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## Simultaneous proximality of vector valued function spaces

*Mona Khandajji, Fadi Awawdeh, Jamila Jawdat*

### Abstract

A characterization of best simultaneous approximation of Köthe spaces of vector-valued functions is given. This characterization is a generalization of some analogous theorems for Orlicz Bochner spaces.

**Key words and phrases:** Simultaneous approximation; Köthe Bochner function space

### 1. Introduction

Through this paper, let  $(T, \Sigma, \mu)$  be a finite complete measure space and  $L^0 = L^0(T)$  denote the space of all (equivalence classes) of  $\Sigma$ -measurable real valued functions. For  $f, g \in L^0$ ,  $f \leq g$  means that  $f(t) \leq g(t)$   $\mu$ -almost every where  $t \in T$ . A Banach space  $(E, \|\cdot\|_E)$  is said to be a Köthe space if

- (1) for  $f, g \in L^0$ ,  $|f| \leq |g|$  and  $g \in E$  imply  $f \in E$  and  $\|f\|_E \leq \|g\|_E$ ;
- (2) for each  $A \in \Sigma$ , if  $\mu(A)$  is finite then  $\chi_A \in E$ . See [7, p. 28].

A Köthe space  $E$  has absolutely continuous norm if for each  $f \in E$  and each decreasing sequence  $(A_n)$  converges to 0, then  $\|\chi_{A_n} f\|_E \rightarrow 0$ . A Köthe space  $E$  is said to be strictly monotone if  $x \geq y \geq 0$  and  $\|x\|_E = \|y\|_E$  imply  $x = y$ . Let  $E$  be a Köthe space on the measure space  $(T, \Sigma, \mu)$  and  $(X, \|\cdot\|_X)$  be a real Banach space then  $E(X)$  is the space (of all equivalence classes) of strongly measurable functions  $f : T \rightarrow X$  such that  $\|f(\cdot)\|_X \in E$  equipped with the norm

$$\| \|f\| \| = \| \|f(\cdot)\|_X \|_E.$$

The space  $(E(X), \| \cdot \|_E)$  is a Banach space called the Köthe Bochner function space [7]. For a function  $F = (f_1, f_2, \dots, f_n) \in (E(X))^n$ , we define the norm of  $F$  by

$$\| \|F\| \| = \left\| \sum_{i=1}^n \|f_i(\cdot)\|_X \right\|_E.$$

The most important classes of Köthe Bochner function spaces are the Lebesgue Bochner spaces  $L^p(X)$ , ( $1 \leq p < \infty$ ) and their generalization the Orlicz-Bochner spaces  $L^\Phi(X)$ . These spaces have been studied by many authors, cf. [2], [4], [5], [6], [8], [10].

Let  $Y$  be a closed subspace of  $X$ . For a set of elements  $x_1, x_2, \dots, x_n \in X$ , define

$$dist(x_1, x_2, \dots, x_n, Y) = \inf_{z \in Y} \left\{ \sum_{i=1}^n \|x_i - z\| \right\}.$$

We say that  $y_0 \in Y$  is a best simultaneous approximation to the set of elements  $x_1, x_2, \dots, x_n \in X$  if, for every  $z \in Y$ , we have

$$\sum_{i=1}^n \|x_i - y_0\| \leq \sum_{i=1}^n \|x_i - z\|.$$

If every set of elements  $x_1, x_2, \dots, x_n \in X$  admits a best simultaneous approximation in  $Y$ , then  $Y$  is said to be simultaneously proximal in  $X$ . In case when  $n = 1$ , we get the usual proximality.

In this paper, for a given closed subspace  $Y$  of  $X$  and  $F = (f_1, f_2, \dots, f_n) \in (E(X))^n$ , we are interested in the existence of n-tuples  $G_0 = (g_0, g_0, \dots, g_0) \in (E(Y))^n$  such that

$$\| \|F - G_0\| \| = \inf_{g \in E(Y)} \| \|F - (g, g, \dots, g)\| \|.$$

If such a function  $g_0$  exists, it is called a best simultaneous approximation of  $F = (f_1, f_2, \dots, f_n)$ . The problem of best simultaneous approximation can be viewed as a special case of vector valued approximation. Recent results in this area are due to Pinkus [9]. Results on best simultaneous approximation in general Banach spaces can also be found in [1], [11], [12].

It is the aim of this work to write and prove a formula for the distance  $dist_E(f_1, f_2, \dots, f_n, E(Y))$ , where  $f_1, f_2, \dots, f_n \in E(X)$ , similar to that of best approximation. This allows us to generalize some recent results in [3].

