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Hypercyclic tuples of the adjoint of the weighted composition operators

Rahmat Soltani, Bahram Khani Robati, Karim Hedayatian

Abstract

An n -tuple of commuting operators, (T_1, T_2, \dots, T_n) on a Hilbert space \mathcal{H} is said to be hypercyclic, if there exists a vector $x \in \mathcal{H}$ such that the set $\{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq 0, i = 1, 2, \dots, n\}$ is dense in \mathcal{H} . In this paper, we give sufficient conditions under which the adjoint of an n -tuple of a weighted composition operator on a Hilbert space of analytic functions is hypercyclic.

Key Words: Hypercyclicity, tuples, weighted composition operators

1. Introduction

An n -tuple of operators is a finite sequence of length n of commuting continuous linear operators T_1, T_2, \dots, T_n acting on a locally convex topological vector space X . Hypercyclic tuples of operators were introduced in [5, 7] and [12]. A tuple (T_1, T_2, \dots, T_n) is said to be hypercyclic, if there exists a vector $x \in X$ such that the set $\{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq 0, i = 1, 2, \dots, n\}$ is dense in X . This definition generalizes the hypercyclicity of a single operator to a tuple of operators. Like Feldman in [7], we denote the semigroup generated by a tuple $T = (T_1, \dots, T_n)$ by $\mathcal{F}_T = \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} : k_i \geq 0, i = 1, 2, \dots, n\}$ and the orbit of x under the tuple T by $orb(T, x) = \{Sx : S \in \mathcal{F}_T\}$.

Consider a Hilbert space \mathcal{H} of functions analytic on the open unit disc \mathbb{D} such that for each $\lambda \in \mathbb{D}$ the linear functional e_λ of evaluation at λ is bounded on \mathcal{H} . Moreover, the constant function 1 and the identity function $f(z) = z$ are in \mathcal{H} . The weighted Hardy space is the well-known example of such \mathcal{H} . Let $(\beta(n))_n$ be a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^2(\beta)$ is defined as the space of analytic functions $f = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ on \mathbb{D} satisfying

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 |\beta(n)|^2 < \infty.$$

The classical Hardy space, the Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1$, $\beta(n) = (n+1)^{-\frac{1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$, respectively. Reference [4] is a good source on properties of

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weighted Hardy spaces. The continuity of point evaluations along with the Riesz representation theorem imply that for each $\lambda \in \mathbb{D}$ there is a unique function $K_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, K_\lambda \rangle$, $f \in \mathcal{H}$. The function K_λ is the reproducing kernel for the point λ .

A complex-valued function ω on \mathbb{D} for which $\omega f \in \mathcal{H}$ for every $f \in \mathcal{H}$ is called a multiplier of \mathcal{H} and the collection of all multipliers is denoted by $M(\mathcal{H})$. Each multiplier ω of \mathcal{H} determines a multiplication operator M_ω on \mathcal{H} by $M_\omega f = \omega f$, $f \in \mathcal{H}$. Each multiplier is a bounded analytic function on \mathbb{D} . In fact, since the constant functions are in \mathcal{H} , every function in $M(\mathcal{H})$ is analytic on \mathbb{D} . Moreover, if $\lambda \in \mathbb{D}$ then

$$|\omega(\lambda)K_\lambda(\lambda)| = |\langle M_\omega K_\lambda, K_\lambda \rangle| \leq \|M_\omega\| \|K_\lambda\|^2.$$

This implies that $|\omega(\lambda)| \leq \|M_\omega\|$ for every $\lambda \in \mathbb{D}$ and so $\omega \in H^\infty$. If $\omega \in M(\mathcal{H})$ and φ is a mapping from \mathbb{D} into \mathbb{D} such that $f \circ \varphi$ is in \mathcal{H} for every $f \in \mathcal{H}$, then an application of the closed graph theorem shows that the weighted composition operator $C_{\omega, \varphi}$ defined by $C_{\omega, \varphi}(f)(z) = M_\omega C_\varphi(f)(z) = \omega(z)f(\varphi(z))$ is bounded. From now on, we assume that ω and φ satisfy these properties. For a positive integer n , the n th iterate of φ , denoted by φ_n , is the function obtained by composing φ with itself n times; also, φ_0 is defined to be the identity function. Moreover, when φ is invertible, we define the iterates $\varphi_{-n} = \varphi^{-1} \circ \varphi^{-1} \circ \dots \circ \varphi^{-1}$ (n times).

