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# Pointwise slant submanifolds in almost Hermitian manifolds

*Bang-Yen Chen and Oscar J. Garay*

## Abstract

An interesting class of submanifolds of almost Hermitian manifolds  $(\tilde{M}, \tilde{g}, J)$  is the class of slant submanifolds. Slant submanifolds were introduced by the first author in [6] as submanifolds  $M$  of  $(\tilde{M}, \tilde{g}, J)$  such that, for any nonzero vector  $X \in T_pM$ ,  $p \in M$ , the angle  $\theta(X)$  between  $JX$  and the tangent space  $T_pM$  is independent of the choice of  $p \in M$  and  $X \in T_pM$ . The first results on slant submanifolds were summarized in the book [7]. Since then slant submanifolds have been studied by many geometers. Many nice results on slant submanifolds have been obtained during the last two decades. The main purpose of this paper is to study pointwise slant submanifolds in almost Hermitian manifolds which extends slant submanifolds in a very natural way. Several basic results in this respect are proved in this paper.

**Key Words:** Slant submanifold, pointwise slant submanifold, slant function, conformal mapping, cohomology

## 1. Introduction

Let  $M$  be an  $n$ -manifold immersed in an almost Hermitian manifold  $(\tilde{M}, J, \tilde{g})$  with an almost complex structure  $J$  and an almost Hermitian metric  $\tilde{g}$ , i.e.,  $\tilde{g}$  is a Riemannian metric satisfying

$$\tilde{g}(JX, JY) = \tilde{g}(X, Y) \tag{1.1}$$

for  $X, Y$  lying in the tangent bundle  $T\tilde{M}$  of  $\tilde{M}$ .

For any vector  $X$  tangent to  $M$  we put

$$JX = PX + FX, \tag{1.2}$$

where  $PX$  and  $FX$  are the tangential and normal components of  $JX$ , respectively. Thus  $P$  is an endomorphism of the tangent bundle  $TM$  of  $M$  and  $F$  a normal-bundle-valued 1-form on  $TM$ .

For any nonzero vector  $X \in T_pM$ ,  $p \in M$ , the angle  $\theta(X)$  between  $JX$  and the tangent space  $T_pM$  is called the *Wirtinger angle* of  $X$ . The Wirtinger angle gives rise to a real-valued function  $\theta : T^*M \rightarrow \mathbf{R}$ , called the *Wirtinger function*, defined on the set  $T^*M$  consisting of all nonzero tangent vectors on  $M$ .

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**Definition 1.1** An immersion  $\phi : M \rightarrow \tilde{M}$  of a manifold  $M$  into an almost Hermitian manifold  $\tilde{M}$  is called *pointwise slant* if, at each given point  $p \in M$ , the Wirtinger angle  $\theta(X)$  is independent of the choice of the nonzero tangent vector  $X \in T_p^*M$ . In this case,  $\theta$  can be regarded as a function on  $M$ , which is called the *slant function* of the pointwise slant submanifold.

**Remark 1.1** *Pointwise slant submanifolds have been studied in [17] by Etayo under the name of quasi-slant submanifolds. It was proved in [17] that a complete, totally geodesic quasi-slant submanifold of a Kähler manifold is a slant submanifold.*

Clearly, every 2-dimensional submanifold (or simply, a surface) in an almost Hermitian manifold is always pointwise slant.

**Definition 1.2** A point  $p$  in a pointwise slant submanifold is called a *totally real point* if its slant function  $\theta$  satisfies  $\cos \theta = 0$  at  $p$ . Similarly, a point  $p$  is called a *complex point* if its slant function satisfies  $\sin \theta = 0$  at  $p$ . A pointwise slant submanifold  $M$  in an almost Hermitian manifold  $(\tilde{M}, J, \tilde{g})$  is called *totally real* if every point of  $M$  is a totally real point. A totally real submanifold  $M$  in  $\tilde{M}$  is called *Lagrangian* if  $\dim M = \dim_{\mathbb{C}} \tilde{M}$ .

