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## On the biharmonic vector fields

*Mustapha Djaa, Hichem Elhendi and Seddik Ouakkas*

### Abstract

The problem studied in this paper is related to the biharmonicity of a vector field from a Riemannian manifold  $(M, g)$  to its tangent bundle  $TM$  equipped with the Sasaki metric  $g^s$ . We show that a vector field on a compact manifold is biharmonic if and only if is harmonic. We also investigate the biharmonicity of vector field of  $M$ , as a map from  $(M, g)$  to  $(TM, g^s)$ .

**Key Words:** Horizontal lift, vertical lift, harmonic maps, biharmonic maps.

### 1. Introduction

Biharmonic maps are critical points of bienergy functional defined on the space of smooth maps between Riemannian manifolds, introduced by Eells and Sampson in 1964, which is a generalization of harmonic maps [3].

If  $\varphi : (M, g) \rightarrow (N, h)$  is a smooth map between Riemannian manifolds, then the tension field of  $\varphi$  is defined as

$$\tau(\varphi) = \text{trace}_g \nabla d\varphi.$$

It is said  $\varphi$  is harmonic if the tension field vanishes. The equivalent definition is that  $\varphi$  is a critical point of the energy functional

$$E(\varphi) = \int_M e(\varphi) v_g,$$

where  $e(\varphi) = \frac{1}{2} \text{trace}_g(\varphi^* h)$  is called energy density of  $\varphi$ .

If  $M$  is not compact then the energy  $E(\varphi)$  may be defined on its compact subsets.

**Definition 1** A map  $\varphi : (M, g) \rightarrow (N, h)$  between Riemannian manifolds is called biharmonic if it is a critical point of the bienergy functional :

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g$$

(or over any compact subset  $K \subset M$ ).

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The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J_\varphi(\tau(\varphi)) = -(\Delta^\varphi \tau(\varphi) + \text{trace}_g R^N(\tau(\varphi), d\varphi)d\varphi), \tag{1}$$

where  $J_\varphi$  is the Jacobi operator defined by

$$\begin{aligned} J_\varphi : \Gamma(\varphi^{-1}(TN)) &\rightarrow \Gamma(\varphi^{-1}(TN)) \\ V &\mapsto \Delta^\varphi V + \text{trace}_g R^N(V, d\varphi)d\varphi. \end{aligned} \tag{2}$$

(One can refer to [6] for more details.)

**2. Some results on horizontal and vertical lifts**

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold and  $(TM, \pi, M)$  be its tangent bundle. A local chart  $(U, x^i)_{i=1\dots n}$  on  $M$  induces a local chart  $(\pi^{-1}(U), x^i, y^i)_{i=1\dots n}$  on  $TM$ . Denote by  $\Gamma_{ij}^k$  the Christoffel symbols of  $g$  and by  $\nabla$  the Levi-Civita connection of  $g$ .

We have two complementary distributions on  $TM$ , the vertical distribution  $\mathcal{V}$  and the horizontal distribution  $\mathcal{H}$ , defined by :

$$\begin{aligned} \mathcal{V}_{(x,u)} &= \text{Ker}(d\pi_{(x,u)}) \\ &= \{a^i \frac{\partial}{\partial y^i} |_{(x,u)}; \quad a^i \in \mathbb{R}\} \\ \mathcal{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i} |_{(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} |_{(x,u)}; \quad a^i \in \mathbb{R}\}, \end{aligned}$$

where  $(x, u) \in TM$ , such that  $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$ .

Let  $X = X^i \frac{\partial}{\partial x^i}$  be a local vector field on  $M$ . The vertical and the horizontal lifts of  $X$  are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{3}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \}. \tag{4}$$

For consequences, we have  $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$  and  $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$ , then  $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1\dots n}$  is a local adapted frame in  $TTM$ .

**Remark 1** 1. If  $w = w^i \frac{\partial}{\partial x^i} + \bar{w}^j \frac{\partial}{\partial y^j} \in T_{(x,u)}TM$ , then its horizontal and vertical parts are defined by

$$\begin{aligned} w^h &= w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)} \\ w^v &= \{ \bar{w}^k + w^i u^j \Gamma_{ij}^k \} \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}. \end{aligned}$$

