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L^p Regularity of some weighted Bergman projections on the unit disc

Yunus E. Zeytuncu

Abstract

We show that weighted Bergman projections, corresponding to weights of the form $M(z)(1 - |z|^2)^\alpha$, where $\alpha > -1$ and $M(z)$ is a radially symmetric, strictly positive and at least C^2 function on $\overline{\mathbb{D}}$, are L^p regular.

Key Words: Weighted Bergman projection, Coefficient multipliers

1. Introduction

Let \mathbb{D} denote the unit disc in \mathbb{C}^1 and $dA(z)$ denote the standard Lebesgue measure on \mathbb{C}^1 . Let $\lambda(r)$ be a strictly positive and continuous function on $[0, 1)$. We consider $\lambda(r)$ as a radially symmetric weight on \mathbb{D} by setting $\lambda(z) := \lambda(|z|)$ and denote the space of square integrable functions with respect to the area element $\lambda(z)dA(z)$ by $L^2(\lambda)$. It is clear that $L^2(\lambda)$ is a Hilbert space with the inner product defined by

$$\langle f, g \rangle_\lambda = \int_{\mathbb{D}} f(z) \overline{g(z)} \lambda(z) dA(z)$$

and the norm defined by

$$\|f\|_\lambda^2 = \int_{\mathbb{D}} |f(z)|^2 \lambda(z) dA(z).$$

The closed subspace of holomorphic functions in $L^2(\lambda)$ is denoted by $A^2(\lambda)$. The orthogonal projection operator between these two spaces is called *the weighted Bergman projection* and denoted by \mathbf{B}_λ , i.e.,

$$\mathbf{B}_\lambda : L^2(\lambda) \rightarrow A^2(\lambda).$$

The Riesz representation theorem indicates that \mathbf{B}_λ is an integral operator. The kernel of this integral operator is called *the weighted Bergman kernel* and denoted by $B_\lambda(z, w)$, i.e. for any $f \in L^2(\lambda)$,

$$\mathbf{B}_\lambda f(z) = \int_{\mathbb{D}} B_\lambda(z, w) f(w) \lambda(w) dA(w).$$

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The monomials $\{z^n\}_{n=0}^\infty$ form an orthogonal basis of $A^2(\lambda)$ and the weighted Bergman kernel is given by the following sum:

$$B_\lambda(z, w) = \sum_{n=0}^\infty a_n(z\bar{w})^n, \text{ where } a_n = \frac{1}{\int_{\mathbb{D}} |z|^{2n} \lambda(z) dA(z)}.$$

The coefficients a_n are called *the Bergman coefficients* of weight λ .

For $1 < p < \infty$, we use the standard notation $L^p(\lambda)$ and $A^p(\lambda)$ to denote the respective Banach spaces of p -integrable functions on \mathbb{D} and we use $\|\cdot\|_{p,\lambda}$ to denote the norm on these spaces.

Let us consider the weights defined by $\lambda_\alpha(r) = (1 - r^2)^\alpha$ for $\alpha > -1$, where we set $z = re^{i\theta}$. The Bergman theory for this family of weights are well investigated and can be found in [4].

In particular, the Bergman coefficients of these weights are computed explicitly and the following explicit expression for the weighted kernel is obtained:

$$B_{\lambda_\alpha}(z, w) = \frac{c_\alpha}{(1 - z\bar{w})^{2+\alpha}},$$

where c_α is a constant that only depends on α .

Furthermore, this explicit expression for the kernel and Schur's lemma together prove the following theorem.

Theorem 1.1 *For $\alpha > -1$, the weighted Bergman projection $\mathbf{B}_{\lambda_\alpha}$ is bounded from $L^p(\lambda_\alpha)$ to $A^p(\lambda_\alpha)$ for any $1 < p < \infty$.*

Proof. See page 12 of [4] and also [6] and [3]. □

The purpose of this note is to extend this theorem to more general weights in the following setup. Let $M(r)$ be a strictly positive and at least C^2 function on $[0, 1]$. Without loss of generality, we assume that $M(1) = 1$. Consider the radially symmetric weight defined by

$$\mu(z) = M(|z|)(1 - |z|^2)^\alpha$$

on \mathbb{D} , for some $\alpha > -1$. By the general theory (see [2] and [3]), there exists the weighted Bergman projection operator $\mathbf{B}_\mu : L^2(\mu) \rightarrow A^2(\mu)$, which is an integral operator with the weighted Bergman kernel $B_\mu(z, w)$, where

$$B_\mu(z, w) = \sum_{n=0}^\infty b_n(z\bar{w})^n, \text{ and } b_n = \frac{1}{\int_{\mathbb{D}} |z|^{2n} \mu(z) dA(z)}.$$

But in this case, it is not easy (unless M is a simple function) to compute the coefficients b_n to get an explicit expression for the weighted kernel and therefore, Schur's lemma is not directly applicable in this case.

