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Lower bounds for the maximum genus of 4-regular graphs*

Ding Zhou, Rongxia Hao, Weili He

Abstract

This paper investigates the maximum genus and upper embeddability of connected 4-regular graphs. We obtain lower bounds on the maximum genus of connected 4-regular simple graphs and connected 4-regular graphs without loops in terms of the Betti number. The definition of the Betti number is referred to [Gross and Tucker, Topological Graph Theory, New York, 1987]. Furthermore, we give examples that show that these lower bounds are tight.

Key Words: Maximum genus, upper embeddable, Betti number

1. Introduction

The *maximum genus* of a connected graph $G = (V, E)$, denoted by $\gamma_M(G)$, is the maximum integer k with the property that there exists a cellular embedding of G on the orientable surface with genus k .

The maximum genus has received considerable attention after Nordhaus et al. [18]. Xuong [22] proved that every 4-edge connected graph is upper embeddable. However, there are many examples of 3-edge connected graphs and 2-edge connected graphs that are not upper embeddable. See, for example, [2, 12, 14]. Therefore, considerable attention is given to the lower bounds on the maximum genus of many kinds of graphs in terms of some graph invariants. See, for example, [3, 5, 7, 10]. Chen et al. [2] proved that for a 2-connected simple graph with all its vertices of degree greater than 2, the maximum genus is at least $\beta(G)/3$. Chen and Huang gave some results on the maximum genus of graphs in [4]. In the paper [15], Nedela and Skoviera proved that any Eulerian graph with at most 2 vertices of degree $0 \pmod{4}$ is necessarily upper embeddable. In particular, any connected regular graph with degree $4k + 2$, for an integer $k \geq 1$, is upper embeddable. Skoviera [20, 21] has obtained the maximum genus of graphs which are 3-regular with 2-factor triangles, and its slightly weaker form of the result was earlier proved in [1]. Naturally, a question on the maximum genus for 4-regular graphs can be posed. More information on upper embeddability of graphs can be found for example in [11]-[19].

In this paper, tight lower bounds on the maximum genus of connected 4-regular simple graphs and connected 4-regular graphs without loops are obtained.

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2. Preliminaries

For simplification, throughout this paper, G is asumed to have n vertices and m edges.

Since any embedding has at least one face, from the Euler formula, it can be seen that $\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor$, where $\beta(G) = m - n + 1$ is called the *Betti number* [11] of the connected graph G . If $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$, G is said to be *upper embeddable*.

For a subset $A \subseteq E(G)$, $c(G \setminus A)$ denotes the number of connected components of $G \setminus A$, and $b(G \setminus A)$ denotes the number of connected components of $G \setminus A$ with an odd Betti number, where $G \setminus A$ means the subgraph obtained from G by deleting all the edges of A from G . Let T be a spanning tree of a connected graph G . Define the *deficiency* $\xi(G, T)$ to be the number of components of $G \setminus E(T)$ that have an odd number of edges. The *deficiency* $\xi(G)$ of the graph G is defined to be the minimum value of $\xi(G, T)$ over all spanning trees T of G . Note that $\xi(G) = \beta(G) \pmod{2}$. For $k \geq 2$, let F_1, F_2, \dots, F_k be k distinct subgraphs of a graph G . Then denote by $E_G(F_1, F_2, \dots, F_k)$ the edges of G with ends in distinct F_i s and $E(F_i, G)$ the edges of $E(G)$ with just one end in F_i .

We follow [6] for terminologies and notations not defined here.

The following theorems are some characterizations on the maximum genus and upper embeddability of a graph, which are due to Xuong and Nebesky, respectively.

Theorem 1 [22, 17] *If G is a connected graph, then*

- (a) G is upper embeddable if and only if $\xi(G) \leq 1$;
- (b) $\gamma_M(G) = (\beta(G) - \xi(G))/2$.

Theorem 2 [16] *If G is a connected graph, then*

- (1) G is upper embeddable if and only if $c(G \setminus A) + b(G \setminus A) - 2 \leq |A|$ for any subset $A \subseteq E(G)$;
- (2) $\xi(G) = \max_{A \subseteq E(G)} \{c(G \setminus A) + b(G \setminus A) - |A| - 1\}$.

The following result, which was proved by Huang, provides a structural characterization of graphs that are not upper embeddable.

