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Flat surfaces in the Minkowski space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map

Uğur Dursun and Emel Coşkun

Abstract

In this article, we obtain all nonplanar cylindrical surfaces in the Minkowski space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind. We also prove that right circular cones and hyperbolic cones in $\mathbb{E}^3_1$ are the only cones in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind. We conclude that there is no tangent developable surface fully lying in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind.

Key Words: Gauss map, Pointwise 1-type, Ruled surface, Cone, Cylinder, Developable surface

1. Introduction

In late 1970’s B.-Y. Chen introduced the notion of finite type submanifolds of Euclidean space [6]. Since then many works were done to characterize or classify submanifolds of Euclidean space or pseudo-Euclidean space in terms of finite type (cf. [7, 8, 12, 16]). Also, B.-Y. Chen and P. Piccinni extended the notion of finite type to differentiable maps, in particular, to Gauss map of submanifolds in [9]. A smooth map $\phi$ of a submanifold $M$ of a Euclidean space or a pseudo-Euclidean space is said to be of finite type if $\phi$ can be expressed as a finite sum of eigenfunctions of the Laplacian $\Delta$ of $M$, that is, $\phi = \phi_0 + \sum_{i=1}^{k} \phi_i$, where $\phi_0$ is a constant map, $\phi_1, \ldots, \phi_k$ nonconstant maps such that $\Delta \phi_i = \lambda_i \phi_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$.

If a submanifold $M$ of a Euclidean space or a pseudo-Euclidean space has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$. In [9], B.-Y. Chen and P. Piccinni studied compact submanifolds of Euclidean spaces with finite type Gauss map. Several articles also appeared on submanifolds with finite type Gauss map (cf. [2, 3, 4, 5, 24, 25]).

However, the Laplacian of the Gauss map of several surfaces and hypersurfaces, such as helicoids of the 1st, 2nd, and 3rd kind, conjugate Enneper’s surface of the second kind and B-scrolls in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ [20], generalized catenoids, spherical n-cones, hyperbolical n-cones and Enneper’s hypersurfaces in $\mathbb{E}^{n+1}_1$ [14], take the form

$$\Delta G = f(G + C)$$

(1)

for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold of a pseudo-Euclidean space is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function $f$ on $M$ and
Therefore we say that a plane in \(E^3\) is a trivial surface with pointwise 1-type Gauss map of the first kind or the second kind.

**Remark 1.** The Gauss map \(G\) of a plane \(M\) in \(E^3\) is a constant vector and \(\Delta G = 0\). For \(f = 0\) if we write \(\Delta G = 0 \cdot G\), then \(M\) has pointwise 1-type Gauss map of the first kind. If we choose \(C = -G\) for any nonzero smooth function \(f\), then (1) holds. In this case \(M\) has pointwise 1-type Gauss map of the second kind. Therefore we say that a plane in \(E^3\) is a trivial surface with pointwise 1-type Gauss map of the first kind or the second kind.

The complete classification of ruled surfaces in \(E^3\) with pointwise 1-type Gauss map of the first kind was obtained in [20]. Also, a complete classification of rational surfaces of revolution in \(E^3\) satisfying (1) was recently given in [19], and it was proved that a right circular cone and a hyperbolic cone in \(E^3\) are the only rational surfaces of revolution in \(E^3\) with pointwise 1-type Gauss map of the second kind. The first author described all nonplanar cylindrical surfaces in the Euclidean space \(E^3\) with pointwise 1-type Gauss map of the second kind [15].

In this article, we study nondegenerate flat surfaces in \(E^3\) with pointwise 1-type Gauss map of the second kind. We describe all nonplanar cylindrical surfaces with pointwise 1-type Gauss map of the second kind, and we also show that right circular cones and hyperbolic cones in \(E^3\) are the only cones in \(E^3\) with pointwise 1-type Gauss map of the second kind. We conclude that there is no tangent developable surface in \(E^3\) with pointwise 1-type Gauss map of the second kind.

Throughout this paper, we assume that all the geometric objects are smooth and all surfaces are connected unless otherwise stated.

2. **Preliminaries**

Let \(E^3\) be a 3-dimensional Minkowski space with the Lorentz metric \(ds^2 = -dx_1^2 + dx_2^2 + dx_3^2\), where \((x_1, x_2, x_3)\) denotes the standard coordinates of \(E^3\). A vector \(X \in E^3\) is said to be space-like if \(\langle X, X \rangle > 0\) or \(X = 0\), time-like if \(\langle X, X \rangle < 0\), and light-like or null if \(\langle X, X \rangle = 0\) and \(X \neq 0\). A curve in \(E^3\) is said to be space-like, time-like or light-like (null) if its tangent vector is, respectively, space-like, time-like or light-like. A time-like or light-like vector in \(E^3\) is said to be **causal**. For the Lorentz vector space it is well known that there are no causal vectors in \(E^3\) orthogonal to a time-like vector [18].

For two vectors \(X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in E^3\), the Lorentz cross-product \(X \times Y\) of \(X\) and \(Y\) is defined by

\[
X \times Y = (-x_2y_3 + x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).
\]

The properties that the Lorentz cross-product satisfies can be seen in [20].

Let \(M\) be a nondegenerate surface in \(E^3\). The map \(G : M \to Q^2(\varepsilon_G) \subset E^3\) which sends each point of \(M\) to the unit normal vector to \(M\) at the point is called the Gauss map of \(M\), where \(\varepsilon_G (= \pm 1)\) denotes the sign of the vector \(G\) and \(Q^2(\varepsilon_G)\) is a 2-dimensional space form given by

\[
Q^2(\varepsilon_G) = \begin{cases} 
    \mathbb{E}^2_1(1) & \text{in } E^3 \quad \text{if } \varepsilon_G = 1 \\
    \mathbb{H}^2(-1) & \text{in } E^3 \quad \text{if } \varepsilon_G = -1,
\end{cases}
\]

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where \( S^2_1(1) \) and \( \mathbb{H}^2(-1) \) are, respectively, the de Sitter space and hyperbolic space in \( \mathbb{E}^3_1 \) centered at the origin.

We denote by \( h, A_G, \tilde{\nabla} \) and \( \nabla \), the second fundamental form, the Weingarten map, the Levi-Civita connection of \( \mathbb{E}^3_1 \) and the induced Riemannian connection on \( M \), respectively. We choose a local oriented orthonormal moving frame \( \{e_1, e_2, e_3\} \) on \( M \) in \( \mathbb{E}^3_1 \) with \( e_i = \langle e_i, e_i \rangle = \mp 1 \), \( i = 1, 2, 3 \), such that \( e_1, e_2 \) are tangent to \( M \) and \( e_3 = G \) is normal to \( M \).

