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Formulas for the Fourier coefficients of cusp form for some quadratic forms (correction to a paper by Ahmet Tekcan with the same title)

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Abstract: In this study $M_1(\Gamma_0(3), \chi_{-3})$, $M_2(\Gamma_0(5), \chi_5)$ and $M_3(\Gamma_0(7), \chi_{-7})$ have been examined and the formulas for the Fourier Coefficients of theta series and the representation number of positive integers by some quadratic forms $3x_1^2+3x_1x_2+x_2^2$, $5(x_1^2+x_1x_2+x_1x_3+x_1x_4+x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4)+2x_4^2$, and $7(x_1^2+x_1x_2+x_1x_3+x_1x_4+x_1x_5+x_2^2+x_2x_3+x_2x_4+x_2x_5+x_3^2+x_3x_4+x_3x_5+x_4^2+x_4x_5+x_5^2+7(x_1x_6+x_2x_6+x_3x_6+x_4x_6+x_5x_6)+3x_6^2$, are determined. This work is a correction to a paper of the same title by Ahmet Tekcan [5].

Key words: Quadratic forms, representation numbers, theta series, cusp forms 11E25, 11E76

1. Fourier Coefficients of Theta Series of $\mathbf{Q}_3$, $\mathbf{Q}_5$ and $\mathbf{Q}_7$

First of all, let's mention the important Theorem about the dimension formulas.

Theorem 1.1 *Let k be an integer and χ a Dirichlet character modulo N with $\chi(-1)=(-1)^k$. For each prime p dividing N, let r_p (respectively, s_p) denote the power of p dividing N (respectively, the conductor of χ). Define*

$$\lambda(r_p, s_p, p) := \begin{cases} p^{r'} + p^{r'-1} \text{ if } 2s_p \le r_p = 2r' \\ 2p_r' \text{ if } 2s_p \le r_p = 2r'+1 \\ 2p^{r_p - s_p} \text{ if } 2s_p > r_p, \end{cases}$$

and

$$v_k := \begin{cases} 0 \text{ if } k \text{ is odd} \\ -1/4 \text{ if } k \equiv 2 \bmod 4 \\ 1/4 \text{ if } k \equiv 0 \bmod 4 \end{cases}, \quad \mu_k := \begin{cases} 0 \text{ if } k \equiv 1 \bmod 3 \\ -1/3 \text{ if } k \equiv 2 \bmod 3 \\ 1/3 \text{ if } k \equiv 0 \bmod 3 \end{cases}.$$

Then we have

$$\dim M_k(\Gamma_0(N), \chi) - \dim S_{2-k}(\Gamma_0(N), \chi) = \frac{(k-1)N}{12} \prod_{p|N}\left(1+\frac{1}{p}\right)$$

$$+\frac{1}{2}\prod_{p|N}\lambda(r_p, s_p, p) - v_{2-k}\alpha(\chi) - \mu_{2-k}\beta(\chi),$$

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where

$$\alpha(\chi) := \sum_{\substack{x \bmod N \\ x^2+1\equiv 0 \bmod N}} \chi(x) \text{ and } \beta(\chi) := \sum_{\substack{x \bmod N \\ x^2+x+1\equiv 0 \bmod N}} \chi(x).$$

Proof See [1]. □

1.1. Case $\mathbf{Q}_3$

For the quadratic form $Q_3 = 3x_1^2 + 3x_1x_2 + x_2^2$ (see [5, page 147, line 5])

$$2Q_3 = 6x_1^2 + 6x_1x_2 + 2x_2^2 = (x_1, x_2)\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

the determinant and a cofactor are $D = 3, A_{11} = 2$. So $\delta = \gcd(\frac{A_{ii}}{2}, A_{ij}, for 1 \le i \le j \le 2) = 1$, the level $N = \frac{D}{\delta} = 3$ and the discriminant is $\Delta = (-1)^{2/2} 3 = -3$. The character of Q_3 is the Kronecker symbol

$$\chi_{-3}(d) = \left(\frac{-3}{d}\right) \text{ for } d \in (\mathbb{Z}/3\mathbb{Z})^{\times}.$$