Also, $C_{w, \varphi}^* K_\lambda = \overline{w(\lambda)} K_{\varphi(\lambda)}$ for every λ in \mathbb{D} which implies that $C_{w, \varphi}^{*n} K_\lambda = \prod_{j=0}^{n-1} \overline{w(\varphi_j(\lambda))} K_{\varphi_n(\lambda)}$. Moreover,

$$C_{w, \varphi}^n(f) = \left(\prod_{k=0}^{n-1} w \circ \varphi_k \right) f \circ \varphi_n \text{ for every } f \in \mathcal{H}.$$

The properties of composition and weighted composition operators on various spaces of analytic functions have been investigated by many authors; see monographs [4, 15] and, for example, the following recent papers [9, 10, 11] and references therein.

In this paper, we give sufficient conditions for the n -tuple of the adjoint of a weighted composition operator to be hypercyclic. Hypercyclicity of operators have been widely studied. It was shown by Rolewicz [13] that twice the backward shift on the space $\ell^2(\mathbb{N})$ is hypercyclic. Many natural operators are hypercyclic. For example, certain operators in the classes of weighted shifts [14], composition operators [2], and the adjoint of subnormal, hyponormal and multiplication operators [6, 3], and the weighted composition operators and their adjoint operators [16, 17, 11] are hypercyclic. A good source on this topic is [1].

Proposition 1 ([7], Proposition 2.4) *Suppose that $T = (T_1, \dots, T_n)$ is a hypercyclic tuple on a separable Banach space X . Then every non-zero orbit of $T^* = (T_1^*, \dots, T_n^*)$ is unbounded.*

Proposition 2 *If φ_1 and φ_2 are analytic maps of the disc into itself then $(C_{\varphi_1}^*, C_{\varphi_2}^*)$ is not hypercyclic on \mathcal{H} .*

Proof. Since $C_{\varphi_1}^{k_1} C_{\varphi_2}^{k_2} 1 = 1$, then the orbit of 1 under $(C_{\varphi_1}, C_{\varphi_2})$ is bounded. Thus, using Proposition 1, the result follows. □

2. Tuples of weighted composition operators

We begin this section with a lemma that gives a necessary and sufficient condition for two weighted composition operators to commute.

Lemma 1 *If $\omega_1(z)$ and $\omega_2(z)$ are nonzero for all $z \in \mathbb{D}$, then C_{ω_1, φ_1} and C_{ω_2, φ_2} commute if and only if $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$ and $\omega_1 \cdot (\omega_2 \circ \varphi_1) = \omega_2 \cdot (\omega_1 \circ \varphi_2)$.*

Proof. Suppose that C_{ω_1, φ_1} and C_{ω_2, φ_2} commute. Then

$$\omega_1 \cdot (\omega_2 \circ \varphi_1) = C_{\omega_1, \varphi_1} C_{\omega_2, \varphi_2} 1 = C_{\omega_2, \varphi_2} C_{\omega_1, \varphi_1} 1 = \omega_2 \cdot (\omega_1 \circ \varphi_2).$$

Moreover, since

$$\begin{aligned} (\omega_1 \cdot (\omega_2 \circ \varphi_1) \cdot (\varphi_2 \circ \varphi_1))(z) &= (C_{\omega_1, \varphi_1} C_{\omega_2, \varphi_2} g)(z) \\ &= (C_{\omega_2, \varphi_2} C_{\omega_1, \varphi_1} g)(z) = (\omega_2 \cdot (\omega_1 \circ \varphi_2) \cdot (\varphi_1 \circ \varphi_2))(z), \end{aligned}$$

where $g(z) = z$ we have $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$. For the converse, take $f \in \mathcal{H}$. Then

$$\begin{aligned} C_{\omega_1, \varphi_1} C_{\omega_2, \varphi_2} f &= \omega_1 \cdot (\omega_2 \cdot (f \circ \varphi_2)) \circ \varphi_1 \\ &= \omega_1 \cdot (\omega_2 \circ \varphi_1) \cdot f \circ \varphi_2 \circ \varphi_1 \\ &= \omega_2 \cdot (\omega_1 \circ \varphi_2) \cdot f \circ \varphi_1 \circ \varphi_2 \\ &= \omega_2 \cdot (\omega_1 \cdot (f \circ \varphi_1)) \circ \varphi_2 \\ &= C_{\omega_2, \varphi_2} C_{\omega_1, \varphi_1} f. \end{aligned}$$

□

Proposition 3 *If $T = (C_{\omega_1, \varphi_1}, C_{\omega_2, \varphi_2})$ is a hypercyclic tuple then*

- (1) $\omega_1(z)$ and $\omega_2(z)$ are both nonzero for every $z \in \mathbb{D}$.
- (2) (φ_1, φ_2) is univalent.