## 2. Distance formula

Through this section,  $X$  is a real Banach space and  $E(X)$  is a Köthe Bochner function space. For  $f_1, f_2, \dots, f_n \in E(X)$ , we define  $dist_E(f_1, \dots, f_n, E(Y))$  by

$$\begin{aligned} dist_E(f_1, f_2, \dots, f_n, E(Y)) &= \inf_{g \in E(Y)} \| \| (f_1, f_2, \dots, f_n) - (g, g, \dots, g) \| \| \\ &= \inf_{g \in E(Y)} \left\| \sum_{i=1}^n \| f_i(\cdot) - g(\cdot) \|_X \right\|_E. \end{aligned}$$

We also define  $B(f_1, f_2, \dots, f_n, E(Y))$  by the set

$$\{g \in E(Y) : \left\| \sum_{i=1}^n \| f_i(\cdot) - g(\cdot) \|_X \right\|_E = dist_E(f_1, f_2, \dots, f_n, E(Y))\}.$$

**Lemma 1** Let  $f_1, f_2, \dots, f_n \in E(X)$ ,  $Y$  a closed subspace of  $X$  and  $g : T \rightarrow Y$  be a strongly measurable function with  $g(t) \in B(f_1(t), f_2(t), \dots, f_n(t), Y)$  for almost all  $t \in T$ . Then  $g \in E(Y) \cap B(f_1, f_2, \dots, f_n, E(Y))$ .

**Proof.** Since  $g(t) \in B(f_1(t), f_2(t), \dots, f_n(t), Y)$ , for almost all  $t \in T$ , we have

$$\|g\| \leq \frac{2}{n} \sum_{i=1}^n \|f_i\|,$$

which shows that  $g \in E(Y)$ . Also, for any  $h \in E(Y)$ , we have

$$\left\| \sum_{i=1}^n \|f_i(\cdot) - g(\cdot)\|_X \right\|_E \leq \left\| \sum_{i=1}^n \|f_i(\cdot) - h(\cdot)\|_X \right\|_E,$$

thus  $g \in B(f_1, f_2, \dots, f_n, E(Y))$ . □

We can now state and prove the main result.

**Theorem 2** Let  $Y$  be a closed subspace of the real Banach space  $X$  and  $E(X)$  be a Köthe Bochner function space with absolutely continuous norm. If  $f_1, f_2, \dots, f_n \in E(X)$ , then the distance function  $dist(f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot))$  belongs to  $E$  and

$$\|dist(f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot), Y)\|_E = dist_E(f_1, f_2, \dots, f_n, E(Y)).$$

**Proof.** Let  $f_1, f_2, \dots, f_n \in E(X)$ , then there exist sequences  $(f_{n,i})$ ,  $1 \leq i \leq n$ , of simple functions in  $E(X)$  such that

$$\|f_{n,i}(t) - f_i(t)\| \rightarrow 0, \quad i = 1, 2, \dots, n, \text{ for almost all } t \text{ in } T.$$

The continuity of the distance function implies that

$$|dist(f_{n,1}(t), f_{n,2}(t), \dots, f_{n,n}(t), Y) - dist(f_1(t), f_2(t), \dots, f_n(t), Y)| \rightarrow 0.$$

Set

$$H_n(t) = dist(f_{n,1}(t), f_{n,2}(t), \dots, f_{n,n}(t), Y),$$

then  $H_n$  is a measurable function. Therefore the  $dist(f_1(\cdot), \dots, f_n(\cdot), Y)$  is measurable and

$$dist(f_1(t), f_2(t), \dots, f_n(t), Y) \leq \sum_{i=1}^n \|f_i(t) - z\|_X, \text{ for all } z \in Y.$$

As a consequence we can write

$$dist(f_1(t), f_2(t), \dots, f_n(t), Y) \leq \sum_{i=1}^n \|f_i(t) - g(t)\|_X, \text{ for all } g \in E(Y),$$

and we obtain, for all  $g \in E(Y)$ ,

$$\|dist(f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot), Y)\|_E \leq \left\| \sum_{i=1}^n \|f_i(\cdot) - g(\cdot)\|_X \right\|_E.$$

Thus,  $dist(f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot), Y) \in E$  and

$$\|dist(f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot), Y)\|_E \leq dist_E(f_1, f_2, \dots, f_n, E(Y)). \tag{1}$$