**Definition 1.3** A pointwise slant submanifold of an almost Hermitian manifold is called *pointwise proper slant* if it contains no totally real points.

A pointwise slant submanifold  $M$  is called *slant* in the sense of [6, 7] if its slant function  $\theta$  is globally constant, i.e.,  $\theta$  is also independent of the choice of the point on  $M$ . In this case, the constant  $\theta$  is called the *slant angle* of the slant submanifold. Since the notion of slant submanifolds was introduced in 1990, slant submanifolds have been studied by many geometers. Consequently, many nice results on slant submanifolds have been obtained during the last two decades (see, for instance, [1]-[3], [7], [8], [10], [14], [15] and [18]-[22] for more details).

The main purpose of this paper is to study pointwise slant submanifolds in almost Hermitian manifolds. The notion of pointwise slant submanifolds extends the notion of slant submanifolds in a very natural way. Several fundamental results on pointwise slant submanifolds are proved in this paper.

## 2. Conformal invariants of pointwise slant submanifolds

A differentiable map  $\phi : M \rightarrow N$  of a differentiable manifold  $M$  into another differentiable manifold  $N$  is called an immersion if the differential  $d\phi : TM \rightarrow TN$  is injective [5]. Let  $\phi : M \rightarrow \tilde{M}$  be an immersion of an  $n$ -manifold into an almost Hermitian manifold  $\tilde{M}$  equipped with an almost complex structure  $J$  and an almost Hermitian metric  $\tilde{g}$ . We denote by  $g$  the metric on  $M$  induced from  $\tilde{g}$  via  $\phi$ .

Let  $P$  and  $F$  be the endomorphism and the normal-bundle-valued 1-form defined by (1.2). Since  $\tilde{M}$  is almost Hermitian, we find from (1.1) and (1.2) that

$$g(PX, Y) = -g(X, PY) \tag{2.1}$$

for vectors  $X, Y$  tangent to  $M$ . Hence  $P^2$  is a self-adjoint endomorphism of the tangent bundle  $TM$  of  $M$ . Thus each tangent space  $T_pM$ ,  $p \in M$ , admits an orthogonal direct decomposition of eigenspaces of  $P^2$ :

$$T_pM = \mathcal{D}_p^1 \oplus \dots \oplus \mathcal{D}_p^{k(p)}. \tag{2.2}$$

Since  $P$  is skew-symmetric and  $J^2 = -I$ , each eigenvalue  $\lambda_i$  of  $P^2$  lies in  $[-1, 0]$ .

Clearly, a point  $p \in M$  is totally real (respectively, complex point) if  $P = 0$  at  $p$  (respectively,  $P = J$  at  $p$ ).

**Lemma 2.1** *An immersion  $\phi : M \rightarrow \tilde{M}$  of a manifold  $M$  into an almost Hermitian manifold  $\tilde{M}$  is a pointwise slant immersion if and only if  $P^2 = -(\cos^2 \theta)I$  for some real-valued function  $\theta$  defined on the tangent bundle  $TM$  of  $M$ .*

**Proof.** If  $\phi : M \rightarrow \tilde{M}$  is a pointwise slant immersion with slant function  $\theta : M \rightarrow \mathbf{R}$ , then we have

$$g(PX, PX) = \cos^2 \theta(p)g(X, X)$$

for  $X \in T_p M$ . Combining this with (2.1) yields

$$g(P^2 X, X) = -\cos^2 \theta(p)g(X, X).$$

Thus, after applying polarization, we obtain  $P^2 = -(\cos^2 \theta)I$  on  $TM$ .

Conversely, if  $M$  is a submanifold of  $\tilde{M}$  satisfying  $P^2 = -(\cos^2 \theta)I$  for some function  $\theta$  on  $M$ , then

$$g(PX, PX) = -g(P^2 X, X) = \cos^2 \theta(p)g(X, X),$$

which implies that the Wirtinger angle is independent of the choice of  $X \in T_p^* M$  at each given point  $p \in M$ . Hence the submanifold is pointwise slant. □

A pointwise slant submanifold is called a *totally real submanifold* if every point of the submanifold is a totally real point (cf. [13]).