We denote by \( \{\omega_1, \omega_2, \omega_3\} \) the dual 1-forms to \( \{e_1, e_2, e_3\} \) defined by \( \omega_A(e_B) = \langle e_A, e_B \rangle = \varepsilon_A \delta_{AB} \) and by \( \{\omega_{AB}\} \), \( A, B = 1, 2, 3 \), the connection 1-forms associated with \( \{\omega_1, \omega_2, \omega_3\} \) satisfying \( \omega_{AB} + \omega_{BA} = 0 \). Thus we have \( \tilde{\nabla}_e e_i = \sum_{j=1}^2 \varepsilon_j \omega_{ij}(ek) e_j + \varepsilon_3 h_{ik} e_3 \), \( \tilde{\nabla}_e e_3 = \sum_{j=1}^2 \varepsilon_j \omega_{3j}(ek) e_j \), where \( h_{ik} \)’s are the coefficients of the second fundamental form \( h \). By Cartan’s Lemma, we also have \( \omega_{j3} = \sum_{k=1}^2 \varepsilon_k h_{jk} \omega_k \), \( h_{jk} = h_{kj} \).

The mean curvature \( H \) and the Gauss curvature \( K \) of \( M \) in \( \mathbb{E}^3_1 \) are, respectively, defined by

\[
H = \frac{1}{2} \text{tr} A_G = \frac{1}{2} \sum_{i=1}^2 \varepsilon_i \langle A_G(e_i), e_i \rangle \quad \text{and} \quad K = \varepsilon_G \det A_G.
\]

A nondegenerate surface in \( \mathbb{E}^3_1 \) with zero Gauss curvature is called a developable surface. The developable surfaces in Minkowski space \( \mathbb{E}^3_1 \) are the same as in Euclidean space. In particular, they are plane, cone, cylinder and tangent developable surfaces.

Let \( I \) and \( J \) be open intervals containing the origin in the real line. Let \( \alpha = \alpha(s) \) be a curve from \( J \) into \( \mathbb{E}^3_1 \) and \( \beta(s) \) a vector field along \( \alpha(s) \) orthogonal to \( \alpha(s) \). A ruled surface \( M \) in \( \mathbb{E}^3_1 \) is defined as a semi-Riemannian surface swept out by the vector \( \beta(s) \) along the curve \( \alpha(s) \). Then \( M \) has always a parametrization

\[
x(s, t) = \alpha(s) + t \beta(s), \quad s \in J, \ t \in I.
\]

The curve \( \alpha = \alpha(s) \) is called a base curve and \( \beta = \beta(s) \) is a director curve. If \( \beta \) is constant, then the ruled surface is said to be cylindrical, and noncylindrical otherwise.

We consider the curve \( \alpha \) is space-like or time-like. As it is explained in [20] we have five different ruled surfaces according to the character of the base curve \( \alpha \) and the director \( \beta \) as follows: If the curve \( \alpha \) is space-like or time-like, the ruled surface \( M \) is said to be of type \( M_+ \) or type \( M_- \), respectively. Also the ruled surface of type \( M_+ \) is divided into three types. When \( \beta \) is space-like, it is said to be of type \( M^1_+ \) or \( M^2_+ \) if \( \beta \) is non-null or light-like, respectively. When \( \beta \) is time-like, then \( \beta \) must be space-like because there is no causal vector in \( \mathbb{E}^3_1 \) orthogonal to a time-like vector. In this case, \( M \) is said to be of type \( M^3_+ \). On the other hand, for the ruled surface \( M_- \) it is said to be of type \( M^1_- \) or \( M^2_- \) if \( \beta \) is non-null or light-like, respectively. The ruled surface type \( M^1_+ \) or \( M^2_+ \) (resp. \( M^1_- \), \( M^2_- \)) is space-like (resp. time-like).

However if the base curve \( \alpha \) is a light-like curve and the vector field \( \beta \) along \( \alpha \) is a light-like vector field, then the ruled surface is called a null scroll. A null scroll with zero Gauss curvature is a plane in \( \mathbb{E}^3_1 \). In particular, a null scroll with Cartan frame is said to be a B-scroll [17] which is a time-like surface. It is known that a B-scroll has 1-type Gauss map of the first kind [20].

For the Frenet equations of a space-like or time-like curve in \( \mathbb{E}^3_1 \) we have the following theorem.

**Theorem 2.1** [25] Let \( \alpha \) be a space-like or time-like curve which we assume to be parametrized by arc length and satisfies \( \langle \alpha'\alpha'' \rangle \neq 0 \). Then this curve induces a Frenet 3-frame \( T = \alpha'(s), \ N = \frac{\alpha''(s)}{\sqrt{\langle \alpha''\alpha'' \rangle}}, \ B = T \times N \)
where \( k = \langle T', N \rangle \) and \( \tau = \langle N', B \rangle \) are called the curvature and torsion of \( \alpha \), and \( \varepsilon_T = \langle T, T \rangle = \mp 1 \), \( \varepsilon_N = \langle N, N \rangle = \mp 1 \).

Let \( M \) be an oriented nondegenerate surface (time-like or space-like) in \( \mathbb{E}^3 \) with corresponding unit normal field \( G \), and let \( \alpha \) be an arc length parametrized curve in \( M \). Let \( V \) be a unit tangent vector field along \( \alpha(s) \) such that \( V(s) = G(\alpha(s)) \times \alpha'(s) \) with \( \varepsilon_V = \langle V, V \rangle = -\varepsilon_G \varepsilon_T \). Then the functions

\[
k_n(s) = \langle \alpha''(s), G(\alpha(s)) \rangle \quad \text{and} \quad k_g(s) = \langle \alpha''(s), V(s) \rangle
\]

are, respectively, called the normal curvature and the geodesic curvature of \( \alpha(s) \) at \( s \) if \( \alpha'' \) is non-null. If \( \alpha'' \) is non-null, then we can write \( \alpha''(s) \) as follows

\[
\alpha''(s) = \varepsilon_N k(s) N(s) = \varepsilon_V k_g(s) V(s) + \varepsilon_G k_n(s) G(\alpha(s))
\]

where \( k(s) \) is the curvature of \( \alpha''(s) \), and thus we have

\[
\varepsilon_N k^2(s) = \varepsilon_G (-\varepsilon_T k_g^2(s) + k_n^2(s)).
\]

Note that there is no a definition of curvature when \( \alpha''(s) \) is null.

3. Cylindrical ruled surfaces with pointwise 1-type Gauss map

Considering Remark 1, a plane in the Minkowski space \( \mathbb{E}^3 \) which is a cylinder has pointwise 1-type Gauss map of the second kind. Here we determine nonplanar cylindrical ruled surfaces in \( \mathbb{E}^3 \) with pointwise 1-type Gauss map of the second kind. A cylindrical ruled surface \( M \) is only of type \( M^1_1 \), \( M^1_2 \) or \( M^1_3 \).

The following lemma is obtained in [14].

**Lemma 3.1** Let \( M_q \) be a hypersurface with index \( q \) in a Lorentz-Minkowski space \( L^{n+1} \). Then the Laplacian of the Gauss map \( G \) is given by

\[
\Delta G = \varepsilon_G \| A_G \|^2 G + n \nabla H,
\]

where \( \| A_G \|^2 = \text{tr}(A_G A_G) \), \( \varepsilon_G = \langle G, G \rangle \) and \( H \) is the mean curvature of \( M_q \).

We prove the following lemma for later use.