Nevertheless, we prove the analog of Theorem 1.1 for \mathbf{B}_μ , without referring to an explicit expression for the kernel or Schur's lemma.

Theorem 1.2 *The weighted Bergman projection \mathbf{B}_μ is bounded from $L^p(\mu)$ to $A^p(\mu)$ for any $1 < p < \infty$.*

The proof is in two steps; first relating \mathbf{B}_μ to $\mathbf{B}_{\lambda_\alpha}$ by a coefficient multiplier operator and then showing that this coefficient multiplier operator is bounded.

For the rest of the note, we denote the boundary of \mathbb{D} by $b\mathbb{D}$ and we write $A \lesssim B$ to mean $A \leq cB$ for some constant c that is clear in context. We also use the Szegő projection $\mathbf{T} : L^2(b\mathbb{D}, d\theta) \rightarrow H^2$, where $d\theta$ is the arc length on the unit circle and H^p is the Hardy space of order p . We refer to [2] for definitions and standard facts about the Szegő projection and Hardy spaces.

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2. Coefficient multipliers and norm convergence

In this section, before giving the details of the proof of Theorem 1.2, we recall a few facts about coefficient multipliers. See [1] and [2] for general account.

Let X be a Banach space of holomorphic functions on \mathbb{D} . Any $f \in X$ has Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

Definition 2.1 A sequence of complex numbers $\{t_n\}$ is called a coefficient multiplier from X to X and denoted by $\{t_n\} \in (X, X)$ if for any function $f \in X$,

$$t(f)(z) := \sum_{n=0}^{\infty} t_n f_n z^n \text{ is also in } X.$$

It is a fairly general question to characterize the coefficient multipliers on an arbitrary Banach space X and there is no full answer to this question.

Definition 2.2 For a holomorphic function f on \mathbb{D} and $N \in \mathbb{N}$, let $S_N f$ denote the Taylor polynomial of f of degree N , i.e., $S_N f(z) = \sum_{n=0}^N f_n z^n$.

If X has the property that for any $f \in X$ the sequence of Taylor polynomials $\{S_N f\}$ converges to f , then a sufficient condition for coefficient multipliers can be formulated as follows.

Proposition 2.3 Let $(X, \|\cdot\|)$ be a Banach space of holomorphic functions on \mathbb{D} such that for every $f \in X$ the sequence $\{S_N f\}$ of Taylor polynomials converges to f in the norm of X . Then any sequence of **bounded variation** is a coefficient multiplier from X to X .

Definition 2.4 A sequence of complex numbers $\{t_n\}$ is said to be of bounded variation if $|t_0| + \sum_{n=1}^{\infty} |t_n - t_{n-1}|$ is finite.

Proposition 2.3 appears in [1, Proposition 3.7]. It follows from summation by parts and we repeat its proof for completeness.

Proof. Since the Taylor polynomials converge, for any given $f \in X$ and $\epsilon > 0$ there exists an N such that for any $k > N$

$$\left\| \sum_{n=k}^{\infty} f_n z^n \right\| < \epsilon.$$

Let $\{t_n\}$ be the sequence of bounded variation and $|t_0| + \sum_{n=1}^{\infty} |t_n - t_{n-1}| \leq K$. Summation by parts and bounded variation hypothesis give

$$\begin{aligned} \left\| \sum_{n=k}^{\infty} t_n f_n z^n \right\| &= \left\| \sum_{n=k}^{\infty} (t_{n+1} - t_n) \sum_{j=n+1}^{\infty} f_j z^j + t_k \sum_{n=k}^{\infty} f_n z^n \right\| \\ &\leq \left[|t_k| + \sum_{n=k}^{\infty} |t_{n+1} - t_n| \right] \epsilon \\ &\leq K\epsilon. \end{aligned}$$

This shows that $t(f)(z) = \sum_{n=0}^{\infty} t_n f_n z^n$ is in X and finishes the proof. □

In order to use this proposition in the proof of Theorem 1.2, we have to check whether Taylor polynomials converge in $A^p(\mu)$. This turns out to be true even in a more general form.