Proof. (1) If $\omega_1(z) = 0$ for some z , then $C_{\omega_1, \varphi_1}^* K_z = \overline{\omega_1(z)} K_{\varphi_1(z)} = 0$. Thus,

$$C_{\omega_2, \varphi_2}^{*n_j} C_{\omega_1, \varphi_1}^{*m_j} K_z = 0$$

for every $m_j \geq 0$ and $n_j \geq 0$ which implies that an orbit of $T^* = (C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is bounded. Therefore, by Proposition 1, $T = (C_{\omega_1, \varphi_1}, C_{\omega_2, \varphi_2})$ is not hypercyclic. Hence, $\omega_1(z) \neq 0$ for every $z \in \mathbb{D}$. Similarly, $\omega_2(z) \neq 0$ for every $z \in \mathbb{D}$.

(2) Let f be a hypercyclic vector for T . Suppose that $(\varphi_1(z), \varphi_2(z)) = (\varphi_1(\lambda), \varphi_2(\lambda))$. Then

$$\begin{aligned} \langle C_{\omega_2, \varphi_2}^m C_{\omega_1, \varphi_1}^n f, K_z \rangle &= \langle C_{\omega_1, \varphi_1}^n f, C_{\omega_2, \varphi_2}^{*m} K_z \rangle \\ &= \left\langle \prod_{i=0}^{n-1} \omega_1 \circ (\varphi_1)_i \cdot f \circ (\varphi_1)_n, \left[\prod_{i=0}^{m-1} \overline{\omega_2 \circ (\varphi_2)_i(z)} \right] K_{(\varphi_2)_m(z)} \right\rangle \\ &= \prod_{i=0}^{m-1} \omega_2((\varphi_2)_i(z)) \prod_{i=0}^{n-1} (\omega_1 \circ (\varphi_1)_i)((\varphi_2)_m(z)) \cdot (f \circ (\varphi_1)_n)((\varphi_2)_m(z)) \\ &= \omega_1((\varphi_2)_m(z)) \omega_2(z) \prod_{i=1}^{m-1} \omega_2((\varphi_2)_i(\lambda)) \prod_{i=1}^{n-1} (\omega_1 \circ (\varphi_1)_i)((\varphi_2)_m(\lambda)) \cdot (f \circ (\varphi_1)_n)((\varphi_2)_m(\lambda)) \\ &= \frac{\omega_1((\varphi_2)_m(z)) \omega_2(z)}{\omega_1((\varphi_2)_m(\lambda)) \omega_2(\lambda)} \prod_{i=0}^{m-1} \omega_2((\varphi_2)_i(\lambda)) \prod_{i=0}^{n-1} (\omega_1 \circ (\varphi_1)_i)((\varphi_2)_m(\lambda)) \cdot (f \circ (\varphi_1)_n)((\varphi_2)_m(\lambda)) \\ &= \frac{\omega_1((\varphi_2)_m(z)) \omega_2(z)}{\omega_1((\varphi_2)_m(\lambda)) \omega_2(\lambda)} \langle C_{\omega_2, \varphi_2}^m C_{\omega_1, \varphi_1}^n f, K_\lambda \rangle, \end{aligned}$$

where m and n are non-negative integers so that $m^2 + n^2 \neq 0$. Thus,

$$\langle g, K_z \rangle = \frac{\omega_1((\varphi_2)_m(z)) \omega_2(z)}{\omega_1((\varphi_2)_m(\lambda)) \omega_2(\lambda)} \langle g, K_\lambda \rangle$$

for every $g \in \mathcal{H}$. Set $g \equiv 1$. Therefore,

$$\langle h, K_z \rangle = \langle h, K_\lambda \rangle$$

for every $h \in \mathcal{H}$. Now, taking $h(s) = s$, we get $z = \lambda$. □

We remark that it follows from the Denjoy-Wolff theorem [4] that if φ is a self map of \mathbb{D} and has a fixed point in \mathbb{D} then it is unique.