Theorem 3 [9, 8, 17] *Let G be a connected graph. If $\xi(G) \geq 2$, namely G is not upper embeddable, then there exists a subset $A \subseteq E(G)$ such that the following properties are satisfied:*

- (1) $c(G \setminus A) = b(G \setminus A) \geq 2$, $\beta(F) = 1 \pmod{2}$ for each component F of $G \setminus A$;
- (2) F is a vertex-induced subgraph of G for each component F of $G \setminus A$;
- (3) for any k distinct components F_1, F_2, \dots, F_k of $G \setminus A$, $|E_G(F_1, F_2, \dots, F_k)| \leq 2k - 3$. In particular, $|E_G(F, H)| \leq 1$ for any 2 distinct components F and H of $G \setminus A$;
- (4) $\xi(G) = 2c(G \setminus A) - |A| - 1$.

3. Main results

Theorem A *If G is a connected 4-regular simple graph with Betti number $\beta = \beta(G)$, then*

$$\gamma_M(G) \geq \lfloor (2\beta + 3)/5 \rfloor.$$

Proof. Let G be a connected 4-regular simple graph with n vertices and m edges. Then by the definition of Betti number, $\beta = m - n + 1 = 4n/2 - n + 1 = n + 1$. The following 2 cases are considered:

Case 1 G is upper embeddable. In this case,

$$\gamma_M(G) = \lfloor \beta/2 \rfloor.$$

Since G is a connected 4-regular simple graph, $n \geq 5$ and $\lfloor \beta/2 \rfloor \geq \lfloor (2\beta + 3)/5 \rfloor$, so $\gamma_M(G) \geq \lfloor (2\beta + 3)/5 \rfloor$.

Case 2 G is not upper embeddable.

By Theorem 3, there exists $A \subseteq E(G)$ such that the properties (1)–(4) of Theorem 3 are satisfied. Then $c(G \setminus A) = b(G \setminus A) \geq 2$ and $\xi(G) = 2c(G \setminus A) - |A| - 1$.

Let a_i^j denote the number of connected components of $G \setminus A$ with order i and Betti number j . Since G is a simple graph and $c(G \setminus A) = b(G \setminus A)$, for each component F of $G \setminus A$, we have $\beta(F) = 1 \pmod{2}$, so there exist $a_i^j \neq 0$ only for $i \geq 3$ and $j = 1 \pmod{2}$, and thus the following Claim 1 is right:

Claim 1 $a_i^j = 0$ for any $i \geq 3$ and $j = 0 \pmod{2}$.

Before finishing the proof of this theorem, we first prove the following, Claim 2.

Claim 2 $a_i^j = 0$ for any positive integer j with $j > i$.

Proof of Claim 2 By the contrary, suppose $a_i^j \neq 0$ for some positive integer j with $j > i$, and then there is at least one connected component, denoted by F , of $G \setminus A$ with order i and Betti number j . Since G is 4-regular, thus $\sum_{v \in F} d_G(v) = 4i$. Because the number of vertices with odd degree in F must be even, $E(F, G) \geq 2$. By $E(F, G) = 4i - 2(i - 1) - 2j = 2i - 2j + 2$, where $i - 1$ and j are the number of spanning tree edges and that of cotree edges of F respectively, then $2i - 2j + 2 \geq 2$, i.e., $i \geq j$, this is a contradiction with $i < j$. \square

We now continue with the proof of the theorem.

Since the number of connected components of $G \setminus A$ is sum of a_i^j for all integers i and j , by Claims 2,

$$c(G \setminus A) = \sum_{j \leq i} a_i^j.$$

Since G is a simple graph, any connected component with order 1 of $G \setminus A$ cannot have Betti number 1; thus $a_1^1 = 0$. Similarly, it can be checked that $a_2^1 = a_3^3 = 0$ because of G being a connected 4-regular simple graph. Also by Claims 1 and 2, we have

$$\begin{aligned} c(G \setminus A) &= \sum_{j \leq i} a_i^j = a_3^1 + a_4^1 + a_5^1 + \cdots + a_4^3 + a_5^3 + a_6^3 + \cdots \\ &\quad + a_5^5 + a_6^5 + a_7^5 + \cdots + \sum_{i \geq j \geq 7} a_i^j. \end{aligned} \tag{1}$$

$j = 1 \pmod{2}$

Since a connected component with order i of $G \setminus A$ contributes to i for the order n of G , thus a_i^j

connected component with order i contributes to ia_i^j for n . So,

$$\begin{aligned} n &= 3a_3^1 + 4a_4^1 + 5a_5^1 + \cdots + 4a_4^3 + 5a_5^3 + 6a_6^3 + \cdots \\ &\quad + 5a_5^5 + 6a_6^5 + 7a_7^5 + \cdots + \sum_{\substack{i \geq j \geq 7 \\ j = 1(\bmod 2)}} ia_i^j. \end{aligned} \tag{2}$$

Let F be a connected component of $G \setminus A$ with order i and Betti number j , then $\sum_{v \in F} d_G(v) = 2[i - 1 + j] + |E(F, G)|$, where $2(i - 1 + j)$ is contributed by the edges in F , and $|E(F, G)|$ is contributed by the edges not in F . Thus $4i = 2[i - 1 + j] + |E(F, G)|$, i.e., $|E(F, G)| = 4i - 2(i - 1) - 2j = 2i - 2j + 2$.