**Lemma 3.2** Let \( M \) be an oriented nondegenerate surface in the Minkowski space \( \mathbb{E}^3 \). Let \( \{e_1, e_2\} \) be a local orthonormal tangent frame on \( M \) with \( \varepsilon_i = \langle e_i, e_i \rangle \), \( i = 1, 2 \). If \( C \) is a constant vector in \( \mathbb{E}^3 \), then the components of \( C = \varepsilon_1 C_1 e_1 + \varepsilon_2 C_2 e_2 + \varepsilon_3 C_3 G \) in the basis \( \{e_1, e_2, G\} \) of \( \mathbb{E}^3 \) satisfy the following equations:

\[
e_1(C_1) + \varepsilon_2 \omega_{21}(e_1) C_2 - \varepsilon_G h_{11} C_3 = 0,
\]

\[
e_2(C_2) + \varepsilon_1 \omega_{12}(e_2) C_1 - \varepsilon_G h_{22} C_3 = 0,
\]

\[
e_3(C_3) = \varepsilon_G h_{33} C_3 = 0.
\]

where \( \omega_{ij} \) are the Christoffel symbols and \( h_{ij} \) are the components of the metric tensor.
\[ \begin{aligned}
e_1(C_2) + \varepsilon_1 \omega_{12}(e_1) C_1 - \varepsilon_G h_{12} C_3 &= 0, \\
e_1(C_3) + \varepsilon_1 h_{11} C_1 + \varepsilon_2 h_{21} C_2 &= 0, \\
e_2(C_1) + \varepsilon_2 \omega_{21}(e_2) C_2 - \varepsilon_G h_{21} C_3 &= 0, \\
e_2(C_2) + \varepsilon_1 \omega_{12}(e_2) C_1 - \varepsilon_G h_{22} C_3 &= 0, \\
e_2(C_3) + \varepsilon_1 h_{12} C_1 + \varepsilon_2 h_{22} C_2 &= 0,
\end{aligned} \]

where \( C_i = \langle C, e_i \rangle, \ i = 1, 2 \) and \( C_3 = \langle C, G \rangle \).

**Proof.** Taking derivative of the vector \( C \) in direction \( e_k \) and using the formulas of Gauss and Weingarten, we obtain

\[ \begin{aligned}
\tilde{\nabla}_{e_k} C &= \varepsilon_1 [e_k(C_1) + \varepsilon_2 \omega_{21}(e_k) C_2 - \varepsilon_G h_{21} C_3] e_1 \\
&\quad + \varepsilon_2 [e_k(C_2) + \varepsilon_1 \omega_{12}(e_k) C_1 - \varepsilon_G h_{22} C_3] e_2 \\
&\quad + \varepsilon_G [e_k(C_3) + \varepsilon_1 h_{11} C_1 + \varepsilon_2 h_{22} C_2] G = 0
\end{aligned} \]

which produces equations (5)–(10) for \( k = 1, 2 \).

\[ \square \]

**Theorem 3.3** A nonplanar cylindrical ruled surface \( M \) in the Minkowski space \( \mathbb{E}^3_1 \) has pointwise 1-type Gauss map of the second kind if and only if \( M \) is congruent to an open part of the following surfaces:

1. **the time-like cylinder** \( M^1_+ \) parametrized by

   \[ x(k, t) = \left( t, \pm \left( \frac{(k + k_0) \sqrt{R(k)}}{2c_0 k_0 k^2} + \frac{c_0}{2k_0} \arctan \left( \frac{k - k_0}{\sqrt{R(k)}} \right) \right), -\frac{k_0}{2c_0 k^2} \right); \]

   where \( R(k) = c_0^2 k^2 - (k - k_0)^2 > 0; \)

2. **the space-like cylinder** \( M^1_+ \) parametrized by

   \[ x(k, t) = \left( \pm \psi(k), \frac{k_0}{2c_0 k^2}, t \right); \]

3. **the space-like cylinder** \( M^1_+ \) parametrized by

   \[ x(k, t) = \left( \frac{k_0}{2c_0 k^2}, \pm \psi(k), t \right); \]

4. **the space-like cylinder** \( M^1_+ \) parametrized by

   \[ x(k, t) = \left( \pm \left( \frac{k_0}{4k^2} - \theta(k) \right), \frac{k_0}{4k^2} + \theta(k), t \right); \]
5. the time-like cylinder $M^1_+$ parametrized by

$$x(k, t) = \left( \pm \psi(k), \frac{k_0}{2c_0k^2}, t \right);$$

(15)

6. the time-like cylinder $M^1_-$ parametrized by

$$x(k, t) = \left( \frac{k_0}{2c_0k^2}, \pm \varphi(k), t \right);$$

(16)

7. the time-like cylinder $M^1_0$ parametrized by

$$x(k, t) = \left( \frac{k_0}{4k^2} + \theta(k), \pm \left( \frac{k_0}{4k^2} - \theta(k) \right), t \right);$$

(17)

where

$$\varphi(k) = \frac{(k + k_0)}{2c_0k^2k^2} \sqrt{c_0^2k^2 + (k - k_0)^2} - \frac{c_0}{2k_0} \ln \left| \frac{k_0 - k + \sqrt{c_0^2k^2 + (k - k_0)^2}}{k} \right|,$$

$$\psi(k) = \frac{(k + k_0)}{2c_0k^2k^2} \sqrt{(k - k_0)^2 - c_0^2k^2} + \frac{c_0}{2k_0} \ln \left| \frac{k_0 - k + \sqrt{(k - k_0)^2 - c_0^2k^2}}{k} \right|,$$

with $(k - k_0)^2 - c_0^2k^2 > 0$,

$$\theta(k) = \frac{1}{2(k - k_0)} - \frac{1}{2k_0} \ln \left| \frac{k}{k - k_0} \right|,$$

and, $p_0$, $c_0$ and $k_0$ are nonzero constants.

**Proof.** Suppose that $M$ has pointwise 1-type Gauss map of the second kind. Then the gradient vector $\nabla H$ of the mean curvature $H$ is nonzero on $M$ because of (3.1). If $\nabla H$ were zero, then the Gauss map would be of pointwise 1-type of the first kind. So the mean curvature $H$ is a nonconstant function on $M$.

Let $\{e_1, e_2\}$ be a local orthonormal tangent frame on $M$ with $\varepsilon_i = \langle e_i, e_i \rangle$, $i = 1, 2$. By equations (1) and (4), we have

$$\varepsilon_G\|A_G\|^2 G + 2 \nabla H = f(G + C)$$

(18) for some nonzero smooth function $f$ on $M$ and some nonzero constant vector $C \in \mathbb{R}^3$. In the basis $\{e_1, e_2, G\}$ we can write

$$C = \varepsilon_1C_1e_1 + \varepsilon_2C_2e_2 + \varepsilon_C C_3G,$$

where $C_i = \langle C, e_i \rangle$, $i = 1, 2$ and $C_3 = \langle C, G \rangle$ which satisfy equations (5)–(10) in Lemma 3.2. Considering $\nabla H = \varepsilon_1e_1(H)e_1 + \varepsilon_2e_2(H)e_2$ equation (18) implies

$$\varepsilon_G\|A_G\|^2 = f(1 + \varepsilon_CC_3),$$

(19)

$$\varepsilon_1(\varepsilon_1h_{11} + \varepsilon_2h_{22}) = fC_1,$$

(20)

$$\varepsilon_2(\varepsilon_1h_{11} + \varepsilon_2h_{22}) = fC_2.$$

(21)
Since \( M \) is a cylindrical surface, then it is parametrized by
\[
x(s, t) = \alpha(s) + t\beta,
\]
where the base curve \( \alpha(s) \) which is a smooth time-like or space-like curve with the arc length parameter \( s \) lies in a plane with a time-like or space-like unit normal vector \( \beta \) which is the director of the cylinder.