Proposition 4 *If $T = (C_{\omega_1, \varphi_1}, C_{\omega_2, \varphi_2})$ is a hypercyclic tuple and a is an interior fixed point of φ_1 or φ_2 , then $|\omega_1(a)| > 1$ or $|\omega_2(a)| > 1$.*

Proof. Suppose that $\varphi_1(a) = a$. Then $\varphi_1(\varphi_2(a)) = \varphi_2(\varphi_1(a)) = \varphi_2(a)$, which implies that $\varphi_2(a) = a$. So

$$C_{\omega_2, \varphi_2}^{*m} C_{\omega_1, \varphi_1}^{*n} K_a = \overline{(\omega_1(a))^n} C_{\omega_2, \varphi_2}^{*m} K_a = \overline{(\omega_1(a))^n (\omega_2(a))^m} K_a$$

Now, if $|\omega_1(a)| \leq 1$ and $|\omega_2(a)| \leq 1$, then $orb(T^*, K_a)$ is bounded. Thus, by Proposition 1, T is not hypercyclic, which is a contradiction. □

Corollary 1 *If φ_1 or φ_2 has an interior fixed point then $(C_{\varphi_1}, C_{\varphi_2})$ is not hypercyclic.*

Proof. Put $\omega_1(z) \equiv 1$ and $\omega_2(z) \equiv 1$ in Proposition 4. □

An argument similar to the proof of Proposition 2.5 of [7] shows the next proposition.

Proposition 5 (*Hypercyclicity Criterion*) Suppose that (T_1, T_2, \dots, T_n) is an n -tuple of operators on a separable Banach space Z . Suppose further that there exist n strictly increasing sequences of positive integers $\{k_{1j}\}_j, \{k_{2j}\}_j, \dots,$ and $\{k_{nj}\}_j$, dense sets X and Y in Z and functions $S_j : Y \rightarrow Z$ such that

- (1) For each $x \in X, T_1^{k_{1j}} T_2^{k_{2j}} \dots T_n^{k_{nj}} x \rightarrow 0$ as $j \rightarrow \infty$;
- (2) for each $y \in Y, S_j y \rightarrow 0$ as $j \rightarrow \infty$;
- (3) for each $y \in Y, T_1^{k_{1j}} T_2^{k_{2j}} \dots T_n^{k_{nj}} S_j y \rightarrow y$ as $j \rightarrow \infty$.

Then (T_1, T_2, \dots, T_n) is hypercyclic.

It follows from Lemma 1 that if

$$\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1, \quad \omega_1 = \omega_1 \circ \varphi_2 \quad \text{and} \quad \omega_2 = \omega_2 \circ \varphi_1, \tag{*}$$

then C_{ω_1, φ_1} and C_{ω_2, φ_2} commute. We give some examples of such functions. Suppose that $\varphi_r(z) = e^{ir\pi} z$ where $r = \frac{p}{q}$, p and q are integers so that $(p, q) = 1$. Define $\omega_r(z) = \sum_{n=0}^{\infty} a_n z^n$, where

$$a_n = \begin{cases} \frac{1}{2^n} & (n = \frac{2kq}{p} \text{ for some } k \in \mathbb{Z}), \\ 0 & \text{otherwise;} \end{cases}$$

then $\omega_r \in H^\infty$. Moreover, $\omega_r \circ \varphi_r(z) = \omega_r(z)$ for all $z \in \mathbb{D}$ and $\varphi_r \circ \varphi_s = \varphi_s \circ \varphi_r$.

Theorem 1 Let φ_1 and φ_2 be two disc automorphism such that $(*)$ holds and

$$\sup_{n \in \mathbb{Z}} \|K_{(\varphi_2)_n \circ (\varphi_1)_n(z)}\| < \infty$$

for every $z \in \mathbb{D}$. If the sets

$$A = \{z \in \mathbb{D} : \lim_{n \rightarrow +\infty} \prod_{j=0}^{n-1} (\omega_1 \circ (\varphi_1)_j)(z) \cdot (\omega_2 \circ (\varphi_2)_j)(z) = 0\}$$

and

$$B = \{z \in \mathbb{D} : \lim_{n \rightarrow +\infty} \prod_{j=0}^n [(\omega_1 \circ (\varphi_1)_{-j})(z) \cdot (\omega_2 \circ (\varphi_2)_{-j})(z)]^{-1} = 0\}$$

have limit points in \mathbb{D} , then $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is hypercyclic.

Proof. We will show that the hypercyclicity criterion holds. To see this take $T_i = C_{\omega_i, \varphi_i}^*$ for $i = 1, 2$. Since

$$T_i^n K_z = \left[\prod_{j=0}^{n-1} \overline{(\omega_i \circ (\varphi_i)_j)(z)} \right] K_{(\varphi_i)_n(z)}$$

for $i = 1, 2$ and $n \geq 1$, we have

$$T_2^n T_1^n K_z = \left[\prod_{j=0}^{n-1} \overline{\omega_1 \circ (\varphi_1)_j(z)} \right] \left[\prod_{j=0}^{n-1} \overline{\omega_2 \circ (\varphi_2)_j \circ (\varphi_1)_n(z)} \right] K_{(\varphi_2)_n \circ (\varphi_1)_n(z)}$$

for every $n \geq 1$.

