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## $p$ -Rank and $p$ -groups in algebraic groups

Adrien Deloro

### Abstract

A few remarks on the measures of the  $p$ -rank of a group equipped with a dimension, including the refutation of a result of Burdges and Cherlin.

**Key Words:** Algebraic groups, groups of finite Morley rank, torsion,  $p$ -groups

Groups of finite Morley rank are abstract analogues of algebraic groups; like them they bear a dimension enabling various genericity arguments. They do not come from geometry but from logic; yet the Cherlin-Zilber conjecture and related work suggest tight relationships between both aspects. My reader may thus view what follows as naive properties of algebraic groups obtained by elementary means; the word “definable” stands for “constructible”. Should he desire more on groups of finite Morley rank, [1] would provide references.

I wish to thank Éric Jaligot for his many suggestions.

A group of finite Morley rank is *connected* if it has no proper definable subgroup of finite index. One lets  $H^\circ$  be the *connected component* of a definable subgroup  $H$ , i.e. its smallest definable connected subgroup of finite index. This extends to arbitrary  $H$ :  $H$  is included in a smallest definable subgroup  $d(H)$ , one takes the connected component  $d(H)^\circ$  and sets  $H^\circ = H \cap d(H)^\circ$ .

Throughout,  $p$  will be a prime (possibly 2). A  $p$ -torus  $T$  is a finite power of the Prüfer quasi-cyclic  $p$ -group  $\mathbb{Z}_{p^\infty}$ ;  $T \simeq \mathbb{Z}_{p^\infty}^d$  is injective among abelian groups. For  $H \leq T$ ,  $H^\circ$  consistently denotes the maximal subtorus of  $H$ .

A group of finite Morley rank is  $U_p^\perp$  if it has no infinite elementary abelian  $p$ -subgroup.  $U_p^\perp$  groups conjugate their *Sylow  $p$ -subgroups* [3, Theorem 4], i.e. their maximal (non-necessarily definable)  $p$ -subgroups; these are finite extensions of  $p$ -tori. Hence, for  $S$  a Sylow  $p$ -subgroup of a  $U_p^\perp$  group,  $S^\circ$  is a  $p$ -torus.

Given a  $U_p^\perp$  group, 3 measures of its Sylow  $p$ -subgroups are available. One can consider the *Prüfer  $p$ -rank*  $\text{Pr}_p(G)$ , which is the number of  $\mathbb{Z}_{p^\infty}$  factors in a Sylow  $p$ -subgroup. One can also estimate the *normal  $p$ -rank*  $n_p(G)$ , which is the maximal  $p$ -rank of an elementary abelian  $p$ -group normal in a Sylow  $p$ -subgroup. Or one can simply compute the  *$p$ -rank*  $m_p(G)$ , which is the maximal  $p$ -rank of an elementary abelian  $p$ -subgroup. All 3 numbers are well defined by conjugacy of the Sylow  $p$ -subgroups, and  $m_p(G) \geq n_p(G) \geq \text{Pr}_p(G)$ .

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**1. The  $n$ -rank**

**Lemma 1** *If  $G$  is a connected,  $U_p^\perp$  group, then  $n_p(G) = \text{Pr}_p(G)$ .*

**Proof.** Let  $S$  be a Sylow  $p$ -subgroup of  $G$ ,  $V \triangleleft S$  an elementary abelian normal subgroup, and  $v \in V$ . As  $V \triangleleft S$ ,  $v^{S^\circ} \subseteq V$  which is finite; by connectedness,  $S^\circ$  centralizes  $v$ . So  $v \in C_S(S^\circ) = S^\circ$  by [3, Corollary 3.1], and  $V \leq S^\circ$ .  $\square$

**2. Not quite a digression**

For a  $p$ -torus  $T \simeq \mathbb{Z}_{p^\infty}^d$ ,  $\Omega_{p^n}(T)$  denotes the set of elements of order at most  $p^n$ .

