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Essential normality for certain weighted composition operators on the Hardy space H^2

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Abstract

We characterize the essentially normal weighted composition operators $C_{\psi,\varphi}$ on the Hardy space H^2, whenever φ is a linear-fractional transformation and $\psi \in A(\mathbb{D})$. Also we investigate the essential normality problem for some other weighted composition operators on H^2.

Key Words: Hardy spaces, essentially normal, weighted composition operator, linear-fractional transformation

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$. The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$.

For an analytic function f on the unit disk and $0 < r < 1$, we define the dilate function f_r by $f_r(e^{i\theta}) = f(re^{i\theta})$. It is easy to see that the functions f_r are continuous on $\partial\mathbb{D}$ for each r, hence they are in $L^p(\partial\mathbb{D}, d\theta/2\pi)$, where $d\theta/2\pi$ is the normalized arc length measure on the unit circle.

For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D}) = H^p$ is the set of all analytic functions on the unit disk for which

$$\|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Also we recall that $H^\infty(\mathbb{D}) = H^\infty$ is the space of all bounded analytic functions defined on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. We know that for $p \geq 1$, H^p is a Banach space (see, e.g., [15, p. 37]).

It is well known that the supremum in the above definition of H^p spaces is actually a limit, that is,

$$\|f\|_p^p = \lim_{r\to 1} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Also we know that if f is in H^p for some $p > 0$, then the radial limit

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}),$$

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exists for almost all θ and the mapping $f \to f^*$ is an isometry of H^p to a closed subspace of $L^p(\partial\mathbb{D}, d\theta/2\pi)$. Therefore

$$\|f\|_p^p = \int_0^{2\pi} |f^*(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

We will also write $f(e^{i\theta})$ for $f^*(e^{i\theta})$. If $p = 2$ and $\hat{f}(n)$ is the nth coefficient of f in its Maclaurin series, then we have another representation for the norm of f on H^2 as follows:

$$\|f\|_2^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$

The formula above defines a norm that turns H^2 into a Hilbert space whose inner product is given by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} \frac{d\theta}{2\pi}$$

for each $f, g \in H^2$. For more information about the Hardy spaces see, for example, [12] and [15].

For each $\psi \in L^\infty(\partial\mathbb{D})$, we define the Toeplitz operator T_ψ on H^2 by $T_\psi(f) = P(\psi f)$, where P denotes the orthogonal projection of $L^2(\partial\mathbb{D})$ onto H^2:

$$P\left(\sum_{-\infty}^{\infty} \hat{f}(n)e^{in\theta}\right) = \sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta}.$$

Since an orthogonal projection has norm 1, clearly T_ψ is bounded.

For any analytic self-map φ of $\mathbb{D}$, the composition operator C_φ on H^2 is defined by the rule $C_\varphi(f) = f \circ \varphi$. It is well known (see, e.g., [15, p. 29] or [28, Theorem 1]) that the composition operators are bounded on each of the Hardy spaces H^p $(0 < p < \infty)$. One of the first papers in this research area is [28], while H. J. Schwartz in [30] begun the research on compact composition operators on H^p. J. H. Shapiro and P. D. Taylor in [32] have studied the role of angular derivative for compactness of C_φ in H^p. For some other classical results see [12] and [31]. If ψ is a bounded analytic function on $\mathbb{D}$ and φ is an analytic map from $\mathbb{D}$ into itself, the weighted composition operator $C_{\psi,\varphi}$ is defined by $C_{\psi,\varphi}(f)(z) = \psi(z)f(\varphi(z))$. The map φ is called the composition map and ψ is called the weight. If ψ is a bounded analytic function on $\mathbb{D}$, then the weighted composition operator can be rewritten as $C_{\psi,\varphi} = T_\psi C_\varphi$. In this case $C_{\psi,\varphi}$ is bounded, but in general every weighted composition operator $C_{\psi,\varphi}$ on H^2 is not bounded. If $C_{\psi,\varphi}$ is bounded, then $C_{\psi,\varphi}(1) = \psi$ belongs to H^2. These operators come up naturally. In 1964, Forelli [16] showed that every isometry on H^p for $1 < p < \infty$ and $p \neq 2$ is a weighted composition operator. A similar result holds for the isometries on the Bergman space L_a^p; see [19]. Recently there has been a great interest in studying weighted composition operators in the unit disk, polydisk, or the unit ball; see, for example, [7, 17, 18, 22, 23, 24, 25, 34].

A mapping of the form

$$\varphi(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{1}$$

is called a linear-fractional transformation. We denote the set of those linear-fractional transformations that take the open unit disk $\mathbb{D}$ into itself by $\mathrm{LFT}(\mathbb{D})$.

For bounded operators A and B on a Hilbert space, we use the notation

$$[A, B] := AB - BA$$

for the commutator of A and B. Recall that an operator A is called normal if $[A, A^*] = 0$ and essentially normal if $[A, A^*]$ is compact. In [35] Nina Zorboska characterized the essentially normal composition operators on the Hardy space H^2 induced by conformal automorphisms of the unit disk. In addition, Zorboska showed that the composition operators on H^2 induced by linear-fractional transformations fixing no point on the unit circle are not nontrivially essentially normal. P. S. Bourdon, D. Levi, S. K. Narayan and J. H. Shapiro in [4] showed that a composition operator induced on H^2 by a linear-fractional self-map of the unit disk is nontrivially essentially normal if and only if it is induced by a parabolic non-automorphism self-map of the unit disk. The essentially normal composition operators on other spaces have been investigated by some authors (see, e.g., [5] and [26]).