2. If  $u = u^i \frac{\partial}{\partial x^i} \in T_x M$  then its vertical and horizontal lifts are defined by

$$u^V = u^i \frac{\partial}{\partial y^i}$$

$$u^H = u^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \right\}.$$

**Proposition 1** (See [10]) Let  $F \in \mathfrak{T}_p^1(M)$  be a tensor of type  $(1,p)$  (respectively,  $G \in \mathfrak{T}_p^0(M)$  a tensor of type  $(0,p)$ ), then there exist a tensor  $\gamma(F) \in \mathfrak{T}_{p-1}^1(TM)$  (respectively,  $\gamma(G) \in \mathfrak{T}_{p-1}^0(TM)$ ), locally defined by

$$\gamma(F) = F_{h_1 \dots h_p}^k y^{h_1} \frac{\partial}{\partial y^k} \otimes dx^{h_2} \otimes \dots \otimes dx^{h_p} \tag{5}$$

$$\gamma(G) = G_{h_1 \dots h_p} y^{h_1} dx^{h_2} \otimes \dots \otimes dx^{h_p}, \tag{6}$$

where  $F = F_{i_1 \dots i_p}^j \frac{\partial}{\partial x^j} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_p}$  and  $G = G_{i_1 \dots i_p} dx^{i_1} \otimes \dots \otimes dx^{i_p}$ .

**Proposition 2** (See [10]). For any  $X, Y \in \Gamma(TM)$  and  $f \in C^\infty(M)$  we have the following relations

$$(X + Y)^h = (X)^h + (Y)^h$$

$$(fX)^v = (f)^v X^v$$

$$(fX)^h = (f)^v X^h$$

$$X^H f^v = (Xf)^v$$

$$X^H f^c = (Xf)^c - \gamma(df \circ \nabla X)$$

$$[X^v, Y^h] = [X, Y]^v - (\nabla_X Y)^v$$

$$[X^h, Y^h] = [X, Y]^h - \gamma R(X, Y),$$

where  $f^v = f \circ \pi$ ,  $f^c = \gamma(\nabla df)$  and  $R$  is the curvature tensor of  $\nabla$ .

**Definition 2** The Sasaki metric  $g^s$  on the tangent bundle  $TM$  of  $M$  is given by

$$1. \quad g^s(X^H, Y^H) = g(X, Y) \circ \pi$$

$$2. \quad g^s(X^H, Y^V) = 0$$

$$3. \quad g^s(X^V, Y^V) = g(X, Y) \circ \pi,$$

for all vector fields  $X, Y \in \Gamma(TM)$ .

In the more general case, Sasaki metrics and their applications were considered in [2], [9].

**Proposition 3** ([10],[4]) *Let  $(M, g)$  be a Riemannian manifold and  $\widehat{\nabla}$  be the Levi-Civita connection of the tangent bundle  $(TM, g^s)$  equipped with the Sasaki metric. Then*

$$\begin{aligned} (\widehat{\nabla}_{X^H} Y^H)_{(x,u)} &= (\nabla_X Y)_{(x,u)}^H - \frac{1}{2}(R_x(X, Y)u)^V \\ (\widehat{\nabla}_{X^H} Y^V)_{(x,u)} &= (\nabla_X Y)_{(x,u)}^V + \frac{1}{2}(R_x(u, Y)X)^H \\ (\widehat{\nabla}_{X^V} Y^H)_{(x,u)} &= \frac{1}{2}(R_x(u, X)Y)^H \\ (\widehat{\nabla}_{X^V} Y^V)_{(x,u)} &= 0, \end{aligned}$$

for all vector fields  $X, Y \in \Gamma(TM)$  and  $(x, u) \in TM$

**Proposition 4** ([10],[4]) *Let  $(M, g)$  be a Riemannian manifold and  $\widehat{R}$  be the Riemann curvature tensor of the tangent bundle  $(TM, g^s)$  equipped with the Sasaki metric. Then the following formulae hold.*

1.  $\widehat{R}_{(x,u)}(X^V, Y^V)Z^V = 0$
2.  $\widehat{R}_{(x,u)}(X^V, Y^V)Z^H = [R(X, Y)Z + \frac{1}{4}R(u, X)(R(u, Y)Z) - \frac{1}{4}R(u, Y)(R(u, X)Z)]_x^H$
3.  $\widehat{R}_{(x,u)}(X^H, Y^V)Z^V = -[\frac{1}{2}R(Y, Z)X + \frac{1}{4}R(u, Y)(R(u, Z)X)]_x^H$
4.  $\widehat{R}_{(x,u)}(X^H, Y^V)Z^H = [\frac{1}{4}R(R(u, Y)Z, X)u + \frac{1}{2}R(X, Z)Y]_x^V + \frac{1}{2}[(\nabla_X R)(u, Y)Z]_x^H$
5.  $\widehat{R}_{(x,u)}(X^H, Y^H)Z^V = [R(X, Y)Z + \frac{1}{4}R(R(u, Z)Y, X)u - \frac{1}{4}R(R(u, Z)X, Y)u]_x^V$   
 $+ \frac{1}{2}[(\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X]_x^H$
6.  $\widehat{R}_{(x,u)}(X^H, Y^H)Z^H = \frac{1}{2}[(\nabla_Z R)(X, Y)u]_x^V$   
 $+ [R(X, Y)Z + \frac{1}{4}R(u, R(Z, Y)u)X$   
 $+ \frac{1}{4}R(u, R(X, Z)u)Y + \frac{1}{2}R(u, R(X, Y)u)Z]_x^H,$

for all vectors  $u, X, Y, Z \in T_x M$ .