Now we take a local orthonormal tangent frame \( \{e_1, e_2\} \) on \( M \) as \( e_1 = \frac{\partial}{\partial s} \) and \( e_2 = \frac{\partial}{\partial t} \) with \( \varepsilon_1 = \langle e_1, e_1 \rangle = \langle \beta, \beta \rangle = \pm 1, \) \( \varepsilon_2 = \langle e_2, e_2 \rangle = \langle \alpha'(s), \alpha'(s) \rangle = \pm 1 \) and \( \langle e_1, e_2 \rangle = \langle \beta, \alpha'(s) \rangle = 0. \) By taking the Gauss map \( G = e_1 \times e_2 \) with \( \varepsilon_G = \langle G, G \rangle = -\varepsilon_1 e_2, \) then the Frenet 3-frame for the curve \( \alpha \) and the frame \( \{e_1, e_2, G\} \) on \( M \) in \( \mathbb{E}^3 \) have the same orientation.

By a direct calculation we obtain \( \nabla e_2 e_2 = \alpha''(s) = \varepsilon_G k(s) G \) because \( \alpha \) is a plane curve and the principal normal vector of the curve \( \alpha \) is the normal of the cylinder, and \( \nabla e_1 e_1 = \nabla e_1 e_2 = \nabla e_2 e_1 = 0, \) where \( k(s) \) is the curvature of \( \alpha(s). \) All these relations imply that \( \omega_2(e_1) = \omega_2(e_2) = 0, \) \( h_{11} = h_{12} = h_{21} = 0, \) and \( h_{22} = k(s). \)

Therefore the mean curvature is \( H = \varepsilon_2 k(s)/2 \) which is the function of \( s, \) and \( \|A_G\|^2 = k^2(s), \) where \( k(s) \neq 0, \) i.e. \( k(s) \) is strictly positive or strictly negative. Without losing generality we suppose that \( k(s) > 0. \) Thus equations (20) and (21) give, respectively
\[
C_1(s) = \frac{\varepsilon_2 C_2(s)}{1 + \varepsilon_G C_3(s)} = 0
\]
and
\[
C_2(s) + \varepsilon_2 k(s) C_2(s) = 0.
\]

It is seen from (21) that \( f \) is also a function of \( s. \) As the vector \( C \) is constant, we have
\[
\varepsilon_2 C_2^2(s) - \varepsilon_1 \varepsilon_2 C_3^2(s) - \langle C, C \rangle = \varepsilon_1 C_3^2 = 0,
\]
where \( c_0 \) is a constant and \( \varepsilon_C = \text{sgn}(\langle C, C \rangle). \)

Equations (19) and (21) yield
\[
\frac{k'(s)}{k^2(s)} = \frac{\varepsilon_2 \varepsilon_G C_2}{1 + \varepsilon_G C_3}
\]
from which and equation (24) we obtain
\[
\frac{k'(s)}{k(s)} = -\frac{\varepsilon_G C_3}{1 + \varepsilon_G C_3}
\]
and from its solution we get
\[
C_3(s) = \varepsilon_1 \varepsilon_2 \left(1 - \frac{k_0}{k(s)}\right),
\]
where \( k_0 \) is a nonzero constant. Also, by using (24) and (28)
\[
C_2(s) = -\varepsilon_1 k_0 k'(s) k^3(s).
\]
Moreover, from (21) and (29) we obtain

\[ f(s) = -\varepsilon_1 \varepsilon_2 \frac{k^3(s)}{k_0}. \quad (30) \]

If \( C \) is a non-null vector, then using (28) and (29) equation (25) yields the differential equation

\[ \frac{d^2}{ds^2}(k^2) = k^4 \left( \varepsilon_2 \varepsilon C C_0 k^2 + \varepsilon_1 (k-k_0)^2 \right). \quad (31) \]

For later use we need

\[ \int C_2(s) ds = -\varepsilon_1 \int \frac{k_0 k'}{k^3} ds + d_2 = \varepsilon_1 \frac{k_0}{2k^2} + d_2 \quad (32) \]

and by considering (31)

\[ \int C_3(s) ds = \varepsilon_1 \varepsilon_2 \int \frac{k-k_0}{k} ds = \pm \varepsilon_1 \varepsilon_2 k_0 \int \frac{(k-k_0)dk}{k^3 \sqrt{\varepsilon_2 \varepsilon C C_0 k^2 + \varepsilon_1 (k-k_0)^2}} + d_3, \]

where \( d_1 \) and \( d_2 \) are integration constants. From the evaluation of the last integral for \( \varepsilon_1 = 1 \) we have

\[ \int C_4(s) ds = \pm \left( \varepsilon_2 \left( \frac{(k+k_0)}{2k_0 k^2} \sqrt{R(k)} - \frac{\varepsilon C C_0}{2k_0} \ln \left( \frac{k_0 - k + \sqrt{R(k)}}{k} \right) \right) \right) + d_3, \quad (33) \]

where \( R(k) = \varepsilon_2 \varepsilon C C_0^2 k^2 + (k-k_0)^2 \), and for \( \varepsilon_1 = -1 \) (in this case \( \varepsilon_2 = \varepsilon C = 1 \)) we have

\[ \int C_5(s) ds = \pm \left( \frac{(k+k_0)}{2k_0 k^2} \sqrt{R(k)} + \frac{C_0^2}{2k_0} \arctan \left( \frac{k-k_0}{\sqrt{R(k)}} \right) \right) + d_3, \quad (34) \]

where \( R(k) = C_0^2 k^2 - (k-k_0)^2 > 0 \).

A cylindrical ruled surface \( M \) is only of type \( M^1_{+} \), \( M^1_{-} \) or \( M^3_{+} \).

**Case 1.** \( M \) is of type \( M^1_{+} \), i.e., the vector \( \beta \) is time-like. Hence \( \varepsilon_1 = -1 \) and \( \varepsilon_2 = \varepsilon G = \varepsilon C = 1 \). Considering equation (25), we may put

\[ C_2(s) = c_0 \sin \lambda(s), \quad C_3(s) = c_0 \cos \lambda(s), \quad (35) \]

which satisfy equations (23) and (24) if \( \lambda'(s) = k(s) \), that is, \( \lambda(s) = \lambda_0 + \int k(s) ds \), where \( \lambda_0 \) is an integration constant. Thus we have

\[ \sin \lambda(s) = \frac{C_2}{c_0} = \frac{k_0 k'}{c_0 k^3} \quad \text{and} \quad \cos \lambda(s) = \frac{C_3}{c_0} = \frac{k_0 - k}{c_0 k} \quad (36) \]

for later use.