The following result is an immediate consequence of Lemma 2.1.

**Corollary 2.1** *Let  $\phi : M \rightarrow \tilde{M}$  be a pointwise slant immersion of an  $n$ -manifold into an almost Hermitian manifold. If  $\phi$  is not a totally real immersion, then  $M$  is even-dimensional.*

The following proposition provides another simple characterization of pointwise slant immersions.

**Proposition 2.1** *Let  $\phi : M \rightarrow \tilde{M}$  be an immersion of a manifold  $M$  into an almost Hermitian manifold  $\tilde{M}$ . Then  $\phi$  is a pointwise slant immersion if and only if  $P : TM \rightarrow TM$  preserves orthogonality, i.e.,  $P$  carries each pair of orthogonal vectors into orthogonal vectors.*

**Proof.** Let  $\phi : M \rightarrow \tilde{M}$  be an immersion of an  $n$ -manifold  $M$  into an almost Hermitian manifold  $\tilde{M}$ . Denote by  $\theta : T^1 M \rightarrow \mathbf{R}$  the Wirtinger function on the unit tangent bundle  $T^1 M$  defined in section 1. Clearly, with respect the induced metric,  $T_p^1 M$  is the unit hypersphere  $\Sigma_p$  in  $T_p M$  centered at  $o$ .

At a given point  $p \in M$ , we have

$$g(PX, PX) = \cos^2 \theta(X) \tag{2.3}$$

for any unit vector  $X \in T_p^1 M$ . For each unit vector  $Y$  tangent to  $\Sigma_p$  at  $X \in \Sigma_p$  (hence  $Y \perp X$ ), we have

$$2g(PX, PY) = Yg(PX, PX) = -(Y\theta) \sin 2\theta(X). \tag{2.4}$$

Consequently,  $P$  carries each pair of orthogonal vectors in  $T_pM$  into a pair of orthogonal vectors in  $T_pM$  if and only if the Wirtinger function  $\theta$  is independent of the choice of  $X \in T_p^1M$ . This implies the proposition.

□

The following proposition shows the important facts that the *notions of pointwise slant submanifolds and slant functions are conformal invariant*.

**Proposition 2.2** *If  $\phi : M \rightarrow (\tilde{M}, J, \tilde{g})$  is a pointwise slant immersion of a manifold  $M$  into an almost Hermitian manifold  $\tilde{M}$ . Then, for any given function  $f$  on  $\tilde{M}$ , the immersion  $\phi : M \rightarrow (\tilde{M}, J, e^{2f}\tilde{g})$  is pointwise slant with the same slant function as the immersion  $\phi : M \rightarrow (\tilde{M}, J, \tilde{g})$ .*

**Proof.** Let  $\phi : M \rightarrow (\tilde{M}, J, \tilde{g})$  be a pointwise slant immersion with slant function  $\theta$ . Then, for any  $X \in T_pM$ , we have

$$\tilde{g}(PX, PX) = (\cos^2 \theta)\tilde{g}(X, X) \tag{2.5}$$

Thus, if we put  $\bar{g} = e^{2f}\tilde{g}$  for a given function  $f$  on  $\tilde{M}$ , then

$$\begin{aligned} \bar{g}(PX, PX) &= e^{2f}\tilde{g}(PX, PX) \\ &= (\cos^2 \theta)e^{2f}\tilde{g}(X, X) \\ &= (\cos^2 \theta)\bar{g}(X, X). \end{aligned} \tag{2.6}$$

Hence,  $\phi : M \rightarrow (\tilde{M}, J, e^{2f}\tilde{g})$  is also a pointwise slant immersion with the same slant function as  $\phi : M \rightarrow (\tilde{M}, J, \tilde{g})$ . □

The following conformal property of slant immersions is an immediate consequence of Proposition 2.2.