**Definition 3** *Let  $(M, g)$  be a Riemannian manifold and  $F \in \mathfrak{T}_1^1(M)$  be a tensor field of type  $(1, 1)$ . Then we define a vertical and horizontal vector fields  $VF, HF$  on  $TM$  by*

$$\begin{aligned} VF : TM &\rightarrow TTM \\ (x, u) &\mapsto (F(u))^V \\ \\ HF : TM &\rightarrow TTM \\ (x, u) &\mapsto (F(u))^H. \end{aligned}$$

Locally we have

$$VF = y^i F_i^j \frac{\partial}{\partial y^j} = y^i (F(\frac{\partial}{\partial x^i}))^V \tag{7}$$

$$HF = y^i F_i^j \frac{\partial}{\partial x^j} - y^i y^k F_i^l \Gamma_{lk}^s \frac{\partial}{\partial y^s} = y^i (F(\frac{\partial}{\partial x^i}))^H. \tag{8}$$

**Proposition 5** *Let  $(M, g)$  be a Riemannian manifold and  $\widehat{\nabla}$  be the Levi-Civita connection of the tangent bundle  $(TM, g^s)$  equipped with the Sasaki metric. If  $F \in \mathfrak{T}_1^1(M)$  is a tensor field of type  $(1, 1)$ , then*

$$\begin{aligned} (\widehat{\nabla}_{X^V} VF)_{(x,u)} &= (F(X))_{(x,u)}^V \\ (\widehat{\nabla}_{X^V} HF)_{(x,u)} &= (F(X))_{(x,u)}^H + \frac{1}{2}(R_x(u, X_x)F(u))^H \\ (\widehat{\nabla}_{X^H} VF)_{(x,u)} &= V(\nabla_X F)(x, u) + \frac{1}{2}(R_x(u, F_x(u))X_x)^H \\ (\widehat{\nabla}_{X^H} HF)_{(x,u)} &= H(\nabla_X F)(x, u) - \frac{1}{2}(R_x(X_x, F_x(u))u)^V, \end{aligned}$$

where  $(x, u) \in TM$  and  $X \in \Gamma(TM)$ .

**Proof.** Locally, using formulas (3) and (4), and the Propositions 2 and 3, we have

$$\begin{aligned} \widehat{\nabla}_{X^V} F^v &= \widehat{\nabla}_{X^V} y^i (F(\frac{\partial}{\partial x^i}))^V = X^V(y^i)(F(\frac{\partial}{\partial x^i}))^V \\ &= X^i(F(\frac{\partial}{\partial x^i}))^V = (F(X))^V \end{aligned}$$

$$\begin{aligned} (\widehat{\nabla}_{X^V} HF)_{(x,u)} &= (\widehat{\nabla}_{X^V} y^i (F(\frac{\partial}{\partial x^i}))^H)_{(x,u)} \\ &= (X^V(y^i)(F(\frac{\partial}{\partial x^i}))^H + y^i \widehat{\nabla}_{X^V} F(\frac{\partial}{\partial x^i})^H)_{(x,u)} \\ &= X^i(F(\frac{\partial}{\partial x^i}))^H + u^i \frac{1}{2}(R_x(u, X)F_x(\frac{\partial}{\partial x^i}))^H \\ &= (F(X))^H + \frac{1}{2}(R_x(u, X)F_x(u))^H, \end{aligned}$$

and

$$\begin{aligned} (\widehat{\nabla}_{X^H} HF)_{(x,u)} &= (\widehat{\nabla}_{X^H} y^k (F(\frac{\partial}{\partial x^k}))^H)_{(x,u)} \\ &= (X^H(y^k)(F(\frac{\partial}{\partial x^k}))^H + y^k \widehat{\nabla}_{X^H} F(\frac{\partial}{\partial x^k})^H)_{(x,u)} \\ &= -X^i u^j \Gamma_{ij}^k (F(\frac{\partial}{\partial x^k}))^H + u^k (\nabla_X F(\frac{\partial}{\partial x^k}))_{(x,u)}^H \\ &\quad - u^k \frac{1}{2}(R_x(X_x, F_x(\frac{\partial}{\partial x^k}))u)^V. \end{aligned}$$