Suppose that φ is an analytic self-map of $\mathbb{D}$ and α is a complex number of modulus 1. Since $\mathrm{Re}\left(\frac{\alpha+\varphi}{\alpha-\varphi}\right)$ is a positive harmonic function on $\mathbb{D}$, there exists a finite positive Borel measure μ_α on $\partial\mathbb{D}$ such that

$$\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \mathrm{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right) = \int_{\partial\mathbb{D}} P_z d\mu_\alpha$$

for each $z \in \mathbb{D}$, where

$$P_z(e^{i\theta}) = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

is the Poisson kernel at z. The measures μ_α are called the Clark measures of φ. Let μ_α^s be the singular parts of μ_α with respect to Lebesgue measure. The singular part μ_α^s is carried by $\varphi^{-1}(\{\alpha\})$, the set of those ζ in $\partial\mathbb{D}$ where $\varphi(\zeta)$ exists and equals α. We write $E(\varphi)$ for the closure in $\partial\mathbb{D}$ of the union of the closed supports of μ_α^s as α ranges over the unit circle. In particular, if φ is a linear-fractional non-automorphism such that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then $\mu_\alpha^s = 0$ when $\alpha \neq \eta$ and $\mu_\eta^s = |\varphi'(\zeta)|^{-1}\delta_\zeta$, where δ_ζ is the unit point mass measure at ζ. The measures μ_α were introduced as an operator-theoretic tool by D. N. Clark [6] and have been further analyzed by A. B. Aleksandrov [1], A. G. Poltoratski [27] and D. E. Sarason [29].

In this paper, we use the results of Bourdon et al. [4] and T. L. Kriete, B. D. MacCluer and J. L. Moorhouse [20] in order to investigate the essential normality problem for certain weighted composition operators $C_{\psi,\varphi}$, whenever φ is a linear-fractional transformation and $\psi \in H^\infty$.

2. Linear-fractional non-automorphism

In this section, we investigate the essential normality problem for certain weighted composition operators $C_{\psi,\varphi}$ on the Hardy space H^2, whenever φ is a linear-fractional non-automorphic self-map of $\mathbb{D}$ and $\psi \in H^\infty$. First, we state some useful results of [4, 6.3] and [33, Proposition 9.1.4] that we use frequently in this paper.

Remark 2.1 *Suppose that φ is an analytic self-map of $\mathbb{D}$ and $\psi \in H^\infty$. Let f and g be two bounded measurable functions on $\partial\mathbb{D}$ and λ be a complex number. Then* *(a)* $C_\varphi T_\psi = T_{\psi\circ\varphi} C_\varphi$,

(b) $T_g T_\psi = T_{g\psi}$,

(c) $T_{f+\lambda g} = T_f + \lambda T_g$, *and*

(d) $T_f^* = T_{\bar{f}}$.

In [11] Carl Cowen showed that if $\varphi \in \mathrm{LFT}(\mathbb{D})$ is given by Equation (1), then

$$C_\varphi^* = T_g C_{\sigma_\varphi} T_h^*, \tag{2}$$

where $\sigma_\varphi(z) := (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$ is a self-map of $\mathbb{D}$, $g(z) := (-\bar{b}z + \bar{d})^{-1}$, $h(z) := cz + d$ and $g, h \in H^\infty$. If $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma_\varphi(\eta) = \zeta$; see [11]. The map σ_φ is called the Krein adjoint of φ; we will write σ for σ_φ except when confusion could arise. For further details see, for example, [4]. From now on, unless otherwise stated, we assume that σ, h and g are given as above.
Now let $\psi \in H^\infty$. Then by Equation (2) and Remark 2.1, we have

$$C_{\psi,\varphi}^* = (T_\psi C_\varphi)^* = C_\varphi^* T_\psi^* = T_g C_\sigma T_{h\psi}^*.$$

Hence

$$C_{\psi,\varphi} C_{\psi,\varphi}^* = T_\psi C_\varphi T_g C_\sigma T_{h\psi}^* = T_{g\circ\varphi} C_{\psi,\varphi} C_\sigma T_{h\psi}^*; \tag{3}$$

and on the other hand, we have

$$C_{\psi,\varphi}^* C_{\psi,\varphi} = T_g C_\sigma T_{h\psi}^* C_{\psi,\varphi}. \tag{4}$$

The set of all bounded operators and the set of all compact operators from H^2 into itself are denoted by $B(H^2)$ and $B_0(H^2)$, respectively. Also we will use the notation $A \equiv B \pmod{B_0(H^2)}$ to indicate that the difference of two bounded operators A and B belongs to $B_0(H^2)$. In [20] Kriete et al. showed that if $\varphi \in \mathrm{LFT}(\mathbb{D})$ is not an automorphism which satisfies $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then

$$C_\varphi^* \equiv |\varphi'(\zeta)|^{-1} C_\sigma \pmod{B_0(H^2)}. \tag{5}$$

Also we will use the following result frequently, which is proved in [20].