Proposition 2.5 *For $1 < p < \infty$ and any integrable radial weight $\lambda(r)$, the Taylor series of every function in $A^p(\lambda)$ converges in norm.*

In particular, the claim is true for $A^p(\lambda_\alpha)$ and $A^p(\mu)$. The statement for $A^p(\lambda_\alpha)$ is in [5]. The general case is obtained by just imitating the proof in [5].

Proof. This is done in three steps.

Step One. The holomorphic polynomials are dense in $A^p(\lambda)$.

For any $f \in A^p(\lambda)$ and for any $0 < \rho < 1$, define $f_\rho(z) = f(\rho z)$. Each f_ρ is holomorphic in a larger disc and the Taylor polynomials of each f_ρ converges uniformly on \mathbb{D} and hence in $A^p(\lambda)$. Therefore it is enough to show that

$$\lim_{\rho \rightarrow 1^-} \|f - f_\rho\|_{p,\lambda} = 0.$$

For any holomorphic f , the averages

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

are well defined and non-decreasing functions of r (see [2, page 26]). Moreover

$$M_p^p(r, f_\rho) = M_p^p(\rho r, f) \leq M_p^p(r, f).$$

Since $f \in A^p(\lambda)$ and

$$\|f\|_{p,\lambda}^p = \int_0^1 r\lambda(r)M_p^p(r, f)dr.$$

$M_p^p(r, f)$ is integrable with respect to the weight $r\lambda(r)dr$.

On the other hand, $f_\rho \rightarrow f$ pointwise on \mathbb{D} as $\rho \rightarrow 1^-$ so by the Lebesgue dominated convergence theorem $\lim_{\rho \rightarrow 1^-} M_p^p(r, f - f_\rho) = 0$. We also have

$$M_p^p(r, f - f_\rho) \leq 2^p (M_p^p(r, f) + M_p^p(r, f_\rho)) \leq 2^{p+1}M_p^p(r, f).$$

Therefore again the Lebesgue dominated convergence theorem implies

$$\begin{aligned} \lim_{\rho \rightarrow 1^-} \|f - f_\rho\|_{p,\lambda}^p &= \lim_{\rho \rightarrow 1^-} \int_0^1 r\lambda(r)M_p^p(r, f - f_\rho)dr \\ &= \int_0^1 r\lambda(r) \lim_{\rho \rightarrow 1^-} M_p^p(r, f - f_\rho)dr \\ &= 0. \end{aligned}$$

This finishes the first step.

Step Two. We show that the operator norms of S_N 's (defined in Definition 2.2) are uniformly bounded. For this we need a well-known result about the Szegő projection. Let $\mathbf{T} : L^2(b\mathbb{D}, d\theta) \rightarrow H^2$ denote the Szegő projection. By using the fact that \mathbf{T} is also bounded from $L^p(b\mathbb{D}, d\theta)$ to H^p for any $1 < p < \infty$, one can prove (see [2, page 27]) that there exists $C > 0$, independent of N and h , such that

$$\int_0^{2\pi} |S_N h(e^{i\theta})|^p d\theta \leq C \int_0^{2\pi} |h(e^{i\theta})|^p d\theta \tag{2.6}$$

for any $h \in H^p$. The proof is only to note that $\overline{S_N f(e^{i\theta})} = e^{-iN\theta} T(e^{iN\theta} \overline{f(e^{i\theta})})$ which is clear for f a polynomial, and follows in general since polynomials are dense in H^p .

Now we calculate the operator norms of S_N 's. For given $f \in A^p(\lambda)$,

$$\begin{aligned} \|S_N f\|_{p,\lambda}^p &= \int_{\mathbb{D}} |S_N f(z)|^p \lambda(z) dA(z) \\ &= \int_0^1 r\lambda(r)dr \int_0^{2\pi} |S_N f(re^{i\theta})|^p d\theta \\ &\leq C \int_0^1 r\lambda(r)dr \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \text{ since } f_r \in H^p \\ &= C \int_{\mathbb{D}} |f(z)|^p \lambda(z) dA(z) \\ &= C \|f\|_{p,\lambda}^p. \end{aligned}$$

This implies that $\sup_N \|S_N\|_{op} \leq C$ and finishes the second step.

Step Three. Next, we show that $\lim_{N \rightarrow \infty} \|S_N f - f\|_{p,\lambda} = 0$ for any $f \in A^p(\lambda)$.