Since \( \alpha \) is a plane curve, acting a Lorentz transformation we can write

\[ \alpha(s) = (0, \alpha_2(s), \alpha_3(s)) \quad \text{and} \quad \beta = (1, 0, 0) \]

without loss of generality. Then the Gauss map of the cylinder \( M^3_{+} \) is

\[ G = e_1 \times e_2 = (0, -\alpha'_2(s), \alpha'_3(s)), \]

\[ 620 \]
as \( e_2 = \alpha'(s) = (0, \alpha'_2(s), \alpha'_3(s)) \). Now we may put \( \alpha'_2(s) = \cos \mu(s) \) and \( \alpha'_3(s) = \sin \mu(s) \) because of \( \alpha'_2^2(s) + \alpha'_3^2(s) = 1 \), where \( \mu(s) \) is a differentiable function.

The equation \( \alpha''(s) = \varepsilon_G k(s) G \) implies that \( \mu'(s) = k(s) \). For simplicity we take \( \mu(s) = \lambda(s) = \lambda_0 + \int k(s) ds \). In view of (32), (34) and (36) the base curve \( \alpha(s) \) of the cylinder \( M_{1+}^3 \) is determined uniquely, up to a rigid motion, by

\[
\alpha(s) = (0, d_3 + \frac{1}{c_0} \int C_3(s) ds, d_2 + \frac{1}{c_0} \int C_2(s) ds),
\]

where \( R(k) = c^2_0 k^2 - (k - k_0)^2 > 0 \). It is seen that the base curve of the cylinder \( M_{1+}^3 \) can be parametrized in terms of the curvature function \( k \), that is, \( \alpha = \alpha(k) \). Therefore we obtain the parametrization (11) for the cylinder \( M_{1+}^3 \) which has pointwise 1-type Gauss map of the second kind for \( f(k) = \frac{k^3}{k_0} \) and \( C = (0, 0, c_0) \).

**Case 2.** \( M \) is of type \( M_{1+}^3 \), i.e., \( \varepsilon_1 = \varepsilon_2 = 1, (\varepsilon_G = -1) \). From equation (25), the vector \( C \) is space-like, time-like or null.

Considering equation (25) we may put

\[
C_2(s) = c_0 \cosh \lambda(s), \quad C_3(s) = c_0 \sinh \lambda(s) \quad \text{for} \quad \varepsilon_G = 1
\]

or

\[
C_2(s) = c_0 \sinh \lambda(s), \quad C_3(s) = c_0 \cosh \lambda(s) \quad \text{for} \quad \varepsilon_G = -1
\]

which hold for equations (23) and (24) if \( \lambda'(s) = -k(s) \), that is, \( \lambda(s) = \lambda_0 - \int k(s) ds \), where \( \lambda_0 \) is an integration constant. Thus we have

\[
\cosh \lambda(s) = \frac{C_2}{c_0} = -\frac{k_0 k'}{c_0 k^3} \quad \text{and} \quad \sinh \lambda(s) = \frac{C_3}{c_0} = \frac{k_0 - k}{c_0 k} \quad \text{for} \quad \varepsilon_G = 1 \tag{38}
\]

or

\[
\cosh \lambda(s) = \frac{C_3}{c_0} = \frac{k_0 - k}{c_0 k} \quad \text{and} \quad \sinh \lambda(s) = \frac{C_2}{c_0} = -\frac{k_0 k'}{c_0 k^3} \quad \text{for} \quad \varepsilon_G = -1 \tag{39}
\]

For the plane curve \( \alpha \), acting a Lorentz transformation we can write

\[
\alpha(s) = (\alpha_1(s), \alpha_2(s), 0) \quad \text{and} \quad \beta = (0, 0, 1)
\]

without loss of generality. The Gauss map of the cylinder \( M_{1+}^3 \) is

\[
G = e_1 \times e_2 = (\alpha'_2(s), \alpha'_1(s), 0)
\]

as \( e_2 = \alpha'(s) = (0, \alpha'_1(s), \alpha'_2(s), 0) \). Considering \(-\alpha'_1^2(s) + \alpha'_2^2(s) = 1 \), we may put \( \alpha'_1(s) = \sin \mu(s) \) and \( \alpha'_2(s) = \cosh \mu(s) \) to determine \( \alpha(s) \), where \( \mu \) is a differentiable function. From the equation \( \alpha''(s) = \varepsilon_G k(s) G \) we obtain \( \mu'(s) = -k(s) \). For simplicity we take \( \mu(s) = \lambda(s) = \lambda_0 - \int k(s) ds \).
Now we suppose that $C$ is space-like, i.e., $\varepsilon_C = 1$. By using (32), (33) and (38) the base curve $\alpha(s)$ of the cylinder $M^1_+$ is determined uniquely, up to a rigid motion, by

$$
\alpha(s) = (d_3 + \frac{1}{c_0} \int C_3(s) ds, \quad d_2 + \frac{1}{c_0} \int C_2(s) ds, \quad 0),
$$

where

$$
R(k) = c_0^2 k^2 + (k - k_0)^2.
$$

It is seen the base curve of the cylinder $M^1_+$ can be parametrized in terms of the curvature function $k$, that is, $\alpha = \alpha(k)$.

Therefore we obtain the parametrization (12) for the cylinder $M^1_+$ which has pointwise 1-type Gauss map of the second kind for $f(k) = -\frac{k^2}{k_0}$ and $C = (0, 0)$.

If $C$ is time-like, i.e., $\varepsilon_C = -1$, then by a similar argument we obtain the base curve of the cylinder $M^1_+$ as

$$
\alpha(k) = (d_2 + \frac{k_0}{2c_0k^2}, \quad d_3 + \frac{(k + k_0)}{2c_0k_0 k^2} \sqrt{R(k)} + \frac{c_0}{2k_0} \ln \left| \frac{k_0 - k + \sqrt{R(k)}}{k} \right|, \quad d_2 + \frac{k_0}{2c_0k^2}, \quad 0),
$$

where $R(k) = (k - k_0)^2 - c_0^2 k^2 > 0$. So we get the parametrization (13) for the cylinder $M^1_+$ which has pointwise 1-type Gauss map of the second kind for $f(k) = -\frac{k^2}{k_0}$ and $C = (-c_0, 0, 0)$.

Now let the vector $C$ be null. From (25) we get $C_2(s) = \pm C_3(s)$. We will consider the case $C_2(s) = C_3(s)$.

Hence, from (23) we get $C_2(s) = e^{\mu(s)}$, where $\mu(s) = \lambda_0 - \int k(s) ds$, and then

$$
\alpha'_1(s) = \sinh \mu(s) = \frac{1}{2} (C_2(s) - \frac{1}{C_2(s)}),
$$

and

$$
\alpha'_2(s) = \cosh \mu(s) = \frac{1}{2} (C_2(s) + \frac{1}{C_2(s)}).
$$

Using (28) and (29) we obtain the first two components of $\alpha(s)$ as

$$
\alpha_i(s) = \frac{k_0}{4k^2} + \frac{(-1)^i}{2} \left( \frac{1}{k - k_0} - \frac{1}{k_0} \ln \left| \frac{k}{k - k_0} \right| \right) + d_i, \quad i = 1, 2.
$$

Similarly if we take $C_2(s) = -C_3(s)$, then the first two components of $\alpha(s)$ are

$$
\alpha_i(s) = (-1)^i \frac{k_0}{4k^2} + \frac{1}{2(k - k_0)} - \frac{1}{2k_0} \ln \left| \frac{k}{k - k_0} \right| + d_i, \quad i = 1, 2.
$$

Therefore, considering (40) and (41) we obtain the parametrization (14) for cylinder $M^1_+$ which has pointwise 1-type Gauss map of the second kind for $f(k) = -\frac{k^2}{k_0}$ and $C = (-1, 1, 0)$ if $C_2(s) = C_3(s)$ or $C = (1, 1, 0)$ if $C_2(s) = -C_3(s)$.