Let  $U = u^i \frac{\partial}{\partial x^i}$  be a constant vector field, then:

$$\begin{aligned} (\widehat{\nabla}_{X^H} HF)_{(x,u)} &= -F(\nabla_X U)_{(x,u)}^H + (\nabla_X F(U))_{(x,u)}^H - \frac{1}{2}(R_x(X_x, F_x(u))u)^V \\ &= ((\nabla_X F)(U))_{(x,u)}^H - \frac{1}{2}(R_x(X_x, F_x(u))u)^V \\ &= (H(\nabla_X F))_{(x,u)} - \frac{1}{2}(R_x(X_x, F_x(u))u)^V. \end{aligned}$$

Similarly, we have

$$\begin{aligned} (\widehat{\nabla}_{X^H} VF)_{(x,u)} &= (X^H(y^k)(F(\frac{\partial}{\partial x^k}))^V + y^k \widehat{\nabla}_{X^H} F(\frac{\partial}{\partial x^k})^V)_{(x,u)} \\ &= V(\nabla_X F)_{(x,u)} + \frac{1}{2}(R_x(u, F_x(u))X_x)^H. \end{aligned}$$

□

### 3. Harmonicity of a vector field $X : (M, g) \longrightarrow (TM, g^S)$

**Lemma 1** *Let  $(M, g)$  be a Riemannian manifold and  $(TM, g^s)$  be the tangent bundle equipped with the Sasaki metric. If  $X, Y \in \Gamma(TM)$  are a vector fields and  $(x, u) \in TM$  such that  $X_x = u$ , then we have*

$$d_x X(Y_x) = Y_{(x,u)}^h + (\nabla_Y X)_{(x,u)}^v.$$

**Proof.** Let  $(U, x^i)$  be a local chart on  $M$  in  $x \in M$  and  $(\pi^{-1}(U), x^i, y^j)$  be the induced chart on  $TM$ , if  $X_x = X^i(x) \frac{\partial}{\partial x^i}|_x$  and  $Y_x = Y^i(x) \frac{\partial}{\partial x^i}|_x$ , then

$$d_x X(Y_x) = Y^i(x) \frac{\partial}{\partial x^i}|_{(x, X_x)} + Y^i(x) \frac{\partial X^k}{\partial x^i}(x) \frac{\partial}{\partial y^k}|_{(x, X_x)},$$

thus the horizontal part is given by

$$\begin{aligned} (d_x X(Y_x))^h &= Y^i(x) \frac{\partial}{\partial x^i}|_{(x, X_x)} - Y^i(x) X^j(x) \Gamma_{ij}^k(x) \frac{\partial}{\partial y^k}|_{(x, X_x)} \\ &= Y_{(x, X_x)}^h, \end{aligned}$$

and the vertical part is given by

$$\begin{aligned} (d_x X(Y_x))^v &= \{Y^i(x) \frac{\partial X^k}{\partial x^i}(x) + Y^i(x) X^j(x) \Gamma_{ij}^k(x)\} \frac{\partial}{\partial y^k}|_{(x, X_x)} \\ &= (\nabla_Y X)_{(x, X_x)}^v. \end{aligned}$$

□

Using the Lemma 1 and Proposition 3, we obtain the following proposition.

**Proposition 6** *Let  $(M, g)$  be a Riemannian manifold and  $(TM, g^S)$  be the tangent bundle equipped with the Sasaki metric, if  $X : M \rightarrow TM$  is a smooth vector field then its tension field is given by*

$$\tau(X) = (tr_g R(X, \nabla_* X))^H + (tr_g \nabla^2 X)^V.$$

Note that if  $X$  is parallel (i.e.,  $\nabla X = 0$ ) then  $X$  is harmonic. Conversely we have the following theorem proved by Ishihara [5], [7].

**Theorem 1** *Let  $(M, g)$  be a compact Riemannian manifold and  $X \in \Gamma(TM)$ , then  $X$  is harmonic with respect to Sasaki metric on  $TM$  if and only if  $X$  is parallel.*

#### 4. Biharmonicities of a vector field $X : (M, g) \rightarrow (TM, g^S)$

For a vector field  $X \in \Gamma(TM)$  we denote

$$\tau^h(X) = tr_g R(X, \nabla_* X) * \tag{9}$$

$$\tau^v(X) = tr_g \nabla^2 X. \tag{10}$$

**Theorem 2** *Let  $(M, g)$  be a compact Riemannian manifold and  $X \in \Gamma(TM)$ , then  $X$  is biharmonic with respect to Sasaki metric on  $TM$  if and only if  $X$  is harmonic.*