Let φ be the set of all the components of $G \setminus A$. From Theorem 3 (2),

$$|A| = \frac{1}{2} \sum_{F \in \varphi} |E(F, G)|. \text{ Also by Claims 1 and 2, then}$$

$$\begin{aligned} |A| &= \sum_{\substack{i \geq 1 \\ j = 1(\bmod 2)}} (i - j + 1)a_i^j, \\ &= 3a_3^1 + 4a_4^1 + 5a_5^1 + \cdots + 2a_4^3 + 3a_5^3 + 4a_6^3 + \cdots \\ &\quad + a_5^5 + 2a_6^5 + 3a_7^5 + \cdots + \sum_{\substack{i \geq j \geq 7 \\ j = 1(\bmod 2)}} (i - j + 1)a_i^j. \end{aligned} \tag{3}$$

From $\beta = n + 1$ and Theorem 1, $\xi(G) = \beta - 2\gamma_M(G) = n + 1 - 2\gamma_M(G)$.

Assume $\gamma_M(G) < \lfloor (2\beta + 3)/5 \rfloor$. Then

$$\xi(G) = n + 1 - 2\gamma_M(G) > n + 1 - 2 \lfloor (2n + 2)/5 + 3/5 \rfloor = n/5 - 1.$$

From Theorem 3 (4), we have

$$\xi(G) = 2c(G \setminus A) - |A| - 1 > n/5 - 1. \text{ Therefore, } 2c(G \setminus A) - |A| - n/5 > 0. \text{ But from (1) - (3),}$$

$$\begin{aligned} 2c(G \setminus A) - |A| - n/5 &= -(8/5)a_3^1 - (14/5)a_4^1 - 4a_5^1 - \cdots \\ &\quad - (4/5)a_4^3 - 2a_5^3 - (16/5)a_6^3 - \cdots - 0a_5^5 - (6/5)a_6^5 - (12/5)a_7^5 \\ &\quad + \cdots + \sum_{\substack{i \geq j \geq 7 \\ j = 1(\bmod 2)}} (1 - 6i/5 + j)a_i^j. \end{aligned} \tag{4}$$

If $i \geq j \geq 7$, then the coefficient of a_i^j in (4) satisfies

$$1 - 6i/5 + j \leq 1 - 6i/5 + i = 1 - i/5 \leq 0. \text{ Thus}$$

$$2c(G \setminus A) - |A| - n/5 \leq 0.$$

There is a contradiction. Consequently, $\gamma_M(G) \geq \lfloor (2\beta + 3)/5 \rfloor$.

From these 2 cases, the theorem follows we complete proof of Theorem A. □

Note 1 In the following, we can find an infinite number of 4-regular graphs whose maximum genus equals the lower bound, that is, $\gamma_M(G) = \lfloor (2\beta + 3)/5 \rfloor$, which means that the lower bound of Theorem A is tight.

Let C_5^3 be the graph shown in Figure 1, which is obtained from K_5 by removing one edge.

Let G_1, \dots, G_k ($k \geq 3$) be copies of the graph C_5^3 . Then G is obtained by adding a new edge from G_i to $G_{i+1} \pmod k$, $i = 1, 2, \dots, k$ as shown in Figure 2.

It can be seen that G is a connected 4-regular simple graph. Let the set of edges from G_i to G_{i+1} be A . Then $G \setminus A$ has k connected components, each with 5 vertices and its Betti number is 5. It can be checked that $n = 5k$, $\beta = n + 1 = 5k + 1$, $\gamma_M(G) = \lfloor (2\beta + 3)/5 \rfloor = 2k + 1$.

It follows from the above examples that there exist an infinite number of connected 4-regular simple graphs whose maximum genus is equal to the lower bounds, so the lower bound of Theorem A is tight.

The following Theorem B is generalized from simple 4-regular graphs to 4-regular graphs without loops.