Theorem 2.2 *[20] Let φ be an analytic self-map of $\mathbb{D}$ such that $|\varphi(e^{i\theta})| < 1$ a.e. with respect to Lebesgue measure on $\partial\mathbb{D}$, and suppose that γ is a bounded measurable function on $\partial\mathbb{D}$ which is continuous at each point of $E(\varphi)$. Then $T_\gamma C_\varphi : H^2 \to H^2$ is compact if and only if $\gamma \equiv 0$ on $E(\varphi)$.*

Proposition 2.3 *Suppose that $\varphi \in LFT(\mathbb{D})$ is not an automorphism and that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. Let $\psi \in A(\mathbb{D})$. Then $[C_{\psi,\varphi}^*, C_{\psi,\varphi}] \equiv |\psi(\zeta)|^2 |\varphi'(\zeta)|^{-1} [C_\sigma, C_\varphi] \ (mod\ B_0(H^2))$.*

Proof. It is easy to see that $E(\sigma\circ\varphi) = \{\zeta\}$, so by Theorem 2.2, [14, Proposition 7.22], Remark 2.1 and Equation (5), we have

$$\begin{aligned}
C_{\psi,\varphi}^* C_{\psi,\varphi} - C_{\psi,\varphi} C_{\psi,\varphi}^* &\equiv |\varphi'(\zeta)|^{-1} (C_\sigma T_\psi^* T_\psi C_\varphi - T_\psi C_\varphi C_\sigma T_\psi^*) \pmod{B_0(H^2)} \\
&\equiv |\varphi'(\zeta)|^{-1} (C_\sigma T_\psi T_\psi^* C_\varphi - T_\psi C_{\sigma\circ\varphi} T_\psi^*) \pmod{B_0(H^2)} \\
&\equiv |\varphi'(\zeta)|^{-1} (\overline{\psi(\zeta)} C_\sigma T_\psi C_\varphi - T_\psi C_{\sigma\circ\varphi} T_\psi^*) \pmod{B_0(H^2)} \\
&\equiv |\varphi'(\zeta)|^{-1} (|\psi(\zeta)|^2 C_{\varphi\circ\sigma} - \psi(\zeta) C_{\sigma\circ\varphi} T_\psi^*) \pmod{B_0(H^2)}.
\end{aligned}$$

Using $\sim$ for the Krein adjoint, by [13, Lemma 12], we see that $\widetilde{\sigma\circ\varphi}=\widetilde{\varphi}\circ\widetilde{\sigma}=\sigma\circ\varphi$. Then by Equation (5), we have

$$\begin{aligned}T_\psi C^*_{\sigma\circ\varphi} &\equiv s_1T_\psi C_{\widetilde{\sigma\circ\varphi}} \pmod{B_0(H^2)}\\ &\equiv s_1T_\psi C_{\sigma\circ\varphi} \pmod{B_0(H^2)}\\ &\equiv s_1\psi(\zeta)C_{\sigma\circ\varphi} \pmod{B_0(H^2)},\end{aligned}$$

where $s_1=|(\sigma\circ\varphi)'(\zeta)|^{-1}$. Hence by [20, Proposition 3.6], $s_1=1$. Therefore $C_{\sigma\circ\varphi}T^*_\psi\equiv\overline{\psi(\zeta)}C^*_{\sigma\circ\varphi} \pmod{B_0(H^2)}$, so by this fact and Equation (5), we have

$$\begin{aligned}C_{\sigma\circ\varphi}T^*_\psi &\equiv \overline{\psi(\zeta)}C_{\widetilde{\sigma\circ\varphi}} \pmod{B_0(H^2)}\\ &\equiv \overline{\psi(\zeta)}C_{\sigma\circ\varphi} \pmod{B_0(H^2)}.\end{aligned}$$

Hence

$$[C^*_{\psi,\varphi},C_{\psi,\varphi}]\equiv|\psi(\zeta)|^2|\varphi'(\zeta)|^{-1}[C_\sigma,C_\varphi] \pmod{B_0(H^2)}.$$

□

A map $\varphi\in\mathrm{LFT}(\mathbb{D})$ whose fixed point set, relative to the Riemann sphere, consists of a single point ζ in $\partial\mathbb{D}$ is termed parabolic. The linear-fractional transformation $\tau(z):=(1+\overline{\zeta}z)/(1-\overline{\zeta}z)$ takes the unit disk onto the right half-plane Π and takes ζ to ∞. Set $\phi:=\tau\circ\varphi\circ\tau^{-1}$. Therefore ϕ is a linear-fractional self-map of Π which fixes only ∞, so it must have the form $\phi(z)=z+t$ for some complex number t, where $\mathrm{Re}t\geq 0$. Let us call t the translation number of either φ or ϕ. When $\mathrm{Re}t=0$ we have a parabolic automorphism; otherwise the map is not automorphism. Also in [31, p. 3] J. H. Shapiro showed that among the linear-fractional transformations fixing $\zeta\in\partial\mathbb{D}$, the parabolic ones are characterized by $\varphi'(\zeta)=1$. For further details see [4] and [31].