**Proof.** Let  $X_t$  be a compactly supported variation of  $X$  defined by  $X_t = (1 + t)X$ . From the formulas (9) and (10) we have

$$\tau^h(X_t) = (1 + t)^2 \tau^h(X)$$

$$\tau^v(X_t) = (1 + t) \tau^v(X)$$

$$\begin{aligned} E_2(X_t) &= \frac{1}{2} \int |\tau(X_t)|_{g^S}^2 v_g \\ &= \frac{1}{2} \int |\tau^h(X_t)|_g^2 v_g + \frac{1}{2} \int |\tau^v(X_t)|_g^2 v_g \\ &= \frac{(1 + t)^4}{2} \int |\tau^h(X)|_g^2 v_g + \frac{(1 + t)^2}{2} \int |\tau^v(X)|_g^2 v_g \end{aligned}$$

then

$$\begin{aligned} \frac{d}{dt} E_2(X_t)|_{t=0} &= \frac{1}{2} \int |\tau^h(X)|_g^2 v_g + \frac{1}{2} \int |\tau^v(X)|_g^2 v_g \\ &= \frac{1}{2} \int |\tau(X)|_{g^S}^2 v_g. \end{aligned}$$

Hence

$$\frac{d}{dt} E_2(X_t)|_{t=0} = 0 \Leftrightarrow \tau(X) = 0.$$



□

As a consequence of Theorems 1 and 2, we get the following corollary.

**Corollary 1** *Let  $(M, g)$  be a compact Riemannian manifold and  $X \in \Gamma(TM)$ , then  $X$  is biharmonic with respect to Sasaki metric on  $TM$  if and only if  $X$  is parallel.*

**Remark 2** *If  $X \in \Gamma(TM)$  is a compactly supported vector field then  $X$  is biharmonic with respect to Sasaki metric on  $TM$  if and only if  $X$  is harmonic.*

**Lemma 2** *Let  $(M, g)$  be a Riemannian manifold and  $(TM, g^s)$  be the tangent bundle equipped with the Sasaki metric. If  $X : M \rightarrow TM$  is a smooth vector field then the Jacobi tensor  $J_X(\tau^v(X)^V)$  is given by*

$$J_X(\tau^v(X)^V)_{(x,u)} = \left\{ tr_g \nabla^2(\tau^v(X)) \right\}_{(x,u)}^V + \left\{ tr_g \left( R(u, \nabla_* \tau^v(X)) * \right. \right. \\ \left. \left. + R(\tau^v(X), \nabla_* X) * + \frac{1}{2} R(u, \tau^v(X)) R(u, \nabla_* X) * \right) \right\}_{(x,u)}^H,$$

for all  $(x, u) \in TM$ .

**Proof.** Let  $(x, u) \in TM$  and  $\{e_i\}_{i=1}^m$  be a local orthonormal frame on  $M$  such that  $(\nabla_{e_i} e_i)_x = 0$ , denote by  $F_i = \frac{1}{2} R(*, \tau^v(X)) e_i$ , we have:

$$\begin{aligned} \nabla_{e_i}^X (\tau^v(X))^V|_{(x,u)} &= \widehat{\nabla}_{e_i^H + (\nabla_{e_i} X)^V} \tau^v(X)^V|_{(x,u)} \\ &= (\nabla_{e_i} \tau^v(X))^V|_{(x,u)} + \frac{1}{2} (R(u, \tau^v(X)) e_i)^H \\ &= (\nabla_{e_i} \tau^v(X))^V|_{(x,u)} + H F_i(x, u). \end{aligned}$$

Then

$$\begin{aligned} tr_g \nabla^2(\tau^v(X))^V_{(x,u)} &= \sum_{i=1}^m \left\{ \nabla_{e_i}^X \nabla_{e_i}^X (\tau^v(X))^V \right\}_{(x,u)} \\ &= \sum_{i=1}^m \left\{ \widehat{\nabla}_{e_i^H + (\nabla_{e_i} X)^V} \left( (\nabla_{e_i} \tau^v(X))^V + H F_i \right) \right\}_{(x,u)} \\ &= \sum_{i=1}^m \left\{ \widehat{\nabla}_{e_i^H} (\nabla_{e_i} \tau^v(X))^V + \widehat{\nabla}_{e_i^H} H F_i + \widehat{\nabla}_{(\nabla_{e_i} X)^V} H F_i \right\}_{(x,u)}. \end{aligned}$$