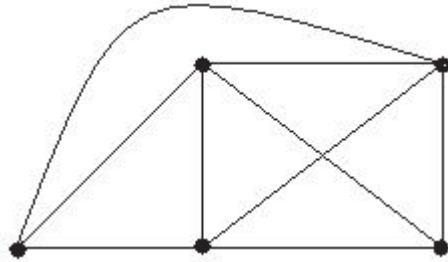


Figure 1. C_5^3 .

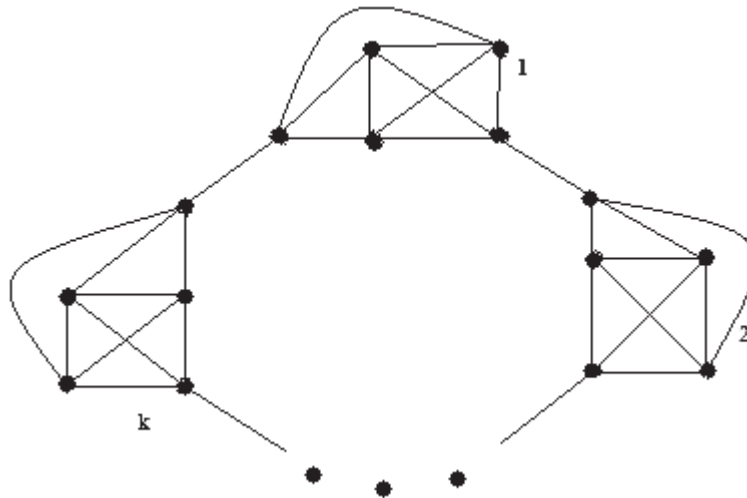


Figure 2. A connected 4-regular simple graph

Theorem B *If G is a connected 4-regular graph without loops, then*

$$\gamma_M(G) \geq \lfloor (\beta + 2)/3 \rfloor.$$

The proof of Theorem B is similar to that of Theorem A, so it is omitted.

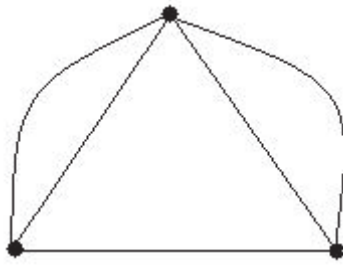


Figure 3. C_3^3 .

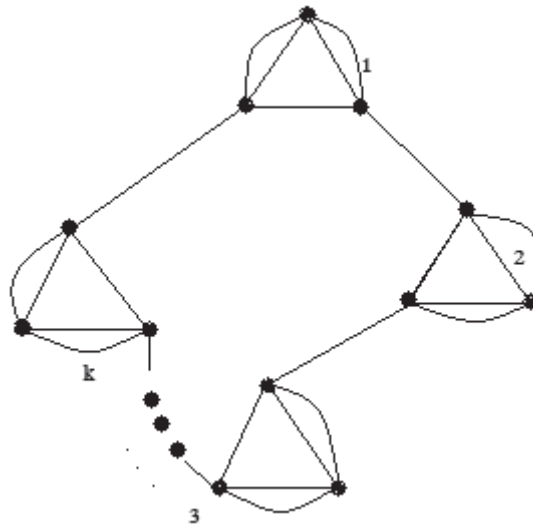


Figure 4. A connected 4-regular graph without a loop.

Note 2 We now give an infinite number of graphs whose maximum genus equal the lower bound, that is, $\gamma_M(G) = \lfloor (\beta + 2)/3 \rfloor$. It means that the lower bound of Theorem B is tight.

Let C_3^3 be the graph shown in Figure 3, which is obtained from K_3 by doubling 2 of its edges. Let G_1, \dots, G_k ($k \geq 3$) be copies of C_3^3 ; G is obtained by adding a new edge from G_i to $G_{i+1} \pmod k$ ($i = 1, 2, \dots, k$) as shown in Figure 4.

Then it can be seen that G is a connected 4-regular graph and has no loops, Let A be the set of edges from G_i to G_{i+1} , then $G \setminus A$ has k connected components, each with 3 vertices and the Betti number is 3. It can be checked that $\beta = n + 1 = 3k + 1$, $\gamma_M(G) = \lfloor (\beta + 2)/3 \rfloor = k + 1$.

This example shows that the lower bound of Theorem B is tight.

Note 3 If G is a graph with loops, Theorems A and B do not hold.

For example, let C_n be the circuit with n vertices; the graph denoted by C_n^1 is obtained from C_n by adding one loop at each vertex. If $n \geq 4$, so $\beta \geq 5$, then $\gamma_M(C_n^1) = 1$, but $\lfloor (2\beta + 3)/5 \rfloor \geq 2$ and $\lfloor (\beta + 2)/3 \rfloor \geq 2$.

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