The corresponding theta function $\Theta_{Q_3}(q)$ is a modular form of weight 1 with character $\chi_{-3}(d) = \left(\frac{-3}{d}\right)$ for $d = 1, 2$, i.e.,$\Theta_{Q_3}(q) \in M_1(\Gamma_0(3), \chi_{-3})$. There is a nonzero Eisenstein series

$$G_{1,3}(\tau) = \frac{L(0, \chi_{-3})}{2} + \sum_{n=1}^{\infty}\left(\sum_{d>0, d|n} \chi_{-3}(d)\right) q^n$$

contained in $M_1(\Gamma_0(3), \chi_{-3})$ and for $k = 1,\ N = 3,\ \chi = \chi_{-3},\ p = 3,$ we have

$$\dim M_1(\Gamma_0(3), \chi_{-3}) - \dim S_1(\Gamma_0(3), \chi_{-3}) = \frac{(1-1)\cdot 3}{12}\left(1 + \frac{1}{3}\right)$$

$$+\frac{1}{2}\lambda(r_3, s_3, 3) - v_1\alpha(\chi) - \mu_1\beta(\chi) = \frac{1}{2}2 \cdot 3^{1-1} - 0 \cdot \alpha(\chi) - 0 \cdot \beta(\chi) = 1,$$

by Theorem [1.1]. On the other hand, we can prove the following theorem.

Theorem 1.2 *There is no nonzero cusp form of level* 3 *with character* χ_{-3} *of weight 1, i.e.,*

$$\dim S_1(\Gamma_0(3), \chi_{-3}) = 0.$$

Proof By [3, (12.76)], we know that

$$\dim S_1(\Gamma_0(3), \chi_{-3}) = \frac{h-1}{2} + 2s + 4a,$$

where h is the class number of $\mathbb{Q}\left(\sqrt{-3}\right)$, $s =$the number of non-isomorphic quartic fields whose Galois closure has Galois group S_4 with discriminant -3, and $a =$the number of non-isomorphic quintic non-real fields whose Galois closure has Galois group A_5 with discriminant 9. It is well known that $h = 1$. Now let $F = \mathbb{Q}(\alpha)$ be a quartic field with primitive α. The discriminant of the minimal polynomial of α is the same as the discriminant of its resolvent cubic polynomial g. Since the Galois group of F is S_4, the polynomial g is irreducible. Moreover, its signature is (1,1) since the discriminant -3 is negative. So if the roots of g are real r, and non-real complexes $c \pm di$, then

$$\begin{aligned} ((r-(c+di))(r-(c-di))(c+di-(c-di)))^2 &= \\ -4d^2\left((r-c)^2+d^2\right) &= -4\left(1+\frac{(r-c)^2}{d^2}\right). \end{aligned}$$

Since it is always smaller than -3, it follows that $s = 0$. Now, let's look at non-real quintic fields. Since the discriminant 9 is positive, the signature of the quintic field should be $(1,2)$. But in this case, the minimum discriminant of such a field is 1609 by [2], so it follows that $a = 0$. □

The theta series $\Theta_{Q_3}(q)$ associated to Q_3 is given by the scalar multiple of Eisenstein series

$$\frac{2}{L(0,\chi_{-3})}\left(\frac{L(0,\chi_{-3})}{2}+\sum_{n=1}^{\infty}\left(\sum_{d>0,d|n}\chi_{-3}(d)\,d^{1-1}\right)q^n\right).$$

Consequently, the representation number of n by Q_3 can be given by the simple formula

$$r(n;Q)=\frac{2}{L(0,\chi_{-3})}\left(\sum_{d>0,d|n}\chi_{-3}(d)\right) \text{ for } n=1,2,\ldots.$$

1.2. Case $\mathbf{Q}_5$

For the quadratic form

$$Q_5 = 5(x_1^2+x_1x_2+x_1x_3+x_1x_4+x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4)+2x_4^2.$$

The determinant of the matrix $D = 125$, $\delta = 25$, the level $N = 125/25 = 5$ and the discriminant is $\Delta = (-1)^{4/2}125 = 125$. The character of Q_5 is the unique Dirichlet character χ such that $\chi(d) = \left(\frac{125}{d}\right)$ for $d \in (\mathbb{Z}/5\mathbb{Z})^{\times}$, where $\left(\frac{125}{d}\right)$ is the Kronecker symbol. Obviously, $\chi(d) = \chi_5(d) = \left(\frac{5}{d}\right)$ for $d \in (\mathbb{Z}/5\mathbb{Z})^{\times}$.