Using Proposition (5), we obtain

$$\begin{aligned} tr_g \nabla^2(\tau^v(X))^V_{(x,u)} &= \sum_{i=1}^m \left\{ (\nabla_{e_i} \nabla_{e_i} \tau^v(X))_{(x,u)} - \frac{1}{4} R_x(e_i, R_x(u, \tau^v(X)) e_i) u \right\}^V \\ &+ \sum_{i=1}^m \left\{ \frac{1}{2} (R_x(u, \nabla_{e_i} \tau^v(X)) e_i + \frac{1}{2} (\nabla_{e_i} R_x(u, \tau^v(X)) e_i) + \frac{1}{2} R_x(\tau^v(X), \nabla_* u) * \right. \\ &\left. + \frac{1}{4} R_x(u, \nabla_{e_i} X) R_x(u, \tau^v(X)) e_i + \frac{1}{2} R_x(\nabla_{e_i} X, \tau^v(X)) e_i \right\}^H. \end{aligned}$$

From Proposition 2 and lemma 1, we have

$$\begin{aligned} tr_g(\widehat{R}(\tau^v(X))^V, dX)dX) &= \sum_{i=1}^m \left\{ \widehat{R}((\tau^v(X))^V, e_i^H)e_i^H + \widehat{R}((\tau^v(X))^V, (\nabla_{e_i}X)^V)e_i^H \right. \\ &\quad \left. + \widehat{R}((\tau^v(X))^V, e_i^H)(\nabla_{e_i}X)^V + \widehat{R}((\tau^v(X))^V, (\nabla_{e_i}X)^V)(\nabla_{e_i}X)^V \right\}. \end{aligned}$$

By calculating at  $(x, u)$ , we obtain

$$\begin{aligned} tr_g(\widehat{R}(\tau^v(X))^V, dX)dX)_{(x,u)} &= \sum_{i=1}^m \left\{ -\frac{1}{4}R(R(u, \tau^v(X))e_i, e_i)u + \frac{1}{2}R(e_i, e_i)\tau^v(X) \right\}_x^V \\ &\quad + \sum_{i=1}^m \left\{ R(\tau^v(X), \nabla_{e_i}X)e_i + \frac{1}{4}R(u, \tau^v(X))R(u, \nabla_{e_i}X)e_i \right. \\ &\quad - \frac{1}{4}R(u, \nabla_{e_i}X)R(u, \tau^v(X))e_i + \frac{1}{2}R(\tau^v(X), \nabla_{e_i}X)e_i \\ &\quad \left. + \frac{1}{4}R(u, \tau^v(X))R(u, \nabla_{e_i}X)e_i - \frac{1}{2}(\nabla_{e_i}R)(u, \tau^v(X))e_i \right\}_x^H. \end{aligned}$$

Considering the formula (2), we deduce

$$\begin{aligned} J_X(\tau^v(X)^V)_{(x,u)} &= \left\{ tr_g \nabla^2(\tau^v(X)) \right\}_{(x,u)}^V + \left\{ \frac{1}{2}R(u, \nabla_{e_i}\tau^v(X))e_i \right. \\ &\quad \left. + \frac{1}{2}\nabla_{e_i}R(u, \tau^v(X))e_i + R(u, \tau^v(X))R(u, \nabla_{e_i}X)e_i - (\nabla_{e_i}R)(u, \tau^v(X))e_i \right\}_x^H. \end{aligned}$$

From the following equality

$$\nabla_{e_i}R(u, \tau^v(X))e_i = (\nabla_{e_i}R)(u, \tau^v(X))e_i + R(\nabla_{e_i}u, \tau^v(X))e_i + R(u, \nabla_{e_i}\tau^v(X))e_i.$$

The proof of Lemma 2 is completed.  $\square$

**Lemma 3** *Let  $(M, g)$  be a Riemannian manifold and  $(TM, g^s)$  be the tangent bundle equipped with the Sasaki metric, if  $X : M \rightarrow TM$  is a smooth vector field then the Jacobi tensor  $J_X(\tau^h(X)^H)$  is given by*

$$\begin{aligned} J_X(\tau^h(X)^H)_{(x,u)} &= tr_g \left\{ 2R(\tau^h(X), *)\nabla_*X - R(*, \nabla_*\tau^h(X))u + \frac{1}{2}R(R(u, \nabla_*)*, \tau^h(X))u \right\}_{(x,u)}^V \\ &\quad + tr_g \left\{ \nabla_*\nabla_*\tau^h(X) + R(u, \nabla_*X)\nabla_*\tau^h(X) + \frac{1}{2}R(u, \nabla_*\nabla_*X)\tau^h(X) \right. \\ &\quad \left. + R(u, R(\tau^h(X), *)u)* + R(\tau^h(X), *)* + (\nabla_{\tau^h(X)}R)(u, \nabla_*X)* \right\}_{(x,u)}^H \end{aligned} \quad (11)$$

for all  $(x, u) \in TM$ .

