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# Generalization of some properties of Banach algebras to fundamental locally multiplicative topological algebras

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**Abstract**

In this article we generalized some properties of Banach algebras, to a new class of topological algebras namely fundamental and fundamental locally multiplicative topological algebras (abbreviated by $FLM$). Also the new notion of sub-multiplicatively metrizable topological algebra is given and some well known spectral properties of Banach algebras are generalized to such kind of algebras.

**Key Words:** $FLM$ algebras, fundamental topological algebras, holomorphic function, multiplicative linear functionals, semi-simple algebras, spectral radius

## 1. Introduction

The notion of fundamental topological spaces (also algebras) has been introduced in [1] in 1990 extending the meaning of both local convexity and local boundedness.

A topological linear space $\mathcal{A}$ is said to be fundamental one if there exists $b > 1$ such that for every sequence $(x_n)$ of $\mathcal{A}$, the convergence of $b^n(x_n - x_{n-1})$ to zero in $\mathcal{A}$ implies that $(x_n)$ is Cauchy.

A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental. The famous Cohen factorization theorem for complete metrizable fundamental topological algebras is proved in [1] and the $n^{\text{th}}$ roots and quasi square roots in fundamental topological algebras are studied in [4].

The fundamental locally multiplicative topological algebras (abbreviated by $FLM$) with a property very similar to the normed algebras is also introduced in [2]. A fundamental topological algebra is called locally multiplicative if there exists a neighborhood $U_0$ of zero such that, for every neighborhood $V$ of zero, the sufficiently large powers of $U_0$ lie in $V$.

Also in [2] a topological structure is defined on the algebraic dual space of an $FLM$ algebra to make it a normed space, and some of the famous theorems of Banach algebras are extended for complete metrizable $FLM$ algebras. In this paper we have studied the linear multiplicative functionals on $FLM$ algebras and proved some results on them in section 2. In section 3, we introduced the new notion of sub-multiplicative metrizable topological algebras and by using it we generalized some properties of Banach algebras to $FLM$ algebras.

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## 2. Multiplicative functional on $FLM$ algebras

A version of the *Gleason, Kahane-Zelazko theorem* is proved for $FLM$ algebras in [3]. In theorem 5.5 of [3], $T:\mathcal{A}\longrightarrow\mathbb{C}$ is assumed a non-zero linear functional on $\mathcal{A}$ and proved that $T$ is multiplicative if and only if $T(a)\in Sp(a)$ for all $a\in\mathcal{A}$. Now by replacing $\mathbb{C}$ by a semi-simple complete metrizable $FLM$ algebra $\mathcal{B}$ we generalized it as follow.

**Theorem 2.1** *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative complete metrizable $FLM$ algebras, with unit elements, and let $\mathcal{B}$ be semi-simple. If $T:\mathcal{A}\longrightarrow\mathcal{B}$ is a linear mapping, such that $Sp(Tx)\subset Sp(x)$, for any $x\in\mathcal{A}$, then $T$ is a multiplicative mapping.*

**Proof.** Let $f$ be a multiplicative and linear functional on $\mathcal{B}$ and put $F(x)=f(Tx)$ for any $x\in\mathcal{A}$. So $F$ is a linear functional on $\mathcal{A}$, and also by theorem 5.5 [3],

$$F(x)=f(Tx)\in Sp(Tx)\subset Sp(x),$$

and so by using again theorem 5.5 [3] $F$ is multiplicative and linear functional on $\mathcal{A}$. It follows that

$$F(xy)=F(x)F(y),$$

or

$$f(Txy)=f(Tx)f(Ty)=f(TxTy).$$

Since $f$ is arbitrary multiplicative linear functional on $\mathcal{B}$ and $\mathcal{B}$ is semi-simple, thus $T$ is multiplicative. □

Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative complete metrizable $FLM$ algebras, with unit elements $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. If $T$ is a multiplicative linear mapping from $\mathcal{A}$ into $\mathcal{B}$, such that $Te_{\mathcal{A}}\neq e_{\mathcal{B}}$, then it may be $Sp(Tx)$ is not a subset of $Sp(x)$. Furthermore $Sp(x)\subset Sp(Tx)$. For example, let $A_1$ and $A_2$ be commutative complete metrizable $FLM$ algebras, $B=A_1\oplus A_2$ and $T:A_1\longrightarrow B$. Then we have the following theorem.