Theorem 2.4 *Let $\varphi\in LFT(\mathbb{D})$ be a parabolic non-automorphism and $\psi\in A(\mathbb{D})$. Then $C_{\psi,\varphi}$ is essentially normal.*

Proof. Suppose that the map φ fixes $\zeta\in\partial\mathbb{D}$, so $\varphi'(\zeta)=1$. Therefore by Proposition 2.3, $C_{\psi,\varphi}$ be essentially normal if and only if $|\psi(\zeta)|^2(C_{\varphi\circ\sigma}-C_{\sigma\circ\varphi})\in B_0(H^2)$. By the proof of Theorem 4.1 in [4], $\sigma\circ\varphi=\varphi\circ\sigma$. So the result follows. □

Theorem 2.5 *Let $\varphi\in LFT(\mathbb{D})$ be a non-automorphism such that $\varphi(\zeta)=\eta$ for some $\zeta,\eta\in\partial\mathbb{D}$ and $\psi\in A(\mathbb{D})$. Also suppose that φ is not a parabolic non-automorphism. Then $C_{\psi,\varphi}$ is essentially normal if and only if $\psi(\zeta)=0$.*

Proof. If $\psi(\zeta)=0$, then by Theorem 2.2, $C_{\psi,\varphi}$ is essentially normal. Conversely, let $C_{\psi,\varphi}$ be essentially normal and $\psi(\zeta)\neq 0$. Then by Proposition 2.3, we have

$$[C^*_{\psi,\varphi},C_{\psi,\varphi}]\equiv|\psi(\zeta)|^2|\varphi'(\zeta)|^{-1}(C_{\varphi\circ\sigma}-C_{\sigma\circ\varphi}) \pmod{B_0(H^2)}.$$

Therefore $C_{\varphi\circ\sigma} - C_{\sigma\circ\varphi} \in B_0(H^2)$. Since a difference of non-compact linear-fractional composition operators is compact only if it is zero (see [3] and [21]), we have $\varphi \circ \sigma = \sigma \circ \varphi$.

Now if $\zeta = \eta$, then by [20, p. 139], φ is a parabolic non-automorphism which is a contradiction, and if $\zeta \neq \eta$, then $\varphi \circ \sigma$ has two fixed points $\zeta, \eta \in \partial\mathbb{D}$. Thus by [20, Proposition 3.4], it is a contradiction. $\square$

A map $\varphi \in \mathrm{LFT}(\mathbb{D})$ is called hyperbolic if it has a fixed point in $\overline{\mathbb{D}}$ and a fixed point outside $\mathbb{D}$. Also φ is an automorphism of $\mathbb{D}$ if and only if these two fixed points lie on $\partial\mathbb{D}$. The case in which φ is a hyperbolic non-automorphism with a fixed point ζ on $\partial\mathbb{D}$ and $\psi \in A(\mathbb{D})$ is an example of Theorem 2.5.

We know that if $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$, then C_φ is compact (see, e.g., [31]) and so $C_{\psi,\varphi}$ is compact too. Since every compact operator is essentially normal, $C_{\psi,\varphi}$ is essentially normal. We investigated the essential normality problem for weighed composition operators $C_{\psi,\varphi}$, whenever φ is a parabolic or hyperbolic non-automorphism. In the rest of this section, we assume that φ is a linear-fractional self-map of $\mathbb{D}$ satisfying the following:

(i) φ is not an automorphism.

(ii) $\varphi(\zeta) = \eta$ for some $\zeta \neq \eta \in \partial\mathbb{D}$.

Conditions (i) and (ii) imply that $\|\varphi \circ \varphi\|_\infty < 1$, therefore C_φ^2 is compact. By Remark 2.1,

$$C_{\psi,\varphi}C_{\psi,\varphi} = T_\psi C_\varphi T_\psi C_\varphi = T_\psi T_{\psi\circ\varphi} C_\varphi^2,$$

hence $C_{\psi,\varphi}^2 \in B_0(H^2)$.

Similar to the notations introduced by Kriete et al. in [20], let x, x^* and e be the cosets of $C_{\psi,\varphi}$, $C_{\psi,\varphi}^*$ and I in the Calkin algebra $B(H^2)/B_0(H^2)$, respectively, so $(xx^*)(x^*x) = (x^*x)(xx^*) = 0$ and therefore the C^*-algebra generated by xx^*, x^*x and e is abelian. In the proof of Proposition 2.6 and Theorem 2.7, we apply these results. Also let $\mathcal{A}$ be an abelian Banach algebra. We denote the collection of all nonzero homomorphisms of $\mathcal{A} \to \mathbb{C}$ by Σ; for further information see, for example, [9].

In [10] C. C. Cowen described the spectrum of a composition operator whose symbol is a parabolic non-automorphism. In [20] Kriete et al. showed that $\sigma \circ \varphi$ is a parabolic non-automorphism and obtained $\sigma(C_{\sigma\circ\varphi}) = \sigma_e(C_{\sigma\circ\varphi}) = [0,1]$. We know that if a and b are two elements of a Banach algebra with identity, then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$; see [9, p. 199]. We use this fact in the following proposition.