**Proof.** Let  $(x, u) \in TM$  and  $\{e_i\}_{i=1}^m$  be a local orthonormal frame on  $M$  such that  $(\nabla_{e_i}e_i)_x = 0$ , if we denote by

$$F_i = \frac{1}{2}R(e_i, \tau^h(X))* \quad (12)$$

and

$$G_i = \frac{1}{2}R(*, \nabla_{e_i}X)\tau^h(X). \tag{13}$$

In the first, using Proposition 3, we calculate

$$\begin{aligned} \text{tr}_g \nabla^2(\tau^h(X))_{(x,u)}^H &= \sum_{i=1}^m \left\{ \nabla_{e_i}^X \nabla_{e_i}^X (\tau^h(X))^H \right\}_{(x,u)} \\ &= \sum_{i=1}^m \left\{ \widehat{\nabla}_{e_i^H + (\nabla_{e_i}X)^V} \left( (\nabla_{e_i} \tau^h(X))^H - VF_i + HG_i \right) \right\}_{(x,u)}. \end{aligned}$$

From Proposition 5, we have

$$\begin{aligned} \text{tr}_g \nabla^2(\tau^h(X))_{(x,u)}^H &= \sum_{i=1}^m \left\{ (\nabla_{e_i} \nabla_{e_i} \tau^h(X))^H + \left( \frac{1}{2}R(u, \nabla_{e_i}X) \nabla_{e_i} \tau^h(X) \right)^H - V(\nabla_{e_i} F_i) \right. \\ &\quad - \left( \frac{1}{2}R(e_i, \nabla_{e_i} \tau^h(X))u \right)^V - \frac{1}{2}(R(u, F_i(u))e_i)^H - (F_i(\nabla_{e_i}X))^V + H(\nabla_{e_i} G_i) \\ &\quad \left. - \frac{1}{2}(R(e_i, G(u))u)^V + (G_i(\nabla_{e_i}X))^H + \frac{1}{2}(R(u, \nabla_{e_i}X)G_i(u))^H \right\}_{(x,u)}. \tag{14} \end{aligned}$$

On substituting (12) and (13) in (14), we arrive at

$$\begin{aligned} \text{tr}_g \nabla^2(\tau^h(X))_{(x,u)}^H &= \sum_{i=1}^m \left\{ \nabla_{e_i} \nabla_{e_i} \tau^h(X) + R(u, \nabla_{e_i}X) \nabla_{e_i} \tau^h(X) + \frac{1}{2}R(u, \nabla_{e_i} \nabla_{e_i} X) \tau^h(X) \right. \\ &\quad + \frac{1}{2}(\nabla_{e_i} R)(u, \nabla_{e_i} X) \tau^h(X) + \frac{1}{4}R(u, \nabla_{e_i} X)(R(u, \nabla_{e_i} X) \tau^h(X)) \\ &\quad \left. - \frac{1}{4}R(u, R(e_i, \tau^h(X))u) e_i \right\}_{(x,u)}^H \\ &\quad - \sum_{i=1}^m \left\{ \frac{1}{2}R(e_i, \tau^h(X)) \nabla_{e_i} X + R(e_i, \nabla_{e_i} \tau^h(X))u + \frac{1}{2}(\nabla_{e_i} R)(e_i, \tau^h(X))u \right. \\ &\quad \left. + \frac{1}{4}R(e_i, R(u, \nabla_{e_i} X) \tau^h(X))u \right\}_{(x,u)}^V. \tag{15} \end{aligned}$$

On the other hand we have

$$\begin{aligned}
 tr_g \left\{ (\widehat{R}(\tau^h(X))^H, dX)dX \right\}_{(x,u)} &= \sum_{i=1}^m \left\{ R(\tau^h(X), e_i)e_i + \frac{3}{4}R(u, R(\tau^h(X), e_i)u)e_i \right. \\
 &+ (\nabla_{\tau^h(X)}R)(u, \nabla_{e_i}X)e_i - \frac{1}{2}(\nabla_{e_i}R)(u, \nabla_{e_i}X)\tau^h(X) \\
 &- \left. \frac{1}{4}R(u, \nabla_{e_i}X)R(u, \nabla_{e_i}X)\tau^h(X) \right\}_{(x,u)}^H \\
 &+ \sum_{i=1}^m \left\{ \frac{1}{2}(\nabla_{e_i}R)(\tau^h(X), e_i)u + \frac{1}{2}R(R(u, \nabla_{e_i}X)e_i, \tau^h(X))u \right. \\
 &+ \left. \frac{3}{2}R(\tau^h(X), e_i)\nabla_{e_i}X - \frac{1}{4}R(R(u, \nabla_{e_i}X)\tau^h(X), e_i)u \right\}_{(x,u)}^V. \tag{16}
 \end{aligned}$$

By summing (15) and (16), we obtain the formula (11). □

From Lemma 2 and Lemma 3, we deduce the next theorem.