Given f and $\epsilon > 0$, by the first step there exists a polynomial Q such that $\|Q - f\|_{p,\lambda}^p < \epsilon$. Then

$$\begin{aligned} \|S_N f - f\|_{p,\lambda}^p &\leq \|S_N f - S_N Q\|_{p,\lambda}^p + \|S_N Q - Q\|_{p,\lambda}^p + \|Q - f\|_{p,\lambda}^p \\ &\leq (C + 1)\epsilon + \|S_N Q - Q\|_{p,\lambda}^p. \end{aligned}$$

Note that $S_N Q = Q$ for large enough N and therefore for sufficiently large N ,

$$\|S_N f - f\|_{p,\lambda}^p \leq (C + 1)\epsilon.$$

Since this is true for any $\epsilon > 0$ we get $\lim_{N \rightarrow \infty} \|S_N f - f\|_{p,\lambda} = 0$. This finishes the last step and the proof of the proposition. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by using Propositions 2.3 and 2.5. Recall that a_n 's are the Bergman coefficients of $(1 - |z|^2)^\alpha$ and b_n 's are the Bergman coefficients of μ . Let \mathcal{R} denote the coefficient multiplier operator for the sequence $\left\{ \frac{b_n}{a_n} \right\}$. The following identity relates the two Bergman projections:

$$\mathbf{B}_\mu f(z) = \mathcal{R} [\mathbf{B}_{\lambda_\alpha} (fM)] (z). \tag{3.1}$$

Indeed, for any $f \in L^2(\mu)$,

$$\begin{aligned} \mathbf{B}_\mu f(z) &= \int_{\mathbb{D}} \sum_{n=0}^{\infty} b_n (z\bar{w})^n f(w) \mu(w) dA(w) = \sum_{n=0}^{\infty} b_n z^n \int_{\mathbb{D}} \bar{w}^n f(w) \mu(w) dA(w) \\ &= \sum_{n=0}^{\infty} a_n z^n \frac{b_n}{a_n} \int_{\mathbb{D}} \bar{w}^n f(w) \mu(w) dA(w) \\ &= \mathcal{R} \left[\sum_{n=0}^{\infty} a_n z^n \int_{\mathbb{D}} \bar{w}^n f(w) \mu(w) dA(w) \right] \\ &= \mathcal{R} [\mathbf{B}_{\lambda_\alpha} (fM)] (z). \end{aligned}$$

Here we change the order of integration and summation but this doesn't cause any problems. We can truncate the summation, which is equivalent to looking at the Taylor polynomials of $\mathbf{B}_\mu f$ and $\mathbf{B}_{\lambda_\alpha} (fM)$, and take limit by using Proposition 2.5. Now, it suffices to prove that the multiplier operator \mathcal{R} is bounded from $A^p(\mu)$ to $A^p(\mu)$ (actually, we have to show that \mathcal{R} is bounded from $A^p(\lambda_\alpha)$ to $A^p(\mu)$ but since M is of class C^2 and thus bounded; the inclusion map $i : A^p(\lambda_\alpha) \rightarrow A^p(\mu)$ is bounded). By the closed graph theorem it is enough

to show that $\mathcal{R}(f) \in A^p(\mu)$ for any $f \in A^p(\mu)$. Moreover, Proposition 2.3 implies that it is enough to show that the sequence $\left\{ \frac{b_n}{a_n} \right\}$ is of bounded variation.

It is immediate that the sequence $\left\{ \frac{b_n}{a_n} \right\}$ is bounded from below and above. Moreover, a direct computation gives that

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{\int_0^1 r^{2n+1}(1-r^2)^\alpha dr}{\int_0^1 r^{2n+1}\mu(r)dr} = M(1)^{-1}.$$

We quantify this computation to get that the sequence $\left\{ \frac{b_n}{a_n} \right\}$ is indeed of bounded variation.

Lemma 3.2 $\left| \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}} \right| \lesssim \frac{1}{n^2}$, i.e., the sequence $\left\{ \frac{b_n}{a_n} \right\}$ is of bounded variation and therefore \mathcal{R} is bounded from $A^p(\mu)$ to $A^p(\mu)$.

Proof.