**Theorem 2.2** *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative complete metrizable $FLM$ algebras, with unit elements $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. Let $T$ be a linear multiplicative mapping from $\mathcal{A}$ to $\mathcal{B}$, such that $Te_{\mathcal{A}}=e_{\mathcal{B}}$. Then for any $x\in\mathcal{A}$, $Sp(Tx)\subset Sp(x)$.*

**Proof.** By assumption we have

$$e_{\mathcal{B}}=Te_{\mathcal{A}}=Txx^{-1}=TxTx^{-1},$$

for any invertible element $x\in\mathcal{A}$. This shows that, for any such $x$, $Tx$ is invertible in $\mathcal{B}$ and

$$T(x^{-1})=(Tx)^{-1}.$$

Now if $\lambda\notin Sp(x)$, then $x-\lambda e_{\mathcal{A}}$ is invertible in $\mathcal{A}$ and so $T(x-\lambda e_{\mathcal{A}})=Tx-\lambda e_{\mathcal{B}}$ is invertible in $\mathcal{B}$. Therefore $\lambda\notin Sp(Tx)$. □

## 3. New results on $FLM$ algebras

In this section, by introducing the new notion of sub-multiplicative metrizable topological algebra, we generalize some well known spectral properties of Banach algebras to complete metrizable $FLM$ algebras.

By $\Omega_{\mathcal{A}}$ we mean the set of all elements $a \in \mathcal{A}$ such that $\rho(a) < 1$, where $\rho(a)$ is the spectral radius of $a \in \mathcal{A}$. We denote the center of topological algebra $\mathcal{A}$, by $Z(\mathcal{A})$, such that

$$Z(\mathcal{A}) = \{a \in \mathcal{A} : ax = xa, \quad \textit{for all } x \in \mathcal{A}\}.$$

**Definition 3.1** *Let $(\mathcal{A}, d)$ be a metrizable topological algebra. We say $\mathcal{A}$ is a sub-multiplicative metrizable topological algebra if*

$$d(0, xy) \leq d(0, x)d(0, y)$$

*for each $x, y \in \mathcal{A}$.*

It is clear that, when $\mathcal{A}$ is a sub-multiplicatively metrizable topological algebra, the meter $d_{\mathcal{A}}$ is not a discrete meter; for example, if $d_{\mathcal{A}}$ is a Dirac meter on some ideal $E$ of $\mathcal{A}$, the sub-multiplicatively of the meter fails. For abbreviation we denote $d_{\mathcal{A}}(0, x)$ by $D_{\mathcal{A}}(x)$ for any $x \in \mathcal{A}$.

The following lemma is proved for Banach algebras and has a similar proof for $FLM$ algebras (see theorem 3.2.6, [5]). Therefore, we remove its proof, because it is well known and clear.

**Lemma 3.2** *Let $\mathcal{A}$ be a complete metrizable $FLM$ algebra and $x \in A$. Then for every nonconstant polynomial $P$ with complex coefficients we have*

$$Sp(P(x)) = P(Sp(x)).$$

Let $\mathcal{A}$ be a complete metrizable fundamental topological algebra with unit $e$ and $x \in \mathcal{A}$. If for some $b > 1$, $b^n x^n \longrightarrow 0$ in $\mathcal{A}$, then $e - x$ is invertible and

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n$$

and also if for some $b > 1$, $b^n (e - x)^n \longrightarrow 0$, then $x$ is invertible (theorem 4.1, [2]). If $\mathcal{A}$ is a complete metrizable $FLM$ algebra with meter $d_{\mathcal{A}}$, then $(e - x)$ is invertible for $d_{\mathcal{A}}(0, x) = D_{\mathcal{A}}(x) < 1$. Now if $\mathcal{A}$ is a complete metrizable $FLM$ algebra with sub-multiplicatively meter $d_{\mathcal{A}}$ and $\lambda \neq 0$, then $(e - \lambda x)$ is invertible for $d_{\mathcal{A}}(0, x) = D_{\mathcal{A}}(x) < |\lambda|$.