**Corollary 2.2** *If  $\phi : M \rightarrow (\tilde{M}, J, \tilde{g})$  is a slant immersion of  $M$  into an almost Hermitian manifold  $\tilde{M}$ . Then, for every given function  $f$  on  $\tilde{M}$ ,  $\phi : M \rightarrow (\tilde{M}, J, e^{2f}\tilde{g})$  is also a slant immersion with the same slant angle.*

Recall that a Hermitian  $n$ -manifold  $(\tilde{M}, J, g)$  is called a *locally conformal Kähler manifold* if there exist an open cover  $\{U_i\}_{i \in I}$  of  $\tilde{M}$  and a family  $\{f_i\}_{i \in I}$  of smooth functions  $f_i : U_i \rightarrow \mathbf{R}$  such that each local metric

$$g_i = e^{-2f_i}g|_{U_i} \tag{2.7}$$

is Kählerian (cf. [16]).

Another interesting application of Proposition 2.2 is the following.

**Corollary 2.3** *For each integer  $n \geq 1$ , there exist infinitely many  $2n$ -dimensional totally umbilical proper slant submanifolds in locally conformal Kähler  $2n$ -manifolds.*

**Proof.** Let  $M$  be an open domain of a proper slant  $2n$ -plane of the complex Euclidean  $2n$ -space  $(\mathbb{C}^{2n}, J, g_0)$  and let  $f$  be a smooth function defined on  $M$ . Then  $M$  is totally geodesic in  $\mathbb{C}^{2n}$ .

Consider the new metric  $g^* = e^{2f}g_0$ . Let  $\xi$  denote the vector field associated with the 1-form  $df$  with respect to  $g_0$ , i.e.,

$$g_0(\xi, Z) = df(Z), \quad \forall Z \in T\mathbb{C}^{2n}.$$

Then it follows from formula (2.8) of [4, p. 381] that the second fundamental  $h^*$  of  $M$  in  $(\mathbb{C}^{2n}, J, g^*)$  satisfies

$$h^*(X, Y) = g_0(X, Y)\xi^N, \tag{2.8}$$

where  $\xi^N$  is the normal component of the vector field  $\xi|_M$ . Thus  $M$  is a non-totally geodesic, totally umbilical submanifold of the locally conformal Kähler  $2n$ -manifold  $(\mathbb{C}^{2n}, J, g^*)$  whenever  $\xi^N \neq 0$ . Therefore it follows from Corollary 2.2 that  $M$  is a proper slant submanifold of  $(\mathbb{C}^{2n}, J, g^*)$ . This proves the corollary.  $\square$

### 3. Examples of pointwise slant submanifolds

In this section, we provide some examples of pointwise slant submanifolds in almost Hermitian manifolds.

**Example 1** Every 2-dimensional submanifold in an almost Hermitian manifold is pointwise slant.

**Remark 3.1** *A pointwise slant surface in an almost Hermitian manifold is called purely real if it contains no complex points. Some basic properties of purely real surfaces in Kähler surfaces have been obtained in [9]-[12].*

**Example 2** Every slant (resp. proper slant) submanifold in an almost Hermitian manifold is pointwise slant (resp. pointwise proper slant).

**Example 3** Let  $\mathbb{E}^{4n} = (\mathbf{R}^{4n}, g_0)$  denote the Euclidean  $4n$ -space endowed with the standard Euclidean metric  $g_0$  and let  $\{J_0, J_1\}$  be a pair of almost complex structures on  $\mathbb{E}^{4n}$  satisfying  $J_0J_1 = -J_1J_0$ . Assume that  $J_0, J_1$  are orthogonal almost complex structures, i.e., they are compatible with the Euclidean metric  $g_0$ ; thus  $g_0(J_iX, J_iY) = g_0(X, Y)$ ,  $i = 0, 1$ , for  $X, Y \in T(\mathbb{E}^{4n})$ .