Proposition 2.6 *Suppose that $\psi \in H^\infty$ and ψ is continuous at ζ. Then $\sigma_e(C_{\psi,\varphi}C_{\psi,\varphi}^*) = \sigma_e(C_{\psi,\varphi}^*C_{\psi,\varphi}) = [0, |\psi(\zeta)|^2|\varphi'(\zeta)|^{-1}]$.*

Proof. By Equation (3), we have

$$\sigma_e(C_{\psi,\varphi}C_{\psi,\varphi}^*) = \sigma_e(T_{g\circ\varphi}C_{\psi,\varphi}C_\sigma T_{h\psi}^*).$$

Since $E(\sigma \circ \varphi) = \{\zeta\}$, by Remark 2.1, Theorem 2.2 and [20, Proposition 3.5], we have

$$\begin{aligned}\sigma_e(T_{g\circ\varphi}C_{\psi,\varphi}C_\sigma T_{h\psi}^*) \cup \{0\} &= \sigma_e(T_{\overline{h\psi}}T_{g\circ\varphi}C_{\psi,\varphi}C_\sigma) \cup \{0\} \\ &= \sigma_e(T_{|\psi|^2\overline{h}g\circ\varphi}C_{\sigma\circ\varphi}) \cup \{0\} \\ &= |\psi(\zeta)|^2\overline{h(\zeta)}g\circ\varphi(\zeta)\sigma_e(C_{\sigma\circ\varphi}) \cup \{0\} \\ &= |\psi(\zeta)|^2\overline{h(\zeta)}g(\eta)[0,1].\end{aligned}$$

We see after some computation that $\overline{h(\zeta)}g(\eta)$ is equal to the number s as in [20, Theorem 3.1], so by [20, Proposition 3.6],

$$\sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi}) \cup \{0\} = [0, |\psi(\zeta)|^2|\varphi^{'}(\zeta)|^{-1}].$$

On the other hand, by Remark 2.1, Theorem 2.2 and Equation (5), we have

$$\begin{aligned}\sigma_e(C^*_{\psi,\varphi}C_{\psi,\varphi}) \cup \{0\} &= \sigma_e(|\varphi^{'}(\zeta)|^{-1}C_\sigma T^*_\psi C_{\psi,\varphi}) \cup \{0\} \\ &= |\varphi^{'}(\zeta)|^{-1}\sigma_e(C_\sigma T_{|\psi|^2}C_\varphi) \cup \{0\} \\ &= |\varphi^{'}(\zeta)|^{-1}\sigma_e(T_{|\psi|^2}C_{\sigma\circ\varphi}) \cup \{0\} \\ &= |\psi(\zeta)|^2|\varphi^{'}(\zeta)|^{-1}\sigma_e(C_{\sigma\circ\varphi}) \cup \{0\}.\end{aligned}$$

Again applying [20, Proposition 3.5], we have

$$\sigma_e(C^*_{\psi,\varphi}C_{\psi,\varphi}) \cup \{0\} = [0, |\psi(\zeta)|^2|\varphi^{'}(\zeta)|^{-1}].$$

Since $\sigma_e(C^*_{\psi,\varphi}C_{\psi,\varphi})$ and $\sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi})$ are compact subsets of $\mathbb{C}$, the result follows. □

Now we apply Proposition 2.6 to obtain a necessary and sufficient condition for $C_{\psi,\varphi}$ to be an essentially normal weighted composition operator. In the following theorem, we use the techniques used in [20, Proposition 4.2].

Theorem 2.7 *Suppose that $\psi \in H^\infty$ and ψ is continuous at ζ. Then $C_{\psi,\varphi}$ is essentially normal if and only if $\psi(\zeta) = 0$.*

Proof. If $\psi(\zeta) = 0$, then Theorem 2.2 implies that $C_{\psi,\varphi}$ is compact. Therefore $C_{\psi,\varphi}$ is essentially normal. Conversely, let $C_{\psi,\varphi}$ be essentially normal. Since $(xx^*)(x^*x) = (x^*x)(xx^*) = 0$, $m(x^*x) = 0$ or $m(xx^*) = 0$ for each $m \in \Sigma$. Therefore by [9, Theorem 8.6, p. 219] and Proposition 2.6, we have $\sigma_e(C^*_{\psi,\varphi}C_{\psi,\varphi} - C_{\psi,\varphi}C^*_{\psi,\varphi}) = [-|\psi(\zeta)|^2|\varphi^{'}(\zeta)|^{-1}, |\psi(\zeta)|^2|\varphi^{'}(\zeta)|^{-1}]$. Hence the proof is complete. □

Since the conditions on ψ in Theorem 2.7 are weaker than the condition on ψ in Theorem 2.5, there are some examples of ψ such that they satisfy the hypotheses of Theorem 2.7, but they do not satisfy the hypothesis of Theorem 2.5.

Theorem 2.8 *Let $\varphi(z) = \frac{i}{2}(1-z)$.*

(a) Suppose that $\psi(z) = e^{\frac{1}{z-1}}$. It is easy to see that $\psi \in H^\infty$ and ψ is continuous at -1. Then by Theorem 2.7, $C_{\psi,\varphi}$ is not essentially normal.