**Theorem 3.3** *Let $\mathcal{A}$ be a complete metrizable $FLM$ algebra with sub-multiplicative meter $d_{\mathcal{A}}$. Then $\rho(x) = \lim_{n \to \infty} D_{\mathcal{A}}(x^n)^{1/n}$.*

**Proof.** From above discussion, we have $\rho(x) \leq D_{\mathcal{A}}(x)$, for any $x \in \mathcal{A}$. Now if applied the previous lemma to $x^n$, we have $\rho(x)^n \leq D_{\mathcal{A}}(x^n)$. Let $f$ be a linear functional on $\mathcal{A}$, then the map from $\mathbb{C} \setminus Sp(x)$ to $\mathbb{C}$, which $\lambda \mapsto f((\lambda e - x)^{-1})$ is holomorphic. By theorem 4.1 of [2], we have

$$f((\lambda e - x)^{-1}) = \frac{1}{\lambda}(f(e) + \frac{f(x)}{\lambda} + \ldots + \frac{f(x^n)}{\lambda^n} + \ldots).$$

Fix $\lambda$, such that $|\lambda| > \rho(x)$. Then for every linear functional $f$ on $\mathcal{A}$, we have $\sup_n |\frac{f(x^n)}{\lambda^n}| < \infty$. By applying the Banach-Steinhaus theorem (theorem 2.8, [7]) to the space of all continuous linear functional on $\mathcal{A}$ and to the sequence of $T_n$ from that to $\mathbb{C}$, defined by $T_n(f) = \frac{f(x^n)}{\lambda^n}$, we conclude that there exists a constant $C$, depending to $\lambda$, such that $D_{\mathcal{A}}(x^n) \leq C|\lambda|^n$ for all $n \geq 1$. Then

$$\lim_{n\longrightarrow\infty} \sup D_{\mathcal{A}}(x^n)^{1/n} \leq |\lambda|,$$

for all $|\lambda| \geq \rho(x)$. Hence we conclude

$$\rho(x) \leq \lim_{n\longrightarrow\infty} \inf D_{\mathcal{A}}(x^n)^{1/n} \leq \lim_{n\longrightarrow\infty} \sup D_{\mathcal{A}}(x^n)^{1/n} \leq \rho(x),$$

therefore $\rho(x) = \lim_{n\to\infty} D_{\mathcal{A}}(x^n)^{1/n}$. □

Let $E(\mathcal{A})$ be the set of all elements $x \in \mathcal{A}$ for which $E(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$, can be defined. If $\mathcal{A}$ be a complete metrizable $FLM$ algebra, then $E(\mathcal{A}) = \mathcal{A}$ (theorem 5.4, [3]).

The following theorem is a version of Zemánek theorem (theorem 5.3.1, [5]) for $FLM$ algebras:

**Theorem 3.4** *Let $\mathcal{A}$ be a complete metrizable FLM algebra with sub-multiplicatively meter $d_{\mathcal{A}}$. Then the following statements are equivalent:*

*(i) $a$ is in the Jacobson radical of $\mathcal{A}$;*

*(ii) $Sp(a+x) = Sp(x)$, for all $x \in \mathcal{A}$;*

*(iii) $\rho(a+x) = 0$, for all quasi-nilpotent elements $x$ in $\mathcal{A}$;*

*(iv) $\rho(a+x) = 0$, for all quasi-nilpotent elements $x$ in a neighborhood $0$ in $\mathcal{A}$;*

*(v) there exists $C > 0$ such that $\rho(x) \leq CD_{\mathcal{A}}(x-a)$, for all $x \in \mathcal{A}$ in a neighborhood of $a$ in $\mathcal{A}$.*

**Proof.** Straightforward. □

**Theorem 3.5** *Let $\mathcal{A}$ be a complete semi-simple metrizable FLM algebra with sub-multiplicative meter. If $g : \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ be a holomorphic map satisfying $g(0) = 0$ and $g'(0) = I$, then $g(c) = c$ for all $c \in \Omega_{\mathcal{A}} \cap Z(\mathcal{A})$.*

To prove this theorem we need to the following lemma.