Let us denote  $(\mathbf{R}^{4n}, J_0, g_0)$  by  $\mathbb{C}_0^{2n}$ . For any real-valued function  $f : \mathbb{E}^{4n} \rightarrow \mathbf{R}$ , we define an almost complex structure  $J_f$  on  $\mathbb{E}^{4n}$  by

$$J_f = (\cos f)J_0 + (\sin f)J_1. \tag{3.1}$$

Then  $\mathbb{C}_f^{2n} = (\mathbf{R}^{4n}, J_f, g_0)$  is an almost Hermitian manifold.

For any given pointwise slant submanifold (resp. slant submanifold)  $M$  in  $(\mathbb{E}^{4n}, J_0, g_0)$ ,  $M$  is also a pointwise slant submanifold (resp. slant submanifold) of  $\mathbb{C}_f^{2n} = (\mathbf{R}^{4n}, J_f, g_0)$ . In particular, if  $M$  is a complex submanifold of  $\mathbb{C}_0^{2n}$ , then  $M$  is a pointwise slant minimal submanifold in  $\mathbb{C}_f^{2n}$  whose slant function  $\theta$  is the restriction of  $f$  on  $M$ , i.e.,  $\theta = f|_M$ .

**Remark 3.2** *There do exist pairs of orthogonal almost complex structures  $\{J_0, J_1\}$  which satisfy the conditions mentioned in Example 3. For instance, let  $J_0$  and  $J_1$  be the two orthogonal almost complex structures on  $\mathbb{E}^{4n}$*

defined by

$$\begin{aligned}
 & J_0(a_1, \dots, a_{2n}, b_1, \dots, b_{2n}) \\
 &= (-b_1, \dots, -b_{2n}, a_1, \dots, a_{2n}), \\
 & J_1(a_1, \dots, a_{2n}, b_1, \dots, b_{2n}) \\
 &= (-a_2, a_1, \dots, -a_{2n}, a_{2n-1}, b_2, -b_1, \dots, b_{2n}, -b_{2n-1}).
 \end{aligned}
 \tag{3.2}$$

Then  $J_0J_1 = -J_1J_0$ . Since there exist many real-valued functions on  $\mathbb{C}^{2n}$  and many complex submanifolds in  $\mathbb{C}^{2n}$ , we may conclude from Example 3 that there are infinitely many examples of pointwise slant minimal submanifolds in almost Hermitian manifolds which are not slant.

#### 4. Pointwise slant submanifolds in Kähler manifolds

For a submanifold  $M$  of an almost Hermitian manifold  $(\tilde{M}, J, \tilde{g})$ , we denote by  $\nabla$  and  $\tilde{\nabla}$  the Levi-Civita connections of  $M$  and  $\tilde{M}$ , respectively. Then the Gauss and Weingarten formulas of  $M$  in  $\tilde{M}$  are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{4.2}$$

where  $X, Y$  are tangent vector,  $\xi$  is a normal vector field,  $h$  is the second fundamental form,  $D$  the normal connection and  $A$  the shape operator of  $M$ .

The second fundamental form  $h$  and the Weingarten map  $A$  are related by

$$g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi). \tag{4.3}$$

The mean curvature vector  $H$  of  $M$  is defined by

$$H = \left(\frac{1}{n}\right) \text{trace } h, \quad n = \dim M. \tag{4.4}$$

For any vector field  $\xi$  normal to the submanifold  $M$ , we put

$$J\xi = t\xi + f\xi, \tag{4.5}$$

where  $t\xi$  and  $f\xi$  are the tangential and the normal components of  $J\xi$ , respectively. It is easy to verify that  $f$  is an endomorphism of the normal bundle and  $t$  is a tangent-bundle-valued 1-form on the normal bundle  $T^\perp N$ .

We have the following result in terms of the operators  $A, F, P$  defined by (1.2) and (4.2).

**Proposition 4.1** *A pointwise slant submanifold  $M$  in a Kähler manifold is slant if and only if the shape operator of  $M$  satisfies*

$$A_{FX}PX = A_{FPX}X \tag{4.6}$$

for  $X \in TM$ .