(b) Suppose that $\psi(z) = (1+z)e^{\frac{1}{z-1}}$. Then again by Theorem 2.7, $C_{\psi,\varphi}$ is essentially normal.

3. Automorphism

It is well known that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the disk onto itself, are just the functions

$$\varphi(z) = \lambda \frac{a - z}{1 - \bar{a}z}, \tag{6}$$

where $|\lambda| = 1$ and $|a| < 1$ (see, e.g., [8]). We denote the class of automorphisms of $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$. Also we know that φ is an automorphism if and only if σ is in this case $\sigma = \varphi^{-1}$; see [4]. In this section, we investigate the essential normality problem for weighted composition operators $C_{\psi,\varphi}$ on the Hardy space H^2, whenever $\varphi \in \mathrm{Aut}(\mathbb{D})$ and $\psi \in A(\mathbb{D})$. Also we assume that φ is given by Equation (6) and $w(z) = (1 - \bar{a}z)\psi(z)$, where $\psi \in A(\mathbb{D})$.

Proposition 3.1 *Suppose that $\varphi \in Aut(\mathbb{D})$ and $\psi \in A(\mathbb{D})$. Then $C_{\psi,\varphi}$ is essentially normal if and only if $[T_{|w|^2}, C_\varphi] \in B_0(H^2)$.*

Proof. Since $\sigma = \varphi^{-1}$, Equations (3) and (4) imply that

$$[C^*_{\psi,\varphi}, C_{\psi,\varphi}] = T_g C_{\varphi^{-1}} T^*_{h\psi} T_\psi C_\varphi - T_{g\circ\varphi} T_\psi T^*_{h\psi}.$$

Thus by [14, Proposition 7.22] and Remark 2.1, we have

$$[C^*_{\psi,\varphi}, C_{\psi,\varphi}] \equiv C_{\varphi^{-1}} T_{\overline{h}|\psi|^2 g\circ\varphi} C_\varphi - T_{\overline{h}|\psi|^2 g\circ\varphi} \ (\mathrm{mod}\ B_0(H^2)).$$

Since $C_\varphi C_{\varphi^{-1}} = I$, we see that $[C^*_{\psi,\varphi}, C_{\psi,\varphi}] \in B_0(H^2)$ if and only if $[T_{\bar{h}|\psi|^2 g\circ\varphi}, C_\varphi] \in B_0(H^2)$. Also after some computation, we conclude that

$$\bar{h}(z) g \circ \varphi(z) = \frac{|1 - \bar{a}z|^2}{1 - |a|^2},$$

which completes the proof. □

Example 3.2 *Let $\varphi \in Aut(\mathbb{D})$ and for some $n \in \mathbb{N} \cup \{0\}$, $\psi(z) = z^n/(1 - \bar{a}z)$. Then $w(z) = z^n$ and $T_{|w|^2} = I$, so Proposition 3.1 implies that $C_{\psi,\varphi}$ is essentially normal.*

Theorem 3.3 *Let $\varphi \in Aut(\mathbb{D})$ and $\psi \in A(\mathbb{D})$. Then $C_{\psi,\varphi}$ is essentially normal if and only if $|w| = |w \circ \varphi|$ on $\partial\mathbb{D}$.*

Proof. By [14, Proposition 7.22], Equation (2) and Remark 2.1, we have

$$\begin{aligned}(T_{|w|^2}C_\varphi - C_\varphi T_{|w|^2})C_\varphi^{-1} &\equiv T_{|w|^2} - C_\varphi T_w T_{\bar{w}} C_{\varphi^{-1}} \pmod{B_0(H^2)}\\ &\equiv T_{|w|^2} - T_{w\circ\varphi} C_\varphi T_{\bar{w}} C_{\varphi^{-1}} \pmod{B_0(H^2)}\\ &\equiv T_{|w|^2} - T_{w\circ\varphi} C_\varphi T_{\bar{w}} T_{\frac{1}{g}} C_\varphi^* T_{\frac{1}{h}}^* \pmod{B_0(H^2)}\\ &\equiv T_{|w|^2} - T_{w\circ\varphi} C_\varphi T_{\frac{1}{g}} T_{\bar{w}} C_\varphi^* T_{\frac{1}{h}}^* \pmod{B_0(H^2)}\\ &\equiv T_{|w|^2} - T_{\frac{w\circ\varphi}{g\circ\varphi}} C_\varphi T_{\bar{w}} C_\varphi^* T_{\frac{1}{h}}^* \pmod{B_0(H^2)}\\ &\equiv T_{|w|^2} - T_{\frac{w\circ\varphi}{g\circ\varphi}} C_\varphi C_\varphi^* T_{w\circ\varphi}^* T_{\frac{1}{h}}^* \pmod{B_0(H^2)}\\ &\equiv T_{|w|^2} - T_{\frac{w\circ\varphi}{g\circ\varphi}} T_{g\circ\varphi} T_h^* T_{\frac{w\circ\varphi}{h}}^* \pmod{B_0(H^2)}\\ &\equiv T_{|w|^2} - T_{|w\circ\varphi|^2} \pmod{B_0(H^2)}.\end{aligned}$$

Now the conclusion follows from Proposition 3.1 and [33, Proposition 9.1.3]. □

In 1969, H. J. Schwartz [30] showed that a composition operator on H^2 is normal if and only if it is induced by a dilation $z \to az$, where $|a| \le 1$. In the following corollary, we assume that C_φ is normal and we obtain a necessary and sufficient condition for $C_{\psi,\varphi}$ to is essentially normal.