Put $S_A = \text{span}\{K_z : z \in A\}$ and $S_B = \text{span}\{K_z : z \in B\}$. Therefore, $\overline{S_A} = \overline{S_B} = \mathcal{H}$ thanks to $(S_A)^\perp = (S_B)^\perp = (0)$.

Since $\sup_{n \in \mathbb{Z}} \|K_{(\varphi_2)_n \circ (\varphi_1)_n(z)}\| < \infty$, $\omega_2 \circ \varphi_1 = \omega_2$ and $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, we conclude that for every $f \in S_A$

$$T_2^m T_1^n f \longrightarrow 0$$

as $n \longrightarrow \infty$.

On the other hand, $\omega_2 \circ \varphi_1^{-1} = \omega_2$, $\omega_1 \circ \varphi_2^{-1} = \omega_1$ and $\varphi_1^{-1} \circ \varphi_2^{-1} = \varphi_2^{-1} \circ \varphi_1^{-1}$; therefore, if $z \in B$ then $\varphi_1^{-1} \circ \varphi_2^{-1}(z) \in B$. So we can define

$$S : \{K_z : z \in B\} \longrightarrow S_B$$

by

$$SK_z = \overline{(\omega_1((\varphi_1)^{-1}(z)) \cdot \omega_2((\varphi_2)^{-1}(z)))}^{-1} K_{\varphi_1^{-1} \circ \varphi_2^{-1}(z)}$$

and extend it linearly to S_B . Now, $T_2 T_1 S K_z = K_z$, and so $T_2^n T_1^n S^n$ is the identity on S_B for every $n \geq 0$. Moreover, it is easily seen that

$$S^n K_z = \prod_{j=1}^n \overline{[(\omega_1 \circ (\varphi_1)_{-j})(z) \cdot (\omega_2 \circ (\varphi_2)_{-j})(z)]}^{-1} K_{((\varphi_1)_{-n} \circ (\varphi_2)_{-n})(z)}$$

for every $n \geq 1$; thus, S^n converges pointwise to zero on the dense subset S_B . Hence, hypercyclicity criterion implies that $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is hypercyclic. □

Corollary 2 *If the sets*

$$\{z \in \mathbb{D} : \lim_{n \rightarrow +\infty} (\omega_1(z)\omega_2(z))^n = 0\}$$

and

$$\{z \in \mathbb{D} : \lim_{n \rightarrow +\infty} \frac{1}{(\omega_1(z)\omega_2(z))^n} = 0\}$$

have limit points in \mathbb{D} then $(M_{\omega_1}^, M_{\omega_2}^*)$ is hypercyclic on \mathcal{H} .*

Proof. Put $\varphi_1(z) = \varphi_2(z) = z$ and apply the preceding theorem. □

Example 1 *Let $\omega_1(z) = z$ and $\omega_2(z) = z + 5$. It is easily seen that*

$$\{x : 0 \leq x \leq \frac{-5 + \sqrt{29}}{2}\} \subseteq \{z \in \mathbb{D} : \lim_{n \rightarrow +\infty} (z(z + 5))^n = 0\}$$

and

$$\{x : -1 < x < \frac{-5 + \sqrt{21}}{2}\} \subseteq \{z \in \mathbb{D} : \lim_{n \rightarrow +\infty} \frac{1}{(z(z + 5))^n} = 0\}$$

hence $(M_{\omega_1}^, M_{\omega_2}^*)$ is hypercyclic on \mathcal{H} . Also, since $\text{ran } \omega_i \cap \partial \mathbb{D} = \emptyset$ for $i = 1, 2$, the operators $M_{\omega_i}^*$, $i = 1, 2$ are not hypercyclic on \mathcal{H} (see [[8], Theorem 4.9]).*

Recall that if φ is a hyperbolic automorphism then by the Denjoy-Wolff Theorem, one of its fixed point is the Denjoy-Wolff point of φ and the other is repulsive; i.e., it is the Denjoy-Wolff point of φ^{-1} . Furthermore, the angular derivative of φ at the Denjoy-Wolff point a , $\varphi'(a)$ is less than 1 (see [[2], Page 24]).