**Case 3.** $M$ is of type $M^1_+$, i.e., $\varepsilon_1 = 1, \varepsilon_2 = -1, (\varepsilon_C = 1)$. From (25) the vector $C$ is space-like, time-like or null.
Considering equation (25) we may put

\[ C_2(s) = c_0 \sinh \lambda(s), \quad C_3(s) = c_0 \cosh \lambda(s) \quad \text{for} \quad \varepsilon_C = 1 \]

or

\[ C_2(s) = c_0 \cosh \lambda(s), \quad C_3(s) = c_0 \sinh \lambda(s) \quad \text{for} \quad \varepsilon_C = -1 \]

which hold equations (23) and (24) if \( \lambda'(s) = k(s) \), that is, \( \lambda(s) = \lambda_0 + \int k(s)ds \), where \( \lambda_0 \) is an integration constant. Thus we have

\[
\sinh \lambda(s) = \frac{C_2}{c_0} = -\frac{k_0 k'}{c_0 k^2} \quad \text{and} \quad \cosh \lambda(s) = \frac{C_3}{c_0} = \frac{k_0 - k}{c_0 k} \quad \text{for} \quad \varepsilon_C = 1 \tag{42}
\]

or

\[
\sinh \lambda(s) = \frac{C_3}{c_0} = \frac{k_0 - k}{c_0 k} \quad \text{and} \quad \cosh \lambda(s) = \frac{C_2}{c_0} = -\frac{k_0 k'}{c_0 k^2} \quad \text{for} \quad \varepsilon_C = -1. \tag{43}
\]

For the plane curve \( \alpha \), acting a Lorentz transformation we can write

\[
\alpha(s) = (\alpha_1(s), \alpha_2(s), 0) \quad \text{and} \quad \beta = (0, 0, 1)
\]

without loss of generality. The Gauss map of the cylinder \( M_1^+ \) is

\[
G = e_1 \times e_2 = (\alpha_2'(s), \alpha_1'(s), 0)
\]

as \( e_2 = \alpha'(s) = (\alpha_1'(s), \alpha_2'(s), 0) \). Considering \(-\alpha_1'^2(s) + \alpha_2'^2(s) = -1 \), we may put \( \alpha_1'(s) = \cosh \mu(s) \) and \( \alpha_2'(s) = \sinh \mu(s) \) to determine \( \alpha(s) \), where \( \mu \) is a differentiable function of \( s \). From the equation \( \alpha''(s) = \varepsilon_C k(s)G \) we obtain \( \mu'(s) = k(s) \). For simplicity we take \( \mu(s) = \lambda(s) = \lambda_0 + \int k(s)ds \).

Now we suppose that \( C \) is space-like, i.e., \( \varepsilon_C = 1 \). By using (32), (33) and (42) the base curve \( \alpha(s) \) of the cylinder \( M_1^+ \) is determined uniquely, up to a rigid motion, by

\[
\alpha(s) = \left( d_3 + \frac{1}{c_0} \int C_3(s)ds, \quad d_2 + \frac{1}{c_0} \int C_2(s)ds, \quad 0 \right),
\]

\[
= \left( d_3 \pm \left( -\frac{(k + k_0)}{2c_0 k_0 k^2} \sqrt{R(k)} - \frac{c_0}{2k_0} \ln \left( \frac{k_0 - k + \sqrt{R(k)}}{k} \right) \right),
\]

where \( R(k) = (k - k_0)^2 - c_0^2 k^2 > 0 \). It is seen that the base curve of the cylinder \( M_1^+ \) can be parametrized in terms of the curvature function \( k \), that is, \( \alpha = \alpha(k) \). Therefore we obtain the parametrization (15) for the cylinder \( M_1^+ \) which has pointwise 1-type Gauss map of the second kind for \( f(k) = \frac{k^3}{k_0} \) and \( C = (0, c_0, 0) \).

If \( C \) is time-like, i.e., \( \varepsilon_C = -1 \), then by a similar argument we obtain the base curve of the cylinder \( M_1^- \) as

\[
\alpha(k) = \left( d_2 + \frac{k_0}{2c_0 k^2}, \quad d_3 \pm \left( -\frac{(k + k_0)}{2c_0 k_0 k^2} \sqrt{R(k)} + \frac{c_0}{2k_0} \ln \left( \frac{k_0 - k + \sqrt{R(k)}}{k} \right) \right), \quad 0 \right), \tag{44}
\]
where $R(k) = c_0^2 k^2 + (k - k_0)^2$. So we have the parametrization (16) for the cylinder $M_{1}^{1}$ which has pointwise 1-type Gauss map of the second kind for $f(k) = \frac{k}{k_0}$ and $C = (0, -c_0, 0)$.

Now let the vector $C$ be null. From equation (25) we get $C_{2}(s) = \pm C_{3}(s)$. We will consider the case $C_{2}(s) = C_{3}(s)$. Hence, from (23) we get $C_{2}(s) = e^{\mu(s)}$ and then

$$
\alpha_{1}'(s) = \cosh \mu(s) = \frac{1}{2}(C_{2}(s) + \frac{1}{C_{2}(s)})
$$

and

$$
\alpha_{2}'(s) = \sinh \mu(s) = \frac{1}{2}(C_{2}(s) - \frac{1}{C_{2}(s)}).
$$

Using (28) and (29) we obtain the first two components of $\alpha(s)$ as

$$
\alpha_{i}(s) = \frac{k_0}{4k^2} + \frac{(-1)^{i-1}}{2} \left( \frac{1}{k - k_0} - \frac{1}{k_0} \ln \left| \frac{k}{k - k_0} \right| \right) + d_{i}, \quad i = 1, 2. \tag{45}
$$

Similarly if we take $C_{2}(s) = -C_{3}(s)$, then the first two components of $\alpha(s)$ are

$$
\alpha_{i}(s) = (-1)^{i-1} \frac{k_0}{4k^2} + \frac{1}{2(k - k_0)} - \frac{1}{2k_0} \ln \left| \frac{k}{k - k_0} \right| + d_{i}, \quad i = 1, 2. \tag{46}
$$

Therefore, considering (45) and (46) we obtain the parametrization (17) for cylinder $M_{1}^{1}$ which has pointwise 1-type Gauss map of the second kind for $f(k) = \frac{k^4}{k_0}$ and $C = (-1, 1, 0)$ if $C_{2}(s) = C_{3}(s)$ or $C = (-1, -1, 0)$ if $C_{2}(s) = -C_{3}(s)$.

4. Noncylindrical flat surfaces with pointwise 1-type Gauss map of the second kind

In this section we study noncylindrical flat surfaces, i.e., cones and tangent developable surfaces with pointwise 1-type Gauss map of the second kind in $\mathbb{R}^{3}_{1}$.