**Proof.** Let  $M$  be a pointwise slant submanifold of a Kähler manifold  $\tilde{M}$  with slant function  $\theta$ . For any unit tangent vector field  $X$  of  $M$ , we may put

$$PX = (\cos \theta)X^*, \tag{4.7}$$

where  $X^*$  is a unit tangent vector field orthogonal to  $X$ . Then, for any tangent vector  $Y$  of  $M$ , we have

$$\begin{aligned} \tilde{\nabla}_Y(JX) &= \tilde{\nabla}_Y((\cos \theta)X^*) + \tilde{\nabla}_YFX \\ &= (\cos \theta)\nabla_YX^* + (\cos \theta)h(Y, X^*) - (\sin \theta)(Y\theta)X^* \\ &\quad - A_{FX}Y + D_Y(FX). \end{aligned} \tag{4.8}$$

On the other hand, we also have

$$\begin{aligned} \tilde{\nabla}_Y(JX) &= J\tilde{\nabla}_YX \\ &= P(\nabla_YX) + F(\nabla_YX) + th(X, Y) + fh(X, Y). \end{aligned} \tag{4.9}$$

Hence, after comparing the tangential components of (4.8) and (4.9), we find

$$(\sin \theta)(Y\theta)X^* = (\cos \theta)\nabla_YX^* - A_{FX}Y - P(\nabla_YX) - th(X, Y). \tag{4.10}$$

Thus, by taking the inner product of (4.10) with  $X^*$ , we obtain

$$(\sin \theta)Y\theta = \tilde{g}(h(X, Y), FX^*) - \tilde{g}(h(Y, X^*), FX). \tag{4.11}$$

Consequently, the pointwise slant submanifold is slant if and only if

$$A_{FX^*}X = A_{FX}X^*$$

holds for any  $X \in TM$ , which implies the proposition. □

Some consequences of Proposition 4.1 are the following.

**Corollary 4.1** *Every totally geodesic pointwise slant submanifold of any Kähler manifold is slant.*

**Proof.** Follows from Proposition 4.1 and the fact that the shape operator  $A$  vanishes identically for totally geodesic submanifolds. □

**Corollary 4.2** *Let  $M$  be a  $2n$ -dimensional totally umbilical pointwise proper slant submanifold of a Kähler  $2n$ -manifold  $\tilde{M}$ . If  $M$  is non-totally geodesic in  $\tilde{M}$ , then  $M$  is always non-slant in  $\tilde{M}$ .*

**Proof.** Assume that  $M$  is a  $2n$ -dimensional totally umbilical pointwise proper slant submanifold of a Kähler  $2n$ -manifold  $\tilde{M}$ . Then the second fundamental form  $h$  of  $M$  satisfies

$$h(X, Y) = g(X, Y)H \tag{4.12}$$

for  $X, Y \in TM$ . Thus we have

$$g(A_{FPX}X, Y) = \tilde{g}(h(X, Y), FPX) = \tilde{g}(FPX, H)g(X, Y), \tag{4.13}$$

$$g(A_{FX}PX, Y) = \tilde{g}(FX, H)g(PX, Y). \tag{4.14}$$



If  $M$  is slant, then Proposition 4.1 and (4.13)–(4.14) imply that

$$\tilde{g}(FPX, H)X = \tilde{g}(FX, H)PX = 0. \tag{4.15}$$

Since  $M$  is non-totally geodesic, totally umbilical in  $\tilde{M}$ ,  $M$  is a non-minimal submanifold. Thus,  $M$  cannot be a complex submanifold of the Kähler manifold. Moreover, by the assumption, there exists a point  $p \in M$  with  $H(p) \neq 0$ . Hence, after applying the condition  $\dim M = \dim_{\mathbb{C}} \tilde{M}$ , we conclude that there exists a vector  $X \in T_p M$  satisfying  $FX = H(p) \neq 0$ . Thus, by using (4.15) we find  $PX = 0$ . Therefore,  $p$  is a totally real point, which contradicts the assumption that  $M$  is pointwise proper slant. Consequently,  $M$  cannot be slant.  $\square$

For totally umbilical surfaces in an arbitrary Kähler manifold, we have the following.