Corollary 3 *Suppose that $\{K_\lambda : \lambda \in \mathbb{D}\}$ is bounded in which K_λ is the reproducing kernel at λ , and φ_1 and φ_2 are two hyperbolic automorphisms with the Denjoy-Wolff points a_1 and a_2 and repulsive fixed points b_1 and b_2 , respectively. Moreover, suppose that $(*)$ holds, ω_1 and ω_2 have non-tangential limits $\omega_1(a_1)$ at a_1 , $\omega_2(a_2)$ at a_2 , $\omega_1(b_1)$ at b_1 and $\omega_2(b_2)$ at b_2 . If $|\omega_1(a_1)\omega_2(a_2)| < 1 < |\omega_1(b_1)\omega_2(b_2)|$ then $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is hypercyclic.*

Proof. Since $\varphi_1'(a_1) < 1$ for every $z \in \mathbb{D}$, there is a non-tangential approach region containing all iterates $(\varphi_1)_n(z)$ (see [[4], Lemma 2.66]), so $\lim_{n \rightarrow \infty} \omega_1((\varphi_1)_n(z)) = \omega_1(a_1)$. Similarly, $\lim_{n \rightarrow \infty} \omega_2((\varphi_2)_n(z)) = \omega_2(a_2)$. Thus,

$$\lim_{n \rightarrow \infty} (\omega_1 \circ (\varphi_1)_n(z)) \cdot (\omega_2 \circ (\varphi_2)_n(z)) = \omega_1(a_1)\omega_2(a_2),$$

which implies that

$$\sum_{j=0}^{\infty} (1 - |(\omega_1 \circ (\varphi_1)_j(z)) \cdot (\omega_2 \circ (\varphi_2)_j(z))|) = \infty.$$

Therefore,

$$\lim_{n \rightarrow \infty} \prod_{j=0}^{n-1} |(\omega_1 \circ (\varphi_1)_j(z)) \cdot (\omega_2 \circ (\varphi_2)_j(z))| = 0.$$

Thus, the set A in Theorem 1, has a limit point in \mathbb{D} . Similarly, since $((\varphi_1)_{-1})'(b_1) < 1$ and $((\varphi_2)_{-1})'(b_2) < 1$,

$$\sum_{j=1}^{\infty} (1 - |(\omega_1 \circ (\varphi_1)_{-j}(z)) \cdot (\omega_2 \circ (\varphi_2)_{-j}(z))|^{-1}) = \infty,$$

and so the set B in Theorem 1, has a limit point in \mathbb{D} . Hence, the proof is completed by applying Theorem 1. □

Note that if φ_1 and φ_2 are two elliptic automorphisms so that C_{φ_1} and C_{φ_2} commute then they have the same interior fixed points. For $\alpha \in \mathbb{D}$, consider an automorphism of \mathbb{D} defined by $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$, ($z \in \mathbb{D}$). Some spaces such as the Hardy, Bergman and Dirichlet spaces contain ψ_α for every $\alpha \in \mathbb{D}$.

Theorem 2 *Suppose that \mathcal{H} contains ψ_α for every $\alpha \in \mathbb{D}$. Let φ_1 and φ_2 be two elliptic automorphisms with an interior fixed point a such that $(*)$ holds. If the sets A and B in Theorem 1 have limits in \mathbb{D} , then $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is hypercyclic.*

Proof. First, assume that $a = 0$. Then $\varphi_1(z) = e^{i\theta_1}z$ and $\varphi_2(z) = e^{i\theta_2}z$ for some $\theta_1, \theta_2 \in [0, 2\pi]$. Thus, for $z \in \mathbb{D}$, $\{(\varphi_2)_n \circ (\varphi_1)_n(z) : n \in \mathbb{Z}\} \subseteq z\partial\mathbb{D}$. But $z\partial\mathbb{D}$ is a compact subset of \mathbb{D} and so for $f \in \mathcal{H}$ the continuity of f implies that

$$(f((\varphi_2)_n \circ (\varphi_1)_n(z)))_{n \in \mathbb{Z}}$$

is a bounded sequence; this along with the principle of uniform boundedness shows that for every $z \in \mathbb{D}$

$$(K_{(\varphi_2)_n \circ (\varphi_1)_n(z)})_{n \in \mathbb{Z}}$$

is also bounded. Hence, by applying Theorem 1, $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is hypercyclic.