The corresponding theta function $\Theta_{Q_5}(q)$ is a modular form of weight 2 with character $\chi_5(d) = \left(\frac{5}{d}\right)$ for $d \in (\mathbb{Z}/5\mathbb{Z})^{\times}$. Since $k = 2, N = 5$, we have

$$\dim M_2(\Gamma_0(5),\chi_5)-\dim S_0(\Gamma_0(5),\chi_5)=\frac{(2-1)\cdot 5}{12}\left(1+\frac{1}{5}\right)$$

$$+\frac{1}{2}\lambda(r_5,s_5,5)-v_0\alpha(\chi)-\mu_0\beta(\chi)$$

$$=\frac{1}{2}+\frac{1}{2}2\cdot 5^{1-1}-\frac{1}{4}\cdot\left(\left(\frac{5}{2}\right)+\left(\frac{5}{3}\right)\right)+\frac{1}{3}\cdot 0=\frac{3}{2}-\frac{1}{4}(-1-1)=2,$$

by Theorem [1.1]. Since $\dim S_k\left(\Gamma_0\left(N\right),\chi\right)=0$ for $k\leq 0,$ the result

$$\dim M_2\left(\Gamma_0\left(5\right),\chi_5\right)=2$$

follows. By Corollary 2.7 in [1], this space is generated by two linearly independent Eisenstein series

$$G_{2,5}\left(q\right)=\frac{L\left(1-2,\chi_5\right)}{2}+\sum_{n=1}^{\infty}\left(\sum_{d>0,d|n}\chi_5\left(d\right)d^{2-1}\right)q^n$$

$$=\frac{L\left(-1,\chi_5\right)}{2}+q+O\left(q^2\right),$$

$$H_{2,5}\left(q\right)=\sum_{n=1}^{\infty}\left(\sum_{d>0,d|n}\chi_5\left(n/d\right)d^{2-1}\right)q^n=q+O\left(q^2\right).$$

Therefore, the theta series $\Theta_{Q_5}\left(q\right)$ associated to Q_5 is given as a linear combination of Eisenstein series

$$\frac{2}{L\left(-1,\chi_5\right)}G_{2,5}+\left(r\left(1,Q_5\right)-\frac{2}{L\left(-1,\chi_5\right)}\right)H_{2,5}=$$

$$\frac{2}{L\left(-1,\chi_5\right)}\left(\frac{L\left(-1,\chi_5\right)}{2}+\sum_{n=1}^{\infty}\left(\sum_{d>0,d|n}\chi_5\left(d\right)d^{2-1}\right)q^n\right)$$

$$-\frac{2}{L\left(-1,\chi_5\right)}\sum_{n=1}^{\infty}\left(\sum_{d>0,d|n}\chi_5\left(n/d\right)d^{2-1}\right)q^n$$

$$=\quad 1+\sum_{n=1}^{\infty}\frac{2}{L\left(-1,\chi_5\right)}\left(\sum_{d>0,d|n}\left(\chi_5\left(d\right)-\chi_5\left(n/d\right)\right)d\right)q^n,$$

since $Q_5=1$ doesn't have any integer solutions. Consequently, the representation number can be given by the simple formula

$$r\left(n;Q\right)=\frac{2}{L\left(-1,\chi_5\right)}\left(\sum_{d>0,d|n}\left(\chi_5\left(d\right)-\chi_5\left(n/d\right)\right)d\right)\quad\text{for } n=1,2,\ldots.$$

1.3. Case $\mathbf{Q}_7$

For the quadratic form

$$Q_7=7(x_1^2+x_1x_2+x_1x_3+x_1x_4+x_1x_5+x_2^2+x_2x_3+x_2x_4+x_2x_5$$

$$+x_3^2+x_3x_4+x_3x_5+x_4^2+x_4x_5+x_5^2)+7\left(x_1x_6+x_2x_6+x_3x_6+x_4x_6+x_5x_6\right)+3x_6^2.$$

The determinant of the matrix $D = 16807,\ \delta = 2401,$ the level $N = 16807/240125 = 7$ and the discriminant is $\Delta = (-1)^{6/2}\, 16807 = -16807.$ The character of Q_7 is the unique Dirichlet character χ such that $\chi(d) = \left(\frac{-16807}{d}\right)$ for $d \in (\mathbb{Z}/7\mathbb{Z})^{\times}$, where $\left(\frac{-16807}{d}\right)$ is the Kronecker symbol. Obviously,

$$\chi(d) = \chi_{-7}(d) = \left(\frac{-7}{d}\right) \text{ for } d \in (\mathbb{Z}/7\mathbb{Z})^{\times}.$$