**Proposition 4.2** *We have the following:*

- (1) *Every totally geodesic surface in any Kähler surface is slant.*
- (2) *Every non-totally geodesic, totally umbilical surface in any Kähler surface is a non-slant, pointwise slant surface, unless it is Lagrangian.*

**Proof.** Statement (1) is a special case of Corollary 4.1. Statement (2) follows from Corollary 4.2 and the fact that every complex surface in a Kähler manifold is minimal.  $\square$

**Remark 4.1** *Statement (2) of Proposition 4.2 implies that every totally umbilical surface in the Euclidean complex plane is a non-slant, pointwise slant surface.*

**Remark 4.2** *It follows from Corollary 2.3 that statement (2) of Proposition 4.2 is false if the Kähler surface were replaced by a locally conformal Kähler surface.*

## 5. A canonical cohomology class

Recall that a  $2n$ -dimensional manifold  $N$  is called a *symplectic manifold* if it has a non-degenerate closed 2-form  $\Phi$ , i.e.,  $\Phi$  is a 2-form satisfying  $d\Phi = 0$  and  $\Phi^n \neq 0$  at each point on  $N$ .

For a given pointwise proper slant  $2n$ -submanifold  $M$  of a Kähler manifold  $(\tilde{M}, J, \tilde{g})$ , we put

$$\Omega(X, Y) = g(X, PY) \tag{5.1}$$

for vectors  $X, Y$  tangent to  $M$ . Then, by applying Lemma 1.1 of [7, p.78], we know that  $\Omega$  is a non-degenerate 2-form on  $M$ .

We recall the following lemma from [6, page 23].

**Lemma 5.1** *Let  $M$  be a submanifold of a Kähler manifold  $\tilde{M}$ . Then for any vectors  $X, Y$  tangent to  $M$ , we have*

$$(\nabla_X P)Y = th(X, Y) + A_{FY}X,$$

where  $h$  is the second fundamental form and  $A$  the shape operator of  $M$ .

The following result extends Theorem 3.4 of [6, page 96].

**Theorem 5.1** *Let  $M$  be a proper pointwise slant submanifold of a Kähler manifold  $\tilde{M}$ . Then  $\Omega$  is closed. Consequently,  $\Omega$  defines a canonical cohomology class of  $M$ :*

$$[\Omega] \in H^2(M; \mathbf{R}). \tag{5.2}$$

**Proof.** By definition of the exterior differentiation, we have

$$\begin{aligned} 3d\Omega(X, Y, Z) &= X\Omega(Y, Z) + Y\Omega(Z, X) + Z\Omega(X, Y) \\ &\quad - \Omega([X, Y], Z) - \Omega([Y, Z], X) - \Omega([Z, X], Y). \end{aligned}$$

Thus, by applying the definition of  $\Omega$ , we obtain

$$\begin{aligned} 3d\Omega(X, Y, Z) &= g(\nabla_X Y, PZ) + g(Y, \nabla_X(PZ)) + g(\nabla_Y Z, PX) \\ &\quad + g(Z, \nabla_Y(PX)) + g(\nabla_Z X, PY) + g(X, \nabla_Z(PY)) \\ &\quad - g([X, Y], PZ) - g([Z, X], PY) - g([Y, Z], PX) \\ &= g(Y, \nabla_X(PZ)) + g(Z, \nabla_Y(PX)) + g(X, \nabla_Z(PY)) \\ &\quad + g(\nabla_X Z, PY) + g(\nabla_Y X, PZ) + g(\nabla_Z Y, PX). \end{aligned}$$