Now, for the general case put $\mathcal{K} = \{f \circ \psi_a^{-1} : f \in \mathcal{H}\}$ endowed with inner product

$$\langle f, g \rangle_{\mathcal{K}} = \langle f \circ \psi_a, g \circ \psi_a \rangle_{\mathcal{H}}.$$

Then \mathcal{K} is a Hilbert space of analytic functions on \mathbb{D} and $C_{\psi_a} : \mathcal{K} \rightarrow \mathcal{H}$ defined by $C_{\psi_a} f = f \circ \psi_a$ is a linear isometric isomorphism. Furthermore, $\tilde{\varphi}_1 = \psi_a \circ \varphi_1 \circ \psi_a^{-1}$ and $\tilde{\varphi}_2 = \psi_a \circ \varphi_2 \circ \psi_a^{-1}$ are automorphisms with the interior fixed point zero, and $\tilde{\omega}_1 = \omega_1 \circ \psi_a^{-1}$ and $\tilde{\omega}_2 = \omega_2 \circ \psi_a^{-1}$ are in $M(\mathcal{K})$. Finally, since by the first step $(C_{\tilde{\omega}_1, \tilde{\varphi}_1}^*, C_{\tilde{\omega}_2, \tilde{\varphi}_2}^*)$ is hypercyclic and $C_{\tilde{\omega}_i, \tilde{\varphi}_i} = C_{\psi_a}^{-1} \circ C_{\omega_i, \varphi_i} \circ C_{\psi_a}$ for $i = 1, 2$, one can see that $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is also hypercyclic. \square

Example 2 Consider $\varphi_1(z) = iz$, $\varphi_2(z) = -iz$, $\omega_1(z) = z^4$ and $\omega_2(z) = z^4 + 3$. Then the sets A and B mentioned in Theorem 1 are

$$A = \{z \in \mathbb{D} : \lim_{n \rightarrow +\infty} z^{4n}(z^4 + 3)^n = 0\}$$

and

$$B = \{z \in \mathbb{D} : \lim_{n \rightarrow +\infty} \frac{1}{z^{4n}(z^4 + 3)^n} = 0\}.$$

It is easily seen that $[0, \frac{1}{2}] \subseteq A$ and $(\frac{1}{\sqrt[4]{2}}, 1) \subseteq B$. Hence, $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is hypercyclic.

Proposition 6 Let φ_1 and φ_2 be two elliptic automorphisms with an interior fixed point a and $\omega_1, \omega_2 : \mathbb{D} \rightarrow \mathbb{C}$ satisfy the inequality

$$|\omega_1(a)\omega_2(a)| < 1 < \liminf_{|z| \rightarrow 1^-} |\omega_1(z)\omega_2(z)|.$$

If $(*)$ holds then $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is a hypercyclic pair.

Proof. As it is seen in the preceding theorem, we can assume that $a = 0$ and $\varphi_1(z) = e^{i\theta_1}z$ and $\varphi_2(z) = e^{i\theta_2}z$ for some $\theta_1, \theta_2 \in [0, 2\pi]$. Therefore, $(\varphi_2)_n \circ (\varphi_1)_n(z) = e^{in\theta_2}e^{in\theta_1}z$ for every $n \in \mathbb{Z}$. This along with the principle of uniform boundedness implies that $\sup_{n \in \mathbb{Z}} \|K_{(\varphi_2)_n \circ (\varphi_1)_n}(z)\| < \infty$ for every $z \in \mathbb{D}$. On the other hand, since $|\omega_1(0)\omega_2(0)| < 1$ there exist a constant λ_1 and a positive number δ_1 such that $|\omega_1(z)\omega_2(z)| < \lambda_1 < 1$ whenever $|z| < \delta_1$. This, in turn, implies that if $|z| < \delta_1$ then

$$\left| \prod_{j=0}^{n-1} \omega_1((\varphi_1)_j(z))\omega_2((\varphi_2)_j(z)) \right| < \lambda_1^n \rightarrow 0$$

as $n \rightarrow +\infty$. Consequently, $\{z : |z| < \delta_1\}$ is a subset of the set A in Theorem 1. Moreover, since $1 < \liminf_{|z| \rightarrow 1^-} |\omega_1(z)\omega_2(z)|$ there exist a constant λ_2 and a positive number $\delta_2 < 1$ such that $|\omega_1(z)\omega_2(z)| > \lambda_2 > 1$ when $|z| > 1 - \delta_2$. Therefore, if $|z| > 1 - \delta_2$ then

$$\prod_{j=1}^n |(\omega_1 \circ (\varphi_1)_{-j})(z)(\omega_2 \circ (\varphi_2)_{-j})(z)|^{-1} < \frac{1}{\lambda_2^n} \rightarrow 0$$

as $n \rightarrow +\infty$. Thus, $\{z : |z| > 1 - \delta_2\}$ is a subset of the set B in Theorem 1. Hence $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is hypercyclic. \square