**Fact 1** *Let  $\varphi$  be an automorphism of finite order of a  $p$ -torus  $T \simeq \mathbb{Z}_{p^\infty}^d$ .*

1. *Suppose  $\Omega_{p^2}(T) \leq C_T(\varphi)$ . Then  $\varphi = \text{Id}$ .*
2. *Suppose  $\Omega_p(T) \leq C_T(\varphi)$ . If  $p = 2$ , then  $\varphi^2 = \text{Id}$ . If  $p \neq 2$ , then  $\varphi = \text{Id}$ .*

**Proof.** This must be classical but I know no reference.

1. Up to taking a power of  $\varphi$ , we may assume that  $\varphi$  has prime order  $q$ . Let  $x \notin C_T(\varphi)$  have minimal order. Then  $\varphi(x^p) = x^p$  so there is  $y \in \Omega_p(T) \setminus \{1\}$  with  $\varphi(x) = xy$ . By assumption  $y \in C_T(\varphi)$ , so  $x = \varphi^q(x) = xy^q$  and  $q = p$ . Let  $\hat{x}$  and  $\hat{y}$  be such that  $\hat{x}^p = x$  and  $\varphi(\hat{x}) = \hat{x}\hat{y}$ . Then  $\hat{y}^p = y$  so  $\hat{y} \in \Omega_{p^2}(T) \leq C_T(\varphi)$  and  $\hat{x} = \varphi^p(\hat{x}) = \hat{x}\hat{y}^p = \hat{x}y$ , a contradiction.
2. Since  $\varphi$  centralizes  $\Omega_p(T)$ , for  $x \in \Omega_{p^2}(T)$  there is  $y \in \Omega_p(T)$  with  $\varphi(x) = xy$ ; hence  $\varphi^p(x) = xy^p = x$  and  $\Omega_{p^2}(T) \leq C_T(\varphi^p)$ . So  $\varphi^p = \text{Id}$ ; we may assume  $p \neq 2$ . Represent  $\varphi|_{\Omega_{p^3}(T)}$  by a matrix  $M \in \text{GL}_d(\mathbb{Z}/p^3\mathbb{Z})$ . As  $\Omega_p(T) \leq C_T(\varphi)$ , the reduction of  $M$  modulo  $p$  is the identity: there is a matrix  $N$  with  $M = \text{Id} + pN$ . Since  $\varphi^p = \text{Id}$ ,

$$0 \equiv \sum_{\ell=1}^p \binom{p}{\ell} p^\ell N^\ell \equiv p^2 N + \frac{p(p-1)}{2} p^2 N^2 \pmod{p^3}.$$

Since  $p \neq 2$ ,  $p$  divides  $\frac{p(p-1)}{2}$ , so  $p^2 N \equiv 0 \pmod{p^3}$  and  $N \equiv 0 \pmod{p}$ . Hence the reduction of  $M = \text{Id} + pN$  modulo  $p^2$  is the identity:  $M$  centralizes  $\Omega_{p^2}(T)$ , and  $\varphi$  is trivial.  $\square$

**Consequence** *If  $p \neq 2$ , the restriction map  $\rho : \text{Aut}(T) \rightarrow \text{Aut}(\Omega_p(T))$  kills no element of finite order. In particular if  $W$  is a finite subgroup of  $\text{Aut}(T)$  then  $W$  embeds into  $\text{Aut}(\Omega_p(T)) \simeq \text{GL}_d(\mathbb{F}_p)$ . If  $p = 2$  then  $\ker \rho|_W \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^d$ .*

**Proof.** The only non-immediate claim is about the rank of  $K = \ker \rho|_W$  when  $p = 2$ . Observe that  $K$  has exponent 2, so it is abelian. We go in a direction that will prove fruitful. Taking automorphism groups changes inductive limits to projective limits, so  $\text{Aut}(T) \simeq \varprojlim \text{Aut}((\mathbb{Z}/p^n\mathbb{Z})^d) = \varprojlim \text{GL}_d(\mathbb{Z}/p^n\mathbb{Z}) = \text{GL}_d(\mathbb{Z}_p)$ . Hence  $K$  embeds into  $\text{GL}_d(\mathbb{Z}_2)$ . Now elements of  $K$  are simultaneously diagonalizable over  $\overline{\mathbb{Q}}_2$  with eigenvalues  $\pm 1$ , so

$K$  embeds into  $\{\pm 1\}^d$ . □

We could use a similar method to get a lazy bound on  $\text{rk } W$  for  $W \leq \text{Aut}(T)$  an elementary abelian  $p$ -group; observe how we are naturally moving towards the  $p$ -adic representation. Anyway, embedding into  $\text{GL}_d(\mathbb{F}_p)$ , i.e. restricting to  $\Omega_p(T)$ , was too clumsy in the first place. For instance, any element of  $\text{GL}_d(\mathbb{F}_p)$  comes from an element of  $\text{GL}_d(\mathbb{Z}_p)$ , but not necessarily from one of finite order. (The reader may check that  $\text{GL}_2(\mathbb{Z}_5)$  has no element of order 5.) Representation-theoretically speaking, embedding into  $\text{GL}_d(\mathbb{Z}_p)$  is more appropriate, and this is what we shall now do.