Fix  $\varepsilon > 0$ . Since  $E(X)$  is a Köthe Bochner function space with absolutely continuous norm, the simple functions are dense in  $E(X)$ , [7]. That is, there exist simple functions  $f_i^*$  in  $E(X)$  such that

$$\|f_i - f_i^*\| < \frac{\varepsilon}{n}, \text{ for } i = 1, \dots, n.$$

Assume that

$$f_i^*(t) = \sum_{k=1}^m x_k^i \chi_{A_k}(t),$$

where the  $A_k$ 's are pairwise disjoint measurable sets of  $T$  with  $\bigcup_{k=1}^m A_k = T$ ,  $\chi_{A_k}$ 's are the characteristic functions related to the  $A_k$ 's and  $x_k^i \in X$ , for  $k = 1, 2, \dots, m$  and  $i = 1, \dots, n$ . Since  $\mu(T)$  is finite, we can put  $\alpha = \|\chi_T\|$ . For each  $k = 1, 2, \dots, m$ , let  $y_k \in Y$  satisfy

$$\sum_{i=1}^n \|x_k^i - y_k\|_X \leq dist(x_k^1, x_k^2, \dots, x_k^n, Y) + \frac{\varepsilon}{\alpha}.$$

By setting  $g(t) = \sum_{k=1}^m y_k \chi_{A_k}(t)$ , we obtain the inequalities

$$\begin{aligned} \left\| \sum_{i=1}^n \|f_i^*(\cdot) - g(\cdot)\|_X \right\|_E &= \left\| \sum_{k=1}^m \chi_{A_k}(\cdot) \left[ \sum_{i=1}^n \|x_k^i - y_k\|_X \right] \right\|_E \\ &\leq \left\| \sum_{k=1}^m \chi_{A_k}(\cdot) \left[ dist(x_k^1, x_k^2, \dots, x_k^n, Y) + \frac{\varepsilon}{\alpha} \right] \right\|_E \\ &\leq \|dist(f_1^*(\cdot), f_2^*(\cdot), \dots, f_n^*(\cdot), Y)\|_E + \left\| \sum_{k=1}^m \chi_{A_k} \right\| \frac{\varepsilon}{\alpha} \\ &\leq \|dist(f_1^*(\cdot), f_2^*(\cdot), \dots, f_n^*(\cdot), Y)\|_E + \frac{\varepsilon}{\alpha} \|\chi_T\| \\ &= \|dist(f_1^*(\cdot), f_2^*(\cdot), \dots, f_n^*(\cdot), Y)\|_E + \varepsilon. \end{aligned}$$

This also gives the following inequalities:

$$\begin{aligned}
 dist_E(f_1, \dots, f_n, E(Y)) &\leq dist_E(f_1^*, \dots, f_n^*, E(Y)) + \sum_{i=1}^n \|f_i - f_i^*\| \\
 &< \left\| \sum_{i=1}^n \|f_i^*(\cdot) - g(\cdot)\|_X \right\|_E + \varepsilon \\
 &\leq \|dist(f_1^*(\cdot), \dots, f_n^*(\cdot), Y)\|_E + 2\varepsilon \\
 &\leq \|dist(f_1(\cdot), \dots, f_n(\cdot), Y)\|_E + \sum_{i=1}^n \|f_i - f_i^*\| + 2\varepsilon \\
 &< \|dist(f_1(\cdot), \dots, f_n(\cdot), Y)\|_E + 3\varepsilon.
 \end{aligned}$$

Thus, we have

$$dist_E(f_1, f_2, \dots, f_n, E(Y)) < \|dist(f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot), Y)\|_E + 3\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, it holds that

$$dist_E(f_1, f_2, \dots, f_n, E(Y)) \leq \|dist(f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot), Y)\|_E. \tag{2}$$

Using inequalities (1) and (2) we get the required results. □

A direct consequence of the previous is the following result

**Corollary 3** *Let  $Y$  be a closed subspace of the real Banach space  $X$  and  $E(X)$  be a Köthe Bochner function space with absolutely continuous and strictly monotone norm. For  $f_1, \dots, f_n \in E(X)$  and  $g \in B(f_1, \dots, f_n, E(Y))$  it is necessary and sufficient that  $g(t) \in B(f_1(t), \dots, f_n(t), Y)$  for almost all  $t \in T$ .*

Next, we give the simultaneous proximality of  $E(Y)$  in  $E(X)$ :

**Theorem 4** *If  $Y$  is simultaneously proximal in the real Banach space  $X$ , then  $B(f_1, f_2, \dots, f_n, E(Y)) \neq \emptyset$ , for every set of simple functions  $f_1, f_2, \dots, f_n$  in  $E(X)$ .*