Example 3 Consider $\varphi_1(z) = iz$, $\varphi_2(z) = e^{i\frac{\pi}{4}}z$, $\omega_1(z) = z^8$ and $\omega_2(z) = z^4 + c$ where $|c| > 2$. Since

$$|\omega_1(0)\omega_2(0)| < 1 < \liminf_{|z| \rightarrow 1^-} |\omega_1(z)\omega_2(z)|$$

$(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$, is a hypercyclic pair.

Proposition 7 Suppose that there are positive integers m_1 and m_2 such that $(\varphi_1)_{m_1}(z) = z$ and $(\varphi_2)_{m_2}(z) = z$ for all $z \in \mathbb{D}$. If

$$\partial\mathbb{D} \cap \bigcap_{j=0}^{m_1 m_2 - 1} \text{ran}(\omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j) \neq \emptyset,$$

and (*) holds then $(C_{\omega_1, \varphi_1}^*, C_{\omega_2, \varphi_2}^*)$ is hypercyclic.

Proof. We observe that $\{(\varphi_1)_n \circ (\varphi_2)_n(z) : n \in \mathbb{Z}\}$ is a subset of $\{(\varphi_1)_j \circ (\varphi_2)_k(z) : j = 0, \dots, m_1 - 1, k = 0, \dots, m_2 - 1\}$. So the sequence $(K_{(\varphi_1)_n \circ (\varphi_2)_n(z)})_{n \in \mathbb{Z}}$ is bounded for every z in \mathbb{D} . Furthermore, since the function

$$\left(\prod_{j=0}^{m_1 m_2 - 1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j \right)(z)$$

is analytic on \mathbb{D} , the open mapping theorem implies that

$$U = \{z \in \mathbb{D} : \left| \left(\prod_{j=0}^{m_1 m_2 - 1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j \right)(z) \right| < 1\}$$

and

$$V = \{z \in \mathbb{D} : \left| \left(\prod_{j=0}^{m_1 m_2 - 1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j \right)(z) \right| > 1\}$$

are non-empty open sets. Fix $z \in U$, and let $P_n = \left(\prod_{j=0}^{n-1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j \right)(z)$ for $n \in \mathbb{N}$. For $\varepsilon > 0$, since

$$Q_k = \left(\prod_{j=0}^{m_1 m_2 - 1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j(z) \right)^k \rightarrow 0$$

as $k \rightarrow \infty$, one can choose $k > 0$ such that $M|Q_k| < \varepsilon$, in which

$$M = \max\left\{ \left| \left(\prod_{j=0}^i \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j \right)(z) \right| : i = 0, 1, \dots, m_1 m_2 - 1 \right\}.$$

Now, if $n > km_1m_2$ then $|P_n| \leq M|Q_k| < \varepsilon$ which implies that $P_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, since for every $z \in \mathbb{D}$

$$\left(\prod_{j=0}^{m_1m_2-1} \omega_1 \circ (\varphi_1)_{-j} \cdot \omega_2 \circ (\varphi_2)_{-j} \right)(z) = \left(\prod_{j=0}^{m_1m_2-1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j \right)(z)$$

by a similar method one can see that

$$\prod_{j=0}^{n-1} [\omega_1 \circ (\varphi_1)_{-j} \cdot \omega_2 \circ (\varphi_2)_{-j}](z)^{-1} \rightarrow 0$$

as $n \rightarrow \infty$ for every $z \in V$. Hence, the result follows using Theorem 1. \square

Corollary 4 *If $\text{ran}(\omega_1, \omega_2) \cap \partial\mathbb{D} \neq \emptyset$, then $(M_{\omega_1}^*, M_{\omega_2}^*)$ is hypercyclic.*

Proof. Let $\varphi_1(z) = z$ and $\varphi_2(z) = z$ for all $z \in \mathbb{D}$ in above proposition. \square

Taking $\omega_2(z) \equiv 1$ in the above corollary, we get the following result from [8], as a special case.

Corollary 5 ([8]) *If $\text{ran}(\omega_1) \cap \partial\mathbb{D} \neq \emptyset$, then $M_{\omega_1}^*$ is hypercyclic.*

Remark. By analogous proofs we can show that the results in this paper are also valid for n -tuples of the adjoint of the weighted composition operators on \mathcal{H} .

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