### 3. Bounding the $m$ -rank

Let  $\varphi$  be an automorphism of order  $p$  of a  $p$ -torus  $T \simeq \mathbb{Z}_p^d$ .

**Fact 2 (Maschke's Theorem)** *Let  $T_1 \leq T$  be a  $\varphi$ -invariant subtorus. Then there is a  $\varphi$ -invariant subtorus  $T_2 \leq T$  such that  $T = T_1 + T_2$  and  $T_1 \cap T_2 \leq \Omega_p(T_1)$ .*

**Proof.** There is a subtorus  $T_0 \leq T$  with  $T = T_1 \oplus T_0$ . Let  $\pi$  be the projection on  $T_1$  along  $T_0$  and  $\hat{\pi} = \sum_{i=0}^{p-1} \varphi^i \pi \varphi^{-i}$ . Then  $\hat{\pi}$  is  $\varphi$ -covariant,  $\text{im } \hat{\pi} = T_1$ , and  $\hat{\pi}(t_1) = pt_1$  for  $t_1 \in T_1$ . Take  $T_2$  to be the maximal subtorus of  $\ker \hat{\pi}$ . □

**Fact 3 ( $\varphi, T$  as above)** *If  $C_T^\circ(\varphi) = 1$  then  $p-1 \mid d$  and  $\text{Id} + \varphi + \dots + \varphi^{p-1} = 0$ .*

**Proof.** (This again must be well known.) We may assume  $p \neq 2$ . Let  $\tau \leq T$  be isomorphic to  $\mathbb{Z}_p^\infty$ , and set  $\Theta = \sum_{i=0}^{p-1} \varphi^i(\tau)$ ;  $\Theta$  is  $\varphi$ -invariant and  $\text{Pr}_p(\Theta) \leq p$ . So by Maschke's Theorem, we may assume  $d \leq p$ . As in the proof of the Consequence above, let us view  $\varphi$  as an element of order  $p$  of  $\text{GL}_d(\mathbb{Z}_p) \leq \text{GL}_d(\mathbb{Q}_p)$ . By assumption, 1 is not an eigenvalue.

The minimal polynomial  $\mu$  of  $\varphi$  over  $\mathbb{Q}_p$  divides  $X^p - 1 = (X - 1)(1 + X + \dots + X^{p-1})$ , so it divides  $1 + X + \dots + X^{p-1}$ . The latter is irreducible over  $\mathbb{Z}_p$  by Eisenstein's criterion, so  $\mu = 1 + X + \dots + X^{p-1}$ . But  $\mu$  divides the characteristic polynomial which has degree  $d$ . So  $p-1 \leq d \leq p$ . Over  $\bar{\mathbb{Q}}_p$ ,  $\varphi$  has  $p-1$  eigenvalues, which sum to  $-1$ . So if  $d = p$ , one of them, say  $j$ , occurs twice: hence  $1 + \text{Tr } \varphi = j \in \mathbb{Q}_p$ , against  $p \neq 2$ . So  $d = p-1$ . □

**Lemma 2** *For  $W \leq \text{Aut } \mathbb{Z}_p^d$  an elementary abelian  $p$ -group,  $\text{rk } W \leq \frac{1}{p-1}d$ .*

**Proof.**  $E = \mathbb{Q}_p^d$  is a sum of  $W$ -irreducible subspaces  $\bigoplus_{i \in I} E_i \bigoplus \bigoplus_{j \in J} F_j$  with  $E_i$ 's the  $W$ -trivial lines. Since  $W$  is abelian, it acts  $W$ -covariantly. Let  $\rho_j : W \rightarrow \text{Aut}_W(F_j)$  be the restriction map, with (non-trivial) image  $W_j$  and kernel  $K_j$ . Each  $\text{End}_W F_j$  is a skew-field by Schur's Lemma, so the abelian group  $W_j$  of exponent  $p$  has order  $p$ . As  $C_W(E) = 1$ ,  $W \hookrightarrow \prod_{j \in J} W/K_j$ , and  $\text{rk } W \leq \#J$ . By Fact 3,  $\dim F_j \geq p-1$ , whence  $\#J \leq \frac{d}{p-1}$ . □