**Theorem 4.1** Let $M$ be a noncylindrical flat surface in the Minkowski space $\mathbb{E}^{3}_{1}$. Then, $M$ has pointwise 1-type Gauss map of the second kind if and only if it is an open part of a right circular cone or a hyperbolic cone in $\mathbb{E}^{3}_{1}$.

**Proof.** Suppose that $M$ has pointwise 1-type Gauss map of the second kind. Since $M$ is a regular noncylindrical flat surface in the Minkowski space $\mathbb{E}^{3}_{1}$, then $M$ is an open part of a cone or an open part of a tangent developable surface in $\mathbb{E}^{3}_{1}$. We consider two cases.

**Case 1.** $M$ is an open part of a cone. Then, by an appropriate rigid motion, $M$ can be parametrized locally by

$$
x(s, t) = \alpha_{0} + t\beta(s), \quad t \neq 0,
$$

where $\langle \beta(s), \beta(s) \rangle = \pm 1$, $\langle \beta'(s), \beta'(s) \rangle = \pm 1$, and $\alpha_{0}$ is a constant vector. The coordinate vector fields $x_{s} = t\beta'(s)$ and $x_{t} = \beta(s)$ are orthogonal because of $\langle \beta(s), \beta(s) \rangle = \pm 1$, and the surface $M$ is regular if $t\beta'(s) \times \beta(s) \neq 0$. So we take the orthonormal tangent frame $\{e_{1}, e_{2}\}$ on $M$ as $e_{1} = \frac{\beta'}{||\beta'||}$ and $e_{2} = \frac{\beta}{||\beta||}$ with
\( \epsilon_1 = \langle e_1, e_1 \rangle = \pm 1 \) and \( \epsilon_2 = \langle e_2, e_2 \rangle = \pm 1 \). The Gauss map of \( M \) is given by \( G = e_1 \times e_2 = \beta'(s) \times \beta(s) \) with \( \epsilon_G = \langle G, G \rangle = -\epsilon_1 \epsilon_2 \).

By a straightforward calculation we obtain

\[
\nabla_{e_1} e_1 = -\frac{\epsilon_1 \epsilon_2}{t} e_2 - \frac{\epsilon_G k_g(s)}{t} G,
\nabla_{e_2} e_2 = \frac{1}{t} e_1,
\n\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = 0,
\]

where \( k_g(s) = \langle \beta''(s), \beta(s) \times \beta'(s) \rangle \neq 0 \) which is the geodesic curvature of \( \beta \) in the hyperbolic space \( \mathbb{H}^2(-1) \) or in the de Sitter space \( S^2_1(1) \). All these relations imply that

\[
\omega_{12}(e_1) = -\frac{\epsilon_1}{t}, \quad \omega_{12}(e_2) = 0,
\]

and thus we have the mean curvature \( H = \frac{k_g(s)}{2t} \) and \( \|A_G\|^2 = \frac{k_g^2(s)}{t^2} \).

Now (8)–(10) imply that \( C_1, C_2, \) and \( C_3 \) are functions of \( s \), and equations (5)–(7) become

\[
C_1'(s) + \epsilon_1 \epsilon_2 C_2(s) + \epsilon_G k_g(s) C_3(s) = 0,
\]

\[
C_2'(s) - C_1(s) = 0,
\]

\[
C_3'(s) - \epsilon_1 k_g(s) C_1(s) = 0.
\]

On the other hand, we have from (19), (20), and (21)

\[
\frac{\epsilon_G k_g(s)}{t^2} = f(1 + \epsilon_G C_3),
\]

\[
-\frac{\epsilon_1}{t^2} \frac{dk_g(s)}{ds} = f C_1,
\]

\[
\frac{\epsilon_1 k_g(s)}{t^2} = f C_2.
\]

It follows from (52) that \( C_2 \neq 0 \). Also equations (50) and (52) give

\[
\epsilon_1 k_g(s) C_2(s) - C_3(s) = \epsilon_G
\]

from which by taking derivative with respect to \( s \), we get

\[
\epsilon_1 k_g'(s) C_2(s) + \epsilon_1 k_g(s) C_2'(s) = C_3'(s)
\]

that gives \( \epsilon_1 k_g'(s) C_2(s) = 0 \) in view of (48) and (49). Hence we obtain \( k_g'(s) = 0 \) as \( C_2 \neq 0 \), that is, \( k_g(s) \) is a nonzero constant.

Now we assume that \( \beta'' \) is non-null. By considering (3) for the curve \( \beta \) in the hyperbolic space \( \mathbb{H}^2(-1) \) (resp., in the de Sitter space \( S^2_1(1) \)) we have \( \epsilon_N k^2(s) = k_g^2(s) - 1 \) (resp., \( \epsilon_N k^2(s) = -\epsilon_1 k_g^2(s) + 1 \)), where \( \epsilon_N \) is the sign of the principal normal vector \( N \) of the curve \( \beta \). Note that we take \( \epsilon_G = -1 \) for \( \mathbb{H}^2(-1) \) and
\( \varepsilon_G = 1 \) for \( S^2_1(1) \) while we use formula (3). Thus, the curvature \( k \) of \( \beta \) is also constant, and \( k \neq 0 \) because if the curvature \( k \) were zero, then \( \beta \) would be a line, and \( M \) would be a part of plane which is a cylindrical surface.

Therefore, taking the derivative of \( k_\gamma(s) = \langle \beta'(s), \beta(s) \times \beta'(s) \rangle = \text{const.} \neq 0 \), and using the Frenet equations (2) it can be shown that the torsion of \( \beta \) is zero, that is, \( \beta \) is a plane curve with nonzero constant curvature. A plane curve in \( E^3_1 \) with nonzero constant curvature is a part of a circle or a hyperbola. Thus the curve \( \beta \) is a part of a circle or a hyperbola in \( \mathbb{H}^2(-1) \) or in the de Sitter space \( S^2_1(1) \) such that the plane containing the curve \( \beta \) does not pass through the origin. Therefore the ruled surface \( M \) is an open part of a right circular cone or a hyperbolic cone in \( E^3_1 \).

Moreover equation (51) implies \( C_1 = 0 \), and equations (48) and (49) imply \( C'_2 = 0 \) and \( C'_3 = 0 \), respectively, i.e., \( C_2 \) and \( C_3 \) are constants. Then we obtain from equations (47) and (53)

\[
C_2 = \frac{\varepsilon_1 \varepsilon_2 k_y}{1 - \varepsilon_1 k_y} \quad \text{and} \quad C_3 = \frac{\varepsilon_1 \varepsilon_2}{1 - \varepsilon_1 k_y}.
\]

Also, we get from (52) \( f = \frac{\varepsilon_2(1 - \varepsilon_1 k_y^3)}{t^2} \). Therefore \( M \) has pointwise 1-type Gauss map of the second kind, that is, equation (1) holds for \( f = \frac{\varepsilon_2(1 - \varepsilon_1 k_y^3)}{t^2} \) and for the constant vector \( C = \frac{1}{1 - \varepsilon_1 k_y}(\varepsilon_2 k_y e_2 - G) \).