**Lemma 3.6** *Let $\mathcal{A}$ be a complete metrizable FLM algebra with sub-multiplicative meter $d_{\mathcal{A}}$. Suppose that $x, y \in \mathcal{A}$ satisfy $xy = yx$. Then $\rho(x+y) \leq \rho(x) + \rho(y)$ and $\rho(xy) \leq \rho(x)\rho(y)$.*

**Proof.** Since $xy = yx$, then $(xy)^n = x^n y^n$ for each integer $n \geq 1$. By theorem 3.3, we have

$$\begin{aligned} \rho(xy) &= \lim_{n\to 0} D_{\mathcal{A}}((xy)^n)^{1/n} = \lim_{n\to 0} D_{\mathcal{A}}(x^n y^n)^{1/n} \\ &\leq \lim_{n\to 0} D_{\mathcal{A}}(x^n)^{1/n} \lim_{n\to 0} D_{\mathcal{A}}(y^n)^{1/n} \\ &= \rho(x)\rho(y). \end{aligned}$$

Let $\rho(x) < \alpha$, $\rho(y) < \beta$ and $a = x/\alpha$, $b = y/\beta$. Then $\rho(a) < 1$ and $\rho(b) < 1$. Therefore there exists some integer $N$ such that for $n \geq N$, we have $\max(D_{\mathcal{A}}(a^{2^n}), D_{\mathcal{A}}(b^{2^n})) < 1$. Now let $\gamma_n = \max_{0\leq k\leq 2^n} D_{\mathcal{A}}(a^k)D_{\mathcal{A}}(b^{2^n-k})$, then we have

$$\begin{aligned}
D_{\mathcal{A}}((a+b)^{2^n})^{1/2^n} &= D_{\mathcal{A}}(\sum_{k=0}^{2^n}\begin{pmatrix} 2^n \\ k \end{pmatrix} x^k y^{2^n-k})^{1/2^n} \\
&\leq (\sum_{k=0}^{2^n}\begin{pmatrix} 2^n \\ k \end{pmatrix} \alpha^k \beta^{2^n-k} D_{\mathcal{A}}(a^k)D_{\mathcal{A}}(b^{2^n-k}))^{1/2^n} \\
&\leq (\alpha+\beta)\gamma_n^{1/2^n}.
\end{aligned}$$

The sequence $(\gamma_n)$ is decreasing and therefore

$$\begin{aligned}
\rho(x+y) &= \lim_{n\longrightarrow\infty}(D_{\mathcal{A}}(x+y)^{2^n})^{1/2^n} \\
&\leq (\alpha+\beta)\lim_{n\longrightarrow\infty}\sup\gamma_n^{1/2^n} \\
&\leq (\alpha+\beta)\lim_{n\longrightarrow\infty}\sup\gamma_N^{1/2^n} \\
&= \alpha+\beta,
\end{aligned}$$

for arbitrary $\rho(x) < \alpha$, $\rho(y) < \beta$. The proof is complete. □

**Proof.** [Proof of theorem 3.5]

Fix $c \in \Omega_A \cap Z(A)$. Define $f : \mathbb{C} \longrightarrow \Omega_A$ with $f(\lambda) = g(\lambda c)$. $f$ is holomorphic on

$$\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\rho(c)}\}.$$

Then $g$ has Taylor expansion about $0$ and we have

$$g(\lambda c) = \lambda c + \sum_{j=2}^{\infty}\lambda^j a_j \qquad (|\lambda| < \frac{1}{\rho(c)}).$$

Now we have to prove that $a_j = 0$, for all $j$; if not the case, suppose for contradiction, that there is some $j$ with $a_j \neq 0$ and let $k$ be the smallest integer such that $a_k \neq 0$. Take $q \in A$ with $\rho(q) = 0$ and let $n \geq 1$. Then, writing $g^n$ for the n-fold composition $g \circ \cdots \circ g$, we have

$$g^n(\lambda c + \lambda^k nq) = \lambda c + \lambda^k n(ak+q) + O(\lambda^{k+1}) \qquad (\lambda \to 0).$$