**Corollary 1** *Let  $G$  be a connected,  $U_p^\perp$  group. Then  $m_p(G) \leq \frac{p}{p-1} \text{Pr}_p(G)$ .*

**Proof.** For  $V \leq S$  an elementary abelian subgroup of a Sylow  $p$ -subgroup  $S$ , write  $V = (V \cap S^\circ) \oplus W$ . By [3, Corollary 3.1],  $C_S(S^\circ) = S^\circ$ ; use Lemma 2.  $\square$

**4. Maximal abelian  $p$ -subgroups**

**Thesis** [2, Theorem 1.2] *Let  $G$  be a connected,  $U_p^\perp$  group with  $m_p(G) \geq 3$ . Then any maximal elementary abelian  $p$ -subgroup  $V < G$  has  $p$ -rank at least 3.*

The flaw in [2] lies at the bottom of page 172. On the very last line, “commutation with  $v$ ” need not in general be “a map from  $\Omega_1(T)/A$  to  $A$ ”. Observe that in [2] Theorem 6.4 relies on Corollary 4.2, which relies on Theorem 1.2.

**Counter-Example** *In  $\text{PSL}_5(\mathbb{C})$  let  $\Theta$  be the usual torus and  $\sigma$  be the Weyl element naturally associated with the 5-cycle (12345). Let  $\theta \in C_\Theta(\sigma) \setminus \{1\}$ . Then  $\langle \theta, \sigma \rangle$  does not extend to an elementary abelian 5-group of rank 3.*

**Proof.** The actual computations will take place in  $\text{SL}_5(\mathbb{C})$ . Let  $\lambda = e^{\frac{2i\pi}{5}} \in \mathbb{C}$ ; then  $Z(\text{SL}_5(\mathbb{C})) = \{\lambda^k \text{Id}\}$ . The matrix  $s = (\delta_{j,i+1}) \in \text{SL}_5(\mathbb{C})$  (equality modulo 5) reduces modulo  $Z(\text{SL}_5(\mathbb{C}))$  to  $\sigma \in \text{PSL}_5(\mathbb{C})$ ; conjugation by  $s$  rotates coefficients of a matrix  $(m_{i,j}) \in \text{SL}_5(\mathbb{C})$  along the 5 (complete) diagonals. So given  $\theta \in \Theta \leq \text{PSL}_5(\mathbb{C})$  and a diagonal matrix  $t \in \text{SL}_5(\mathbb{C})$  representing it, one sees that  $[\sigma, \theta] = 1$  iff  $t_{i,i} = \lambda^{k+\ell i}$  for some integers  $k$  and  $\ell$ ; thus  $C_\Theta(\sigma)$  has order 5. Fix  $\theta \in C_\Theta(\sigma) \setminus \{1\}$ . Conjugation by  $t$  on  $(m_{i,j})$  multiplies  $m_{i,j}$  by  $\lambda^{\ell(j-i)}$ . So  $C(\theta) = \Theta \rtimes \langle \sigma \rangle$ , and  $\langle \theta, \sigma \rangle$  is maximal.  $\square$

The following merely serves the purpose of exposing an important method.

**Observation** *Let  $G$  be a connected,  $U_p^\perp$  group, and  $S \leq G$  a Sylow  $p$ -subgroup. Then  $S$  is connected iff abelian iff nilpotent.*

**Proof.** Only one claim is non-trivial; we prove it by induction on the Morley rank (read: dimension) of  $G$ . Suppose  $S$  nilpotent; let  $\omega \in S$ . Then by nilpotence,  $\tau = C_{S^\circ}^\circ(\omega) \neq 1$ . By [3, Corollary 3.1],  $\omega$  lies in any maximal  $p$ -torus of  $C^\circ(\omega)$ , so  $\omega \in C^\circ(\tau)$ . Hence  $\langle S^\circ, \omega \rangle \leq C^\circ(\tau)$ . If  $C^\circ(\tau) < G$  we are done by induction. Otherwise  $\tau$  is central and we can factor by  $Z^\circ(G)$ , pursuing by induction.  $\square$

I shall now bring my reader some comfort.

**Lemma 3** *The thesis of [2, Theorem 1.2] holds for  $p = 2$ , and so does [2, Corollary 6.5].*

**Proof.** Suppose  $m_2(G) \geq 3$ ; clearly  $\text{Pr}_2(G) \geq 2$ . Let  $i, j$  be 2 commuting involutions; by totality [3, Theorem 3] there is a Sylow 2-subgroup  $S$  with  $i \in S^\circ$  and  $j \in S$ .