Corollary 3.4 *Let $\varphi \in Aut(\mathbb{D})$, $\psi \in A(\mathbb{D})$ and suppose that C_φ is normal. Then $C_{\psi,\varphi}$ is essentially normal if and only if $|\psi(\varphi(z))| = |\psi(z)|$ for each $z \in \partial\mathbb{D}$.*

Corollary 3.5 *Let $\varphi \in Aut(\mathbb{D})$, $\psi \in A(\mathbb{D})$ and for each $z \in \bar{\mathbb{D}}$, $\psi(z) \neq 0$. Then $C_{\psi,\varphi}$ is essentially normal if and only if w is an eigenvector for the operator C_φ and the modulus of the corresponding C_φ-eigenvalue for w is 1.*

Proof. Suppose that $C_{\psi,\varphi}$ is essentially normal. Since $|\frac{(1-\bar{a}z)\psi}{(1-\bar{a}\varphi)\psi\circ\varphi}|$ and $|\frac{(1-\bar{a}\varphi)\psi\circ\varphi}{(1-\bar{a}z)\psi}|$ are subharmonic on $\mathbb{D}$, by [2, Corollary A.1.4] and Theorem 3.3, $|\frac{(1-\bar{a}z)\psi}{(1-\bar{a}\varphi)\psi\circ\varphi}|$ is constant on $\mathbb{D}$. Therefore $|\frac{(1-\bar{a}z)\psi}{(1-\bar{a}\varphi)\psi\circ\varphi}| = 1$ on $\bar{\mathbb{D}}$. Hence by the Maximum Modulus Theorem, there is a $\beta \in \mathbb{C}$ such that $|\beta| = 1$ and

$$((1-\bar{a}z)\psi)\circ\varphi = \beta(1-\bar{a}z)\psi.$$

Hence $(1-\bar{a}z)\psi$ is an eigenvector for C_φ. The other direction follows from Theorem 3.3. □

Example 3.6 *Let $\varphi_a(z) = (a-z)/(1-\overline{a}z)$, where $a \neq 0$. Suppose that $\psi(z) = (\varphi_a(z) + z + b)/(1-\overline{a}z)$, where $|b| > 2$. We see that $w(z) = \varphi_a(z) + z + b$ and $C_{\varphi_a}(w) = w$. Hence by Corollary 3.5, C_{ψ,φ_a} is essentially normal.*

In the rest of this section, we investigate the essential normality problem for weighted composition operators $C_{\psi,\varphi}$, whenever $|\psi| = t$ almost everywhere on $\partial\mathbb{D}$ for a positive constant t.

Lemma 3.7 *Let $\varphi \in Aut(\mathbb{D})$ and $\psi \in H^\infty$. Also $|\psi| = t$ almost everywhere on $\partial\mathbb{D}$, where t is a positive constant. Then $\{\frac{\psi}{t}z^n : n \in \mathbb{N}\cup\{0\}\}$ is an orthonormal set in H^2.*

Proof. We have

$$\left\|\frac{\psi}{t}z^n\right\|_2^2=\int_0^{2\pi}\left|\frac{\psi(e^{i\theta})}{t}\right|^2|e^{in\theta}|^2\frac{d\theta}{2\pi}=1.$$

Also, we have

$$\begin{aligned}\langle\frac{\psi}{t}z^n,\frac{\psi}{t}z^m\rangle &= \frac{1}{t^2}\langle\psi z^n,\psi z^m\rangle\\ &= \frac{1}{t^2}\int_0^{2\pi}|\psi(e^{i\theta})|^2e^{in\theta}e^{-im\theta}\frac{d\theta}{2\pi}\\ &= 0,\end{aligned}$$

where $n\neq m\in\mathbb{N}$, so the result follows. □

The idea behind Proposition 3.8 is similar to one found in [26, Proposition 2].

Proposition 3.8 *Suppose that φ is given by Equation (6) such that $a\neq 0$. Then $\|C_\varphi^*z^n\|_2=\|C_\varphi^*z^m\|_2>1$ for each $n,m\in\mathbb{N}$.*

Proof. Let $\varphi_a(z)=(a-z)/(1-\bar{a}z)$. Then there is a constant λ of modulus 1 such that $C_\varphi=C_{\varphi_a}C_{\lambda z}$. Since $C_{\lambda z}$ is a surjective isometry, we have

$$C_\varphi^*=C_{\lambda z}^*C_{\varphi_a}^*=C_{\lambda z}^{-1}C_{\varphi_a}^*=C_{\frac{1}{\lambda}z}C_{\varphi_a}^*.$$