First, we consider the difference between elements of the sequence $\left\{ \frac{b_n}{a_n} \right\}$. Here, all the integrals are taken with respect to r and from 0 to 1.

$$\begin{aligned} \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}} &= \frac{\int r^{2n+1}(1-r^2)^\alpha}{\int r^{2n+1}\mu(r)} - \frac{\int r^{2n-1}(1-r^2)^\alpha}{\int r^{2n-1}\mu(r)} \\ &= \frac{\int r^{2n+1}(1-r^2)^\alpha \int r^{2n-1}\mu(r) - \int r^{2n-1}(1-r^2)^\alpha \int r^{2n+1}\mu(r)}{\int r^{2n+1}\mu(r) \int r^{2n-1}\mu(r)} \\ &=: \frac{B(n)}{A(n)}. \end{aligned}$$

We can rewrite the numerator as

$$\begin{aligned} B(n) &= \int r^{2n+1}(1-r^2)^\alpha \int r^{2n-1}(1-r^2)\mu(r) - \int r^{2n-1}(1-r^2)^{\alpha+1} \int r^{2n+1}\mu(r) \\ &= \int r^{2n+1} [(1-M(r))(1-r^2)^\alpha] \int r^{2n-1}(1-r^2)\mu(r) \\ &\quad - \int r^{2n-1} [(1-M(r))(1-r^2)^{\alpha+1}] \int r^{2n+1}\mu(r) \\ &=: B_1(n) - B_2(n). \end{aligned}$$

Next, we integrate $B_1(n)$ and $B_2(n)$ by parts twice to obtain

$$\begin{aligned} B_1(n) &= \frac{1}{(2n+2)2n} \int r^{2n+2} [(1-M(r))(1-r^2)^\alpha]' \int r^{2n} [M(r)(1-r^2)^{\alpha+1}]' \\ &=: \frac{1}{(2n+2)2n} C_1(n)C_2(n) \end{aligned}$$

$$\begin{aligned} B_2(n) &= \frac{1}{2n(2n+1)} \int r^{2n+1} [(1-M(r))(1-r^2)^{\alpha+1}]'' \int r^{2n+1} [M(r)(1-r^2)^\alpha] \\ &=: \frac{1}{2n(2n+1)} C_3(n)C_4(n). \end{aligned}$$

Here, C_1, C_2, C_3, C_4 denote the respective integrals. Note that we don't get any boundary terms after integration by parts since M is of class C^2 on $[0, 1]$ and $M(1) = 1$.

In order to finish the proof, it suffices to show that

$$\sup_n \left\{ n^2 \left| \frac{B_1(n)}{A(n)} \right| \right\} \quad \text{and} \quad \sup_n \left\{ n^2 \left| \frac{B_2(n)}{A(n)} \right| \right\} \quad \text{are finite.}$$

Thus, it is enough to show that

$$\sup_n \left\{ \left| \frac{C_1(n)C_2(n)}{A(n)} \right| \right\} \quad \text{and} \quad \sup_n \left\{ \left| \frac{C_3(n)C_4(n)}{A(n)} \right| \right\} \quad \text{are finite.}$$

We start with the first one.

$$\begin{aligned} \frac{C_1(n)C_2(n)}{A(n)} &= \frac{\int r^{2n+2} [(1-M(r))(1-r^2)^\alpha]' \int r^{2n} [M(r)(1-r^2)^{\alpha+1}]'}{\int r^{2n+1} M(r)(1-r^2)^\alpha \int r^{2n-1} M(r)(1-r^2)^\alpha} \\ &\rightarrow \frac{[(1-M(r))(1-r^2)^\alpha]' [M(r)(1-r^2)^{\alpha+1}]'}{M(r)(1-r^2)^\alpha M(r)(1-r^2)^\alpha} \Big|_{r=1} \quad \text{as } n \rightarrow \infty \\ &= 2(\alpha+1)^2 M'(1). \end{aligned}$$

This shows that the first supremum is indeed finite. Note that the condition $M(1) = 1$ is used here.

We argue the same way for the second one.

$$\begin{aligned} \frac{C_3(n)C_4(n)}{A(n)} &= \frac{\int r^{2n+1} [(1-M(r))(1-r^2)^{\alpha+1}]'' \int r^{2n+1} [M(r)(1-r^2)^\alpha]}{\int r^{2n+1} M(r)(1-r^2)^\alpha \int r^{2n-1} M(r)(1-r^2)^\alpha} \\ &\rightarrow \frac{[(1-M(r))(1-r^2)^{\alpha+1}]'' [M(r)(1-r^2)^\alpha]}{M(r)(1-r^2)^\alpha M(r)(1-r^2)^\alpha} \Big|_{r=1} \quad \text{as } n \rightarrow \infty \\ &= 2(\alpha+2)(\alpha+1)M'(1) - 2(1+\alpha)(1-M(1)). \end{aligned}$$

This shows that the second supremum is indeed finite. Again, note that the condition $M(1) = 1$ is used here.

This finishes the proof Lemma 3.2.

Since Lemma 3.2 is established, we conclude the proof of Theorem 1.2.

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