**Proof.** Let  $f_i$  ( $1 \leq i \leq n$ ) be simple functions in  $E(X)$ . Then  $f_i$  ( $1 \leq i \leq n$ ) can be written as

$$f_i(t) = \sum_{k=1}^m u_k^i \chi_{A_k}(t), \quad i = 1, \dots, n,$$

where  $A_k$ 's are pairwise disjoint measurable sets of  $T$  with  $\bigcup_{k=1}^m A_k = T$ . Also, we may take  $\mu(A_k) > 0$ , for each  $k = 1, 2, \dots, m$ . By assumption we know that for each  $k = 1, 2, \dots, m$ , there exists a best simultaneous approximation  $w_k$  in  $Y$  of the  $n$ -tuples  $(u_k^1, u_k^2, \dots, u_k^n) \in X^n$  such that

$$dist(x_k^1, x_k^2, \dots, x_k^n, Y) = \sum_{i=1}^n \|u_k^i - w_k\|_X.$$

Set

$$g(t) = \sum_{k=1}^n w_k \chi_{A_k}(t),$$

then for any  $\alpha > 0$  and  $h \in E(Y)$ , we obtain that

$$\begin{aligned} & \left\| \sum_{i=1}^n \|f_i(\cdot) - h(\cdot)\|_X \right\|_E \\ & \geq \left\| \sum_{k=1}^m \chi_{A_k}(\cdot) \left[ \sum_{i=1}^n \|u_k^i - w_k\|_X \right] \right\|_E \\ & = \left\| \sum_{i=1}^m \|f_i(\cdot) - g(\cdot)\|_X \right\|_E. \end{aligned}$$

By taking infimum over all  $h \in E(Y)$ , it results that

$$dist_E(f_1, f_2, \dots, f_n, E(Y)) = \left\| \sum_{i=1}^n \|f_i(\cdot) - g(\cdot)\|_X \right\|_E.$$

This implies that the set of simple functions  $f_1, f_2, \dots, f_n$  in  $E(X)$  admits a best simultaneous approximation.  $\square$

**Theorem 5** *Let  $Y$  be a closed subspace of the real Banach space  $X$  and  $E(X)$  be a Köthe Bochner function space with absolutely continuous and strictly monotone norm. If  $E(Y)$  is simultaneous proximal in  $E(X)$ , then  $Y$  is simultaneous proximal in  $X$ .*

**Proof.** Let  $x_1, x_2, \dots, x_n \in X$ . Set  $f_i(t) = x_i$  ( $1 \leq i \leq n$ ) for almost all  $t \in T$ . Since

$$\begin{aligned} \| \|f_i\| \| &= \| \|f_i(\cdot)\|_X \|_E = \| \|x_i \chi_T(\cdot)\|_X \|_E \\ &= \|x_i\|_X \| \chi_T \|, \text{ for } 1 \leq i \leq n, \end{aligned}$$

which is finite, then  $f_i \in E(X)$ , ( $1 \leq i \leq n$ ). By assumption there exists  $g \in E(Y)$  such that

$$\left\| \sum_{i=1}^n \|f_i(\cdot) - g(\cdot)\|_X \right\|_E < \left\| \sum_{i=1}^n \|f_i(\cdot) - h(\cdot)\|_X \right\|_E, \text{ for all } h \in E(Y)$$

Because  $E(X)$  is a Köthe Bochner function space with a strictly monotone norm, for almost  $t \in T$ , we thus have

$$\sum_{i=1}^n \|f_i(t) - g(t)\|_X \leq \sum_{i=1}^n \|f_i(t) - h(t)\|_X.$$

Fix  $t_0 \in T$  and  $y = g(t_0)$ , then  $y \in Y$  and

$$\sum_{i=1}^n \|x_i - y\|_X \leq \sum_{i=1}^n \|x_i - h(t_0)\|_X, \text{ for all } h \in E(Y).$$

Since  $Y$  is embedded isometrically into  $E(Y)$ , we obtain

$$\sum_{i=1}^n \|x_i - y\|_X \leq \sum_{i=1}^n \|x_i - z\|_X, \text{ for all } z \in Y.$$

□

**Theorem 6** *Let  $Y$  be a closed separable subspace of the real Banach space  $X$  and  $E(X)$  be a Köthe Bochner function space with absolutely continuous and strictly monotone norm. Then  $E(Y)$  is simultaneous proximal in  $E(X)$  if and only if  $Y$  is simultaneous proximal in  $X$ .*

**Proof.** The necessity is included in Theorem 5. Now for sufficiency suppose that  $Y$  is simultaneous proximal in  $X$  and let  $f_1, f_2, \dots, f_n$  be a set of functions in  $E(X)$ , then Theorem 3.4 in [1] guarantees that there exists a measurable function  $g : T \rightarrow X$  such that  $g(t) \in B(f_1(t), f_2(t), \dots, f_n(t), Y)$  for almost all  $t \in T$ . We conclude that  $E(Y)$  is simultaneous proximal in  $E(X)$  on account of Lemma 1. Thus

$$g \in B(f_1, f_2, \dots, f_n, E(Y)), \text{ for almost all } t \in T,$$

thereby completing the proof of the theorem. □

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