geometry: Methods and its Applications, 6 (2010), 008, 9 page.

[19] Tayebi, A., Rafie. Rad, M.: S-curvature of isotropic Berwald metrics, Science in China, Series A: Mathematics. 51, 2198–2204 (2008).

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