Therefore, by using the definition of  $\nabla P$ , we find

$$3d\Omega(X, Y, Z) = \langle X, (\nabla_Z P)Y \rangle + \langle Y, (\nabla_X P)Z \rangle + \langle Z, (\nabla_Y P)X \rangle, \tag{5.3}$$

where  $\nabla P$  is defined by

$$(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$$

for any vector fields  $X, Y$  tangent to  $M$ . Therefore, after applying Lemma 5.1 and formula (5.3), we find

$$\begin{aligned} 3d\Omega(X, Y, Z) &= g(X, th(Y, Z)) + g(X, A_{FY}Z) + g(Y, th(Z, X)) \\ &\quad + g(Y, A_{FZ}X) + g(Z, th(X, Y)) + g(Z, A_{FX}Y). \end{aligned} \tag{5.4}$$

Consequently, by applying formulas (1.1), (1.4), (1.5) of [6, pages 13-14] and formula (5.4), we obtain (5.2). This proves the theorem.  $\square$

**Corollary 5.1** *Every  $2n$ -dimensional proper pointwise slant submanifold  $M$  of a Kähler manifold is a symplectic manifold.*

**Proof.** Under the hypothesis, it follows from Theorem 5.1 that  $d\Omega = 0$ . On the other hand, it follows from (5.1) that  $\Omega^n$  is a positive multiple of the volume element of  $M$ . Thus,  $\Omega^n \neq 0$ . Consequently,  $\Omega$  defines a symplectic structure on  $M$ .  $\square$

## 6. A topological obstruction to pointwise slant immersions

As an immediate consequence of Theorem 5.1 we have the following *topological obstruction to pointwise proper slant immersions*.

**Theorem 6.1** *Let  $M$  be a compact  $2n$ -dimensional differentiable manifold with  $H^{2i}(N; \mathbf{R}) = 0$  for some  $i \in \{1, \dots, n\}$ . Then  $M$  cannot be immersed in any Kähler manifold as a pointwise proper slant submanifold.*

**Proof.** Assume that  $M$  can be immersed in a Kähler manifold  $\tilde{M}$  as a pointwise proper slant submanifold. Then we have  $d\Omega = 0$  according to Theorem 5.1.

If  $H^{2i}(N; \mathbf{R}) = 0$  holds for some  $i \in \{1, \dots, n\}$ , then  $\Omega^i$  is an exact form. Thus, we may put  $\Omega^i = d\omega$  for some  $(2i - 1)$ -form  $\omega$ . Hence, we find

$$\Omega^n = (d\omega) \wedge \Omega^{n-i} = d(\omega \wedge \Omega^{n-i}).$$

But this is impossible, since  $\Omega^n$  is a positive multiple of the volume form according to (5.1).  $\square$

As an important immediate consequence of Theorem 6.1, we have the following optimal non-existence result.

**Corollary 6.1** *Every topological  $2n$ -sphere with  $n > 1$  cannot be immersed in any Kähler manifold as a pointwise proper slant submanifold.*

**Proof.** Follows from Theorem 6.1 and the fact that the Betti numbers  $\beta_i$  of a topological  $2n$ -sphere satisfy  $\beta_1 = \dots = \beta_{2n-1} = 0$ .  $\square$

Corollary 6.1 is sharp. This can be seen from the following two remarks.

**Remark 6.1** *If  $n = 1$ , Corollary 6.1 is false. For instance, the diagonal embedding of the 2-sphere  $S^2$  into the Hermitian symmetric space*

$$Q_2 = SO(4)/SO(2) \times SO(2) = S^2 \times S^2$$

*is a Lagrangian surface, which is by definition a pointwise proper slant submanifold.*

**Remark 6.2** *The condition “proper” in Corollary 6.1 cannot be dropped. Since the unit  $2n$ -sphere  $S^{2n}(1)$  can be isometrically immersed in the complex projective  $2n$ -space  $CP^{2n}(4)$  with constant holomorphic sectional curvature 4 as a totally geodesic Lagrangian submanifold.*

*The immersed  $S^{2n}(1)$  is a pointwise non-proper slant submanifold of  $CP^{2n}(4)$  for any  $n \geq 2$ .*

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