Therefore it is enough to prove the result for φ_a. We know that $C_{\varphi_a}^*=T_gC_{\sigma_{\varphi_a}}T_h^*$, where $\sigma_{\varphi_a}=\varphi_a$, $g(z)=(1-\overline{a}z)^{-1}$ and $h(z)=1-\overline{a}z$. Hence by Equation (2), we obtain

$$\begin{aligned}C_{\varphi_a}^*(z^n) &= T_gC_{\sigma_{\varphi_a}}T_h^*(z^n)\\ &= T_gC_{\varphi_a}T_{1-a\bar{z}}(z^n)\\ &= T_gC_{\varphi_a}(z^n-az^{n-1})\\ &= (\varphi_a-a)g\varphi_a^{n-1}.\end{aligned}$$

Then

$$\begin{aligned}\|C_{\varphi_a}^*(z^n)\|_2^2 &= \int_0^{2\pi}|g(e^{i\theta})|^2|\varphi_a^{n-1}(e^{i\theta})|^2|\varphi_a(e^{i\theta})-a|^2\frac{d\theta}{2\pi}\\ &= \int_0^{2\pi}|g(e^{i\theta})|^2|\varphi_a(e^{i\theta})-a|^2\frac{d\theta}{2\pi}\\ &= \|(\varphi_a-a)g\|_2^2.\end{aligned}$$

By some computation, we obtain

$$\begin{aligned}(\varphi_a-a)g &= \frac{(|a|^2-1)z}{1-\bar{a}z}\cdot\frac{1}{1-\bar{a}z}\\ &= (|a|^2-1)\sum_{n=0}^{\infty}(n+1)(\bar{a})^nz^{n+1}.\end{aligned}$$

Therefore

$$\begin{aligned}
\|(\varphi_a - a)g\|_2^2 &= (1-|a|^2)^2 \left\| \sum_{n=0}^{\infty} (n+1)(\bar{a})^n z^{n+1} \right\|_2^2 \\
&= (1-|a|^2)^2 \sum_{n=0}^{\infty} (n+1)^2 |a|^{2n} \\
&> (1-|a|^2)^2 \sum_{n=0}^{\infty} (n+1) |a|^{2n} \\
&= 1,
\end{aligned}$$

so the proof is complete. □

The idea of the proof of the following theorem is similar to the proof of Theorem 6.2 in [4].

Theorem 3.9 *Let $\varphi \in Aut(\mathbb{D})$ and $\psi \in H^\infty$. Also assume that $|\psi| = t$ almost everywhere on $\partial\mathbb{D}$, where t is a positive constant. If $C_{\psi,\varphi}$ is essentially normal, then C_φ is normal.*

Proof. Suppose that $C_{\psi,\varphi}$ is essentially normal and C_φ is not normal, so φ is not a rotation. By some computation, we see that

$$\begin{aligned}
\langle [C_{\psi,\varphi}, C^*_{\psi,\varphi}] \left(\frac{\psi}{t} z^n\right), \left(\frac{\psi}{t} z^n\right) \rangle &= \left\| C^*_{\psi,\varphi} \left(\frac{\psi}{t} z^n\right) \right\|_2^2 - \left\| C_{\psi,\varphi} \left(\frac{\psi}{t} z^n\right) \right\|_2^2 \\
&= \left\| C^*_{\varphi} T^*_{\psi} \left(\frac{\psi}{t} z^n\right) \right\|_2^2 \\
&- \int_0^{2\pi} |\psi(e^{i\theta})|^2 \frac{|\psi(\varphi(e^{i\theta}))|^2}{t^2} |\varphi^n(e^{i\theta})|^2 \frac{d\theta}{2\pi} \\
&= t^2 \|C^*_\varphi(z^n)\|_2^2 - t^2 \\
&= t^2(\|C^*_\varphi(z^n)\|_2^2 - 1).
\end{aligned}$$

Since $\{\frac{\psi}{t} z^n\}$ converges weakly to 0 as $n \to \infty$, we conclude that $t^2(\|C^*_\varphi(z^n)\|_2^2 - 1) \to 0$ as $n \to \infty$. Thus, by Proposition 3.8, it is a contradiction. □

Remark 3.10 *If φ and ψ satisfy the hypotheses of Theorem 3.9 and also $1/\psi \in H^\infty$, then the converse of the proceeding theorem is true. Let $\varphi(z) = \lambda z$, where $|\lambda| = 1$. Then by Equations (3) and (4), we have*

$$C^*_{\psi,\varphi} C_{\psi,\varphi} = C_{\bar{\lambda} z} T^*_\psi T_\psi C_{\lambda z} = t^2 C_{\bar{\lambda} z} C_{\lambda z} = t^2 I = T^*_\psi T_\psi$$

and

$$C_{\psi,\varphi} C^*_{\psi,\varphi} = T_\psi C_{\lambda z} C_{\bar{\lambda} z} T^*_\psi = T_\psi I T^*_\psi = T_\psi T^*_\psi.$$

Therefore $[C^*_{\psi,\varphi}, C_{\psi,\varphi}] = T^*_\psi T_\psi - T_\psi T^*_\psi$. Since $1/\psi \in H^\infty$, we have

$$T_{\frac{1}{\psi}} [C^*_{\psi,\varphi}, C_{\psi,\varphi}] = T_{t^2 \frac{1}{\psi}} - T^*_\psi = T_{\frac{t^2}{\psi} - \bar{\psi}}.$$

Thus, by [33, Proposition 9.1.3], $C_{\psi,\varphi}$ is essentially normal.

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