The corresponding theta function $\Theta_{Q_7}(q)$ is a modular form of weight 3 with character $\left(\frac{-7}{d}\right)$ for $d \in (\mathbb{Z}/7\mathbb{Z})^{\times}$. Here $k = 3,\ N = 7$ and we have

$$\dim M_3(\Gamma_0(7), \chi_{-7}) - \dim S_{-1}(\Gamma_0(7), \chi_{-7}) = \frac{(3-1)\cdot 7}{12}\left(1 + \frac{1}{7}\right)$$

$$+\frac{1}{2}\lambda(r_7, s_7, 7) - v_{-1}\alpha(\chi) - \mu_{-1}\beta(\chi)$$

$$= \frac{4}{3} + \frac{1}{2}2\cdot 7^{1-1} - 0\cdot\alpha(\chi) + \frac{1}{3}\cdot\left(\left(\frac{-7}{2}\right) + \left(\frac{-7}{4}\right)\right) = \frac{7}{3} + \frac{1}{3}(1+1) = 3,$$

by Theorem [1.1]. Since $\dim S_k(\Gamma_0(N), \chi) = 0$ for $k \le 0$, the result $\dim M_3(\Gamma_0(7), \chi_{-7}) = 3$ follows. By Corollary 2.7 in [1], this space is generated by two linearly independent Eisenstein series:

$$G_{3,7}(q) = \frac{L(1-3, \chi_{-7})}{2} + \sum_{n=1}^{\infty}\left(\sum_{d>0, d|n}\chi_{-7}(d)\, d^{3-1}\right) q^n = \frac{L(-2, \chi_{-7})}{2} + q + O\left(q^2\right),$$

$$H_{3,7}(q) = \sum_{n=1}^{\infty}\left(\sum_{d>0, d|n}\chi_{-7}(n/d)\, d^{3-1}\right) q^n = q + O\left(q^2\right).$$

On the other hand, Kachakhidze constructed a basis of cusp forms of

$$S_k(\Gamma_0(7), \chi_{-7}), 3 \le k \le 5$$

in [4, page 66]. Taking $k = 3,$ we see that there is only one element in the basis, i.e.,

$$\Theta_{F_1,\varphi}(q), \text{ where } F_1(x_1, x_2) = x_1^2 + x_1x_2 + 2x_2^2,\ \varphi(x_1, x_2) = \left(x_1^2 - 2x_2^2\right).$$

Obviously,

$$\Theta_{F_1,\varphi}(q) = \sum_{n=1}\left(\sum_{F_1=n}\left(x_1^2 - 2x_2^2\right)\right) q^n.$$

Now after the calculation of

$$F_1 = x_1^2 + x_1x_2 + 2x_2^2 = n$$

for $n = 1, 2, \cdots, 19,$ we get

$$\Theta_{F_1,\varphi}(q) = q - 4q^2 + 10q^4 - 14q^7 - 6q^8 + 18q^9 - 12q^{11} + 42q^{14} - 22q^{16} - 54q^{18} + \cdots.$$

The theta series $\Theta_{Q_7}(q)$ associated to Q_7 is given as linear combination of two Eisenstein series and cusp forms as

$$\frac{2}{L(-2,\chi_{-7})}G_{3,7}+c_2H_{3,7}+c_3\Theta_{F_1,\varphi}.$$

Now,

$$\begin{aligned}\frac{2}{L(-2,\chi_{-7})}\sum_{d>0,d|1}\chi_{-7}(d)\,d^2+c_2+c_3 &= 0\\ \frac{2}{L(-2,\chi_{-7})}\sum_{d>0,d|2}\chi_{-7}(d)\,d^2+\left(\sum_{d>0,d|2}\chi_{-7}(2/d)\,d^2\right)c_2-4c_3 &= 0,\end{aligned}$$

since $Q_7=1$ and $Q_7=2$ doesn't have any integer solutions. We immediately obtain that

$$c_2=-\frac{2}{L(-2,\chi_{-7})}\text{ and }c_3=0.$$

Hence, the theta series $\Theta_{Q_7}(q)$ associated to Q_7 is a linear combination of two Eisenstein series as

$$\Theta_{Q_7}(q)=\frac{2}{L(-2,\chi_{-7})}\left(G_{3,7}-H_{3,7}\right)=$$

$$\frac{2}{L(-2,\chi_{-7})}\left(\frac{L(-2,\chi_{-7})}{2}+\sum_{n=1}^{\infty}\left(\sum_{d>0,d|n}\left(\chi_{-7}(d)-\chi_{-7}(n/d)\right)d^2\right)q^n\right).$$

So, the representation number $\mathrm{r}(Q_7,n)$ is given by the following simple formula

$$r(n,Q_7)=\frac{2}{L(-2,\chi_{-7})}\sum_{d>0,d|n}\left(\chi_{-7}(d)-\chi_{-7}(n/d)\right)d^2.$$

The values

$$L(0,\chi_{-3})=\frac{1}{3},L(-1,\chi_5)=-\frac{1}{2}B_{2,\chi_5}=-2/5,L(-2,\chi_{-7})=-\frac{1}{3}B_{3,\chi_{-7}}=-16/7$$

follow from direct calculations.

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