Suppose  $\text{Pr}_2(G) \geq 3$ . If  $j \in S^\circ$  we are done. If not, consider the map  $\varphi(k) = [j, k] : \Omega_2(S^\circ) \rightarrow \Omega_2(S^\circ)$ . Then  $\text{im } \varphi \leq \ker \varphi$  and  $\text{rk im } \varphi + \text{rk ker } \varphi \geq 3$ , so  $\text{rk ker } \varphi \geq 2$  and we are done. From now on, suppose  $\text{Pr}_2(G) = 2$  (so  $m_2(G) \leq 4$ ) and let  $V = \Omega_2(S^\circ)$ .

Assume first that  $j \in S \setminus S^\circ$ . If  $j$  inverts  $S^\circ$  then  $\langle i, j \rangle \leq \langle V, j \rangle$ : we are done. Otherwise  $\tau = C_{S^\circ}^\circ(j) \neq 1$ . If  $i \notin \tau$  then  $\langle i, j \rangle \leq \langle i, \Omega_2(\tau), j \rangle$ : we are done. So assume  $i \in \tau \leq C^\circ(j)$ . By totality [3, Theorem 3 and Corollary 3.1],  $i$  lies in a 2-torus of  $C^\circ(j)$  and  $j$  lies in any 2-torus of  $C^\circ(j)$ , so  $i$  and  $j$  are cotoral.

So assume that  $j \in S^\circ$ , that is  $V = \langle i, j \rangle$ . By assumption there is an elementary abelian 2-subgroup of rank 3:  $A = \langle r, s, t \rangle \leq S$ ; clearly  $A \cap S^\circ \neq 1$ , say  $r \in V$ . If  $s$  or  $t$  is in  $V$  then  $\langle i, j \rangle = V \leq A$ : we are done. Suppose that  $s$  and  $t$  (hence  $st$  as well) lie in  $S \setminus S^\circ$ . Since  $|\text{Aut}(V)| = 6$ , one of  $s, t, st$  must centralize  $V = \langle i, j \rangle$ : we are done again.  $\square$

Here is a final word on counter-examples.

**Lemma 4** *Let  $G$  be a counter-example to [2, Theorem 1.2]. Then  $\text{Pr}_p(G) = p - 1$ . In particular, [2, Theorem 1.2] also holds for  $p = 3$ .*

**Proof.** By Lemma 3,  $p \geq 3$ . As  $m_p(G) \geq 3$ , one sees with Corollary 1 that  $\text{Pr}_p(G) \geq 2$ . Equality can hold only for  $p = 3$ ; as there is an elementary 3-group of rank 3, there is an automorphism of order 3 fixing  $\Omega_3(\mathbb{Z}_3^\infty)$ , against Fact 1: equality cannot hold.

Hence  $\text{Pr}_p(G) \geq 3$ . Let  $V = \langle \alpha, \omega \rangle$  be a maximal abelian  $p$ -group and  $S \geq V$  a Sylow  $p$ -subgroup. By totality we may assume  $\alpha \in S^\circ$ , so  $\omega \in S \setminus S^\circ$ . If  $C_{S^\circ}(\omega) \neq 1$  then by maximality,  $\alpha \in C_{S^\circ}(\omega) \leq C^\circ(\omega)$ , and as in the proof of Lemma 3,  $\alpha$  and  $\omega$  are cotoral, a contradiction. Hence  $C_{S^\circ}(\omega) = 1$ . Let  $\varphi \in \text{End}_{\Omega_p}(S^\circ)$  map  $x$  to  $[x, \omega]$ ; writing  $\omega$  as an automorphism,  $\varphi(x) = \omega(x) - x$  and  $\varphi^n(x) = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \omega^i(x)$ . As  $(-1)^i \binom{p-1}{i} \equiv 1 \pmod{p}$ ,  $\varphi^{p-1} = \text{Id} + \omega + \dots + \omega^{p-1}$ . But  $C_{S^\circ}(\omega) = 1$ , so Fact 3 applied to  $\omega$  implies  $\varphi^{p-1} = 0$ . Since  $\ker \varphi = C_{\Omega_p(S^\circ)}(\omega) = \langle \alpha \rangle$ , one has  $\text{rk } \Omega_p(S^\circ) \leq p - 1$ .  $\square$

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