**Theorem 3** *Let  $(M, g)$  be a Riemannian manifold and  $(TM, g^s)$  be the tangent bundle equipped with the Sasaki metric, if  $X : M \rightarrow TM$  is a smooth vector field then the bitension field of  $X$  is given by*

$$\begin{aligned}
 \tau_2(X)_{(x,u)} &= tr_g \left\{ \nabla^2(\tau^v(X)) + 2R(\tau^h(X), *)\nabla_*X - R(*, \nabla_*\tau^h(X))u \right. \\
 &+ \left. \frac{1}{2}R(R(u, \nabla_*)*, \tau^h(X))u \right\}_{(x,u)}^V \\
 &+ tr_g \left\{ R(u, \nabla_*\tau^v(X)) * + R(\tau^v(X), \nabla_*X) * + \frac{1}{2}R(u, \tau^v(X))R(u, \nabla_*X) * \right. \\
 &+ \nabla_*\nabla_*\tau^h(X) + R(u, \nabla_*X)\nabla_*\tau^h(X) + \frac{1}{2}R(u, \nabla_*\nabla_*X)\tau^h(X) \\
 &+ \left. R(u, R(\tau^h(X), *)u) * + R(\tau^h(X), *) * + (\nabla_{\tau^h(X)}R)(u, \nabla_*X) * \right\}_{(x,u)}^H,
 \end{aligned}$$

for all  $(x, u) \in TM$ .

By Theorem 3 we have the following theorem.

**Theorem 4** *Let  $(M, g)$  be a Riemannian manifold and  $(TM, g^s)$  its tangent bundle equipped with the Sasaki metric. A vector field  $X : M \rightarrow TM$  is biharmonic if and only if the following conditions are verified*

$$0 = tr_g \left\{ \nabla^2(\tau^v(X)) + 2R(\tau^h(X), *)\nabla_*X - R(*, \nabla_*\tau^h(X))u + \frac{1}{2}R(R(u, \nabla_*)*, \tau^h(X))u \right\}_x$$

and

$$\begin{aligned}
0 &= \operatorname{tr}_g \left\{ R(u, \nabla_* \tau^v(X)) * + R(\tau^v(X), \nabla_* X) * + \frac{1}{2} R(u, \tau^v(X)) R(u, \nabla_* X) * + \nabla_* \nabla_* \tau^h(X) \right. \\
&+ R(u, \nabla_* X) \nabla_* \tau^h(X) + \frac{1}{2} R(u, \nabla_* \nabla_* X) \tau^h(X) + R(u, R(\tau^h(X), *)u) * + R(\tau^h(X), *) * \\
&\left. + (\nabla_{\tau^h(X)} R)(u, \nabla_* X) * \right\}_x
\end{aligned}$$

for all  $(x, u) \in TM$ .

### References

- [1] Bejan, C.L., Benyounes, M.: Harmonic  $\varphi$  Morphisms. Beitrge zur Algebra und Geometry, 44, no. 2, 309–321 (2003).
- [2] Cengiz, N., Salimov, A.A.: Diagonal lift in the tensor bundle and its applications. Appl. Math. Comput. 142, no. 2–3, 309–319 (2003).
- [3] Eells, J., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Amer. J. Maths. 86, 109–160 (1964).
- [4] Gudmundsson, S., Kappos, E.: On the Geometry of Tangent Bundles. Expo. Math. 20, 1–41 (2002).
- [5] Ishihara, T.: Harmonic sections of tangent bundles. J. Math. Tokushima Univ. 13, 23–27 (1979).
- [6] Jiang, G.Y.: Harmonic maps and their first and second variational formulas. Chinese Ann. Math. Ser. A. 7, 389–402 (1986).
- [7] Konderak, J.J.: On Harmonic Vector Fields, Publications Matmatiques. 36, 217–288 (1992).
- [8] Oproiu, V.: Harmonic Maps Between Tangent Bundles. Rend. Sem. Mat. Univ. Polit. Torino. 47, vol.1, 47–55 (1989).
- [9] Salimov, A.A., Gezer, A., Akbulut, K.: Geodesics of Sasakian metrics on tensor bundles. Mediterr. J. Math. 6, no.2, 135–147 (2009).
- [10] Yano, K., Ishihara, S.: Tangent and Cotangent Bundles. Marcel Dekker. INC. New York 1973.

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