Now as $c$ and $q$ commute, it follows that $\rho(\lambda c+\lambda^k nq) \leq \rho(\lambda c) + \rho(\lambda^k nq) = |\lambda|\rho(c)$ (lemma 3.4), and so we can define a holomorphic function $h : \{0 < |\lambda| < 1/\rho(c)\} \longrightarrow \mathcal{A}$ by

$$h(\lambda) = \frac{g^n(\lambda c+\lambda^k nq) - \lambda c}{n\lambda^k} \qquad (0 < |\lambda| < 1/\rho(c)),$$

isolated singularity at $\lambda = 0$ can be removed by setting $h(0) = a_k + q$. By Vesentini's theorem (theorem 3.4.7, [5]), the composition $\rho \circ h$ is a subharmonic function on $\{0 < |\lambda| < 1/\rho(c)\}$, and so by the maximum principle

$$\rho(h(0)) \leq \max_{|\lambda|=1} \rho(h(\lambda)).$$

Making use of lemma 3.4 again to estimate the right-hand side, it follows that

$$\rho(a_k + q) \leq 2/n.$$

As this is true for each $n$, we can let $n \longrightarrow \infty$ deduce that $\rho(a_k + q) = 0$. And as this holds for each $q \in \mathcal{A}$ with $\rho(q) = 0$, Zemánek's characterization of the radical (theorem 3.5), implies that $a_k$ belongs to the radical of $\mathcal{A}$, which is zero since $\mathcal{A}$ is semi-simple. Thus $a_k = 0$, and we have arrived at a contradiction. We conclude that indeed $a_j = 0$ for all $j \geq 2$, and hence from (1) that $g(c) = c$. □

In theorem 3.5 the property of sub-multiplicativity of $FLM$ algebras is essential but in the next theorem we do not need it.

**Theorem 3.7** *Let $\mathcal{A}$ be a semi-simple complete metrizable $FLM$ algebra. Given $a \in \Omega_A \backslash Z_{\mathcal{A}}$, then there exists a holomorphic map $g : \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ satisfying $g(0) = 0$ and $g'(0) = I$ such that $g(a) \neq a$.*

**Proof.** Let $a \in \Omega_A \backslash Z_{\mathcal{A}}$. Then there exists $u \in \mathcal{A}$, such that $au \neq ua$. Suppose that $d_{\mathcal{A}}(0, u) < 1$, where $d_{\mathcal{A}}$ is a meter on $\mathcal{A}$. Then $v := \log(e - u)$ satisfies $e^{-v} a e^{v} \neq a$. Define $g : \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ by

$$g(x) = e^{-\frac{xv}{a}} x e^{\frac{xv}{a}} \qquad (x \in \Omega_{\mathcal{A}}).$$

Then $g$ is a holomorphic function, $g(0) = 0$ and $g'(0) = I$, but $g(a) = e^{-v} a e^{v} \neq a$. □

By combination of theorems 3.5 and 3.7, we have the following theorem.

**Theorem 3.8** *Let $\mathcal{A}$ and $\mathcal{B}$ be semi-simple complete metrizable $FLM$ algebras with sub-multiplicatively meter. If $f : \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{B}}$ is a biholomorphic map, then $f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}) = \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$.*

**Proof.** Let $c \in \Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}$ be an arbitrary. Without loss of generality suppose that $c \neq 0$. Then from assumption about $f$, $f(c) \neq f(0)$. Take $b \in \mathcal{B}$, and define $h : \Omega_{\mathcal{B}} \longrightarrow \Omega_{\mathcal{B}}$ by

$$h(y) = e^{-(\frac{y-f(0)}{f(c)-f(0)})^2 b} y e^{(\frac{y-f(0)}{f(c)-f(0)})^2 b} \qquad (y \in \Omega_{\mathcal{B}}).$$

By above definition $h$ is a holomorphic function, $h(f(0)) = f(0)$ and $h'(f(0)) = I$. Now set $g = f^{-1} \circ h \circ f$, then $g$ is a holomorphic function from $\Omega_{\mathcal{A}}$ into $\Omega_{\mathcal{A}}$, such that $g(0) = 0$ and $g'(0) = I$. Therefore $g(c) = c$ (theorem 2.1), and from definition of $g$, we have $h(f(c)) = f(c)$. By proof of theorem 2.2, $f(c) \in Z_{\mathcal{B}}$. In case $c = 0$, by continuity of $f$, the proof remains true. Hence $f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}) \subset \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$.

Let $c \in \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$, with applying the same argument to $f^{-1}$, we have $c \in f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}})$. Therefore $\Omega_{\mathcal{B}} \cap Z_{\mathcal{B}} \subset f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}})$. □

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