Now let \( \beta'' \) be null. If \( \beta \) lies in \( \mathbb{H}^2(-1) \), we have \( k_y = 1 \), and also \( \varepsilon_2 = -1, \varepsilon_1 = \varepsilon_G = 1 \). Then equation (51) implies \( C_1 = 0 \), and equations (50) and (52) imply \( k_y C_2 - C_3 = 1 \). Also, from (47) we have \( C_2 = k_y C_3 \). In view of the last two equations we obtain \( (k_y^2 - 1)C_3 = 1 \), which is not valid as \( k_y^2 = 1 \). If \( \beta \) lies in \( S^2_1(1) \) we have \( \varepsilon_2 = 1 \) and \( k_y = \varepsilon_1 \), which holds if \( \varepsilon_1 = 1 \). By a similar argument given above we have \( (1 - k_y^2)C_3 = 1 \), which is not valid as \( k_y^2 = 1 \). As a result, if \( \beta'' \) is null, then the Gauss map of the cone \( M \) is not of pointwise 1-type of the second kind.

**Case 2.** \( M \) is an open part of a tangent developable surface fully lying in \( E^3_1 \). We will show that there is no tangent developable surface in \( E^3_1 \) with pointwise 1-type Gauss map of the second kind. The surface \( M \) is locally parametrized by

\[
x(s, t) = \alpha(s) + t\alpha'(s),
\]

where \( \alpha(s) \) is a unit speed curve with nonzero curvature \( k(s) \). Note that if \( \alpha \) is a null curve or \( \alpha' \) is null, then the tangent surface is degenerate. We assume that the torsion \( \tau(s) \) of \( \alpha(s) \) is nonzero. If \( \tau = 0 \), then the tangent surface is a part of a plane which is a cylindrical.

Let \( T(s), N(s), \) and \( B(s) \) denote the unit tangent vector, principal normal vector and binormal vector of the curve \( \alpha \) with signatures \( \varepsilon_T, \varepsilon_N \) and \( \varepsilon_B = -\varepsilon_T \varepsilon_N \), respectively. The coordinate vector fields of \( M \) are \( x_s = \alpha'(s) + t\alpha''(s) = T(s) + \varepsilon_N tk(s)N(s) \) and \( x_t = \alpha'(s) = T(s) \) which are not orthogonal. The parametrization \( x \) is regular if \( tk(s) \neq 0 \). We take the orthonormal tangent frame \( \{e_1, e_2\} \) on \( M \) as \( e_1 = \frac{\partial}{\partial t} \) and \( e_2 = \frac{\partial}{\partial s}(\frac{\partial}{\partial t} - \frac{\partial}{\partial s}) \) with \( e_1 = (\varepsilon_1, \varepsilon_1) = \pm 1 \) and \( e_2 = (\varepsilon_2, \varepsilon_2) = \pm 1 \). It is seen that \( e_1 = T, e_2 = N, e_1 = \varepsilon_T \) and \( e_2 = \varepsilon_N \). Then the Gauss map of \( M \) is given by \( G = e_1 \times e_2 = T \times N = B \) with \( \varepsilon_G = -\varepsilon_1 \varepsilon_2 \).

By a direct calculation we obtain

\[
\nabla e_1 e_2 = \nabla e_1 e_2 = 0, \quad \nabla e_2 e_1 = \frac{1}{t} e_2, \quad \nabla e_2 e_2 = \frac{\varepsilon_1 \varepsilon_2}{t} e_1 - \frac{\varepsilon_1 \tau(s)}{tk(s)} G.
\]
So we have \( \omega_{21}(e_1) = 0, \omega_{21}(e_2) = -\frac{\tau}{r}, \) \( h_{11} = h_{12} = h_{21} = 0 \) and \( h_{22} = \frac{\tau r(s)}{tk(s)}. \) Therefore the mean curvature is \( H(s,t) = \frac{\tau(s)}{2tk(s)}, \) and \( \|A_G\| = \frac{(\tau(s))^2}{tk(s)}. \)

Now, it follows from equations (5)–(7) that \( C_1, C_2, \) and \( C_3 \) are functions of \( s, \) and thus equations (8)–(10) become

\[
C_1'(s) - \varepsilon_2 k(s) C_2(s) = 0, \tag{55}
\]

\[
C_2'(s) + \varepsilon_1 k(s) C_1(s) + \varepsilon_1 \varepsilon_2 \tau(s) C_3(s) = 0, \tag{56}
\]

\[
C_3'(s) + \varepsilon_2 \tau(s) C_2(s) = 0. \tag{57}
\]

On the other hand, we have from (19), (20), and (21) that

\[
\varepsilon G \tau^2 = f(1 + \varepsilon G C_3), \tag{58}
\]

\[
-\frac{\tau}{t^2 k} = f C_1, \tag{59}
\]

\[
\frac{\varepsilon_2}{t^2 k} \left( \frac{d}{ds} \left( \frac{\tau}{k} \right) + \frac{\tau}{tk} \right) = f C_2. \tag{60}
\]

Equation (59) implies that \( C_1 \neq 0 \) as \( \tau \neq 0. \) Also, from (58) and (59) we obtain

\[
\tau(s) C_1(s) + k(s) C_3(s) = \varepsilon_1 \varepsilon_2 k(s) \tag{61}
\]

from which, by taking the derivative we get

\[
\tau'(s) C_1(s) + \tau(s) C_1'(s) + k'(s) C_3(s) + k(s) C_3'(s) = \varepsilon_1 \varepsilon_2 k'(s). \tag{62}
\]

Using equations (55) and (57), equation (62) turns into

\[
\tau'(s) C_1(s) + k'(s) C_3(s) = \varepsilon_1 \varepsilon_2 k'(s). \tag{63}
\]

If \( \tau'(s) k(s) - k'(s) \tau(s) \neq 0, \) then equations (61) and (63) give \( C_1 = 0 \) and \( C_3 = -\varepsilon G. \) Hence, we have \( \tau = 0 \) from (58) or (59), which is a contradiction.

Now suppose that \( \tau'(s) k(s) - k'(s) \tau(s) = 0, \) which means that \( \tau(s)/k(s) = r_0 \) is a nonzero constant. In this case, by (59) and (60) we get

\[
tk(s) C_2(s) + \varepsilon_2 C_1(s) = 0
\]

which implies that \( C_1 = C_2 = 0, \) that is, \( \tau = 0 \) by (59). This is a contradiction. Therefore the torsion \( \tau \) is zero, and there is no tangent developable surface fully lying in \( \mathbb{E}_1^3 \) with pointwise 1-type Gauss map of the second kind.

The converse of the proof follows from a straightforward calculation. □

We then have the following.
Corollary 4.2 Right circular cones and hyperbolic cones in Minkowski space $E^3_1$ are the only cones in $E^3_1$ with pointwise 1-type Gauss map of the second kind.

Corollary 4.3 There is no tangent developable surface fully lying in Minkowski space $E^3_1$ with pointwise 1-type Gauss map of the second kind.

Combining Theorem 3.3 and Theorem 4.1 we have

Theorem 4.4 Let $M$ be a flat ruled surface in the Minkowski space $E^3_1$. Then, $M$ has pointwise 1-type Gauss map of the second kind if and only if it is a part of a plane, cylinders given by (11)–(17), a right circular cone or a hyperbolic cone.

References


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