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G-frames as special frames

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Abstract: G-frames are generalizations of ordinary frames for Hilbert spaces. In the present paper we study frames, and operators on a special separable Hilbert C^*-module, $B(H,K)$, where H and K are Hilbert spaces, and we prove that every g-frame for H is a frame for $B(H,K)$ and vice versa. Also, we derive some relationships between g-Riesz bases for H and Riesz bases in $B(H,K)$. Similar results for orthogonal bases will be discussed.

Key words: Hilbert C^*-module, Frame, g-Frame, Riesz basis, g-Riesz basis, Orthogonal basis, g-Orthonormal basis

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [5] in the study of nonharmonic Fourier series. More than thirty years later, Young [17] and Daubechies et al. [4] reintroduced frames and used them as bases in Hilbert spaces, especially $L^2(\mathrm{R})$. Recent research has shown that frame theory has applications in pure [2, 8] and applied mathematics [7], harmonic analysis [3] and even quantum communication [1].

Generalizations of frames have also been used in many applications. The best-known generalizations of frames, called g-frames, were defined by Sun [15]. The class of g-frames includes the class of ordinary frames. Also, frames in Hilbert C^*-modules were extended to unital C^*-algebras by Frank and Larson [6].

For Hilbert spaces H and K, the Banach space $B(H,K)$ of all bounded linear operators from H into K is a Hilbert $B(K)$-module.

The goal of this paper is to show that a sequence of operators in $B(H,K)$ is a g-frame for H if and only if it is a frame for $B(H,K)$. We then conclude that g-frames are frames. Also, we illustrate some differences between g-orthonormal and g-Riesz bases. We show that the set of Riesz bases in $B(H,K)$ contains the set of g-Riesz bases, but that the sets are not equal. The same relation is true for orthogonal bases and g-orthonormal bases.

The rest of the paper is organized as follows. In Section 2, we review Hilbert C^*-modules and some properties of the operators on $B(H,K)$, which will be used in Section 3. In Section 3, we offer a necessary and sufficient condition for a sequence of operators in $B(H,K)$ to be a g-frame. Also, we study the relations between orthogonal and Riesz bases in $B(H,K)$ considered as a Hilbert C^*-module with g-orthogonal and g-Riesz bases for H.

Throughout the paper, I and $\mathbb{C}$ denote the sets of all integers and all complex numbers, respectively.

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2. Frames in Hilbert C^*-modules and $B(H,K)$

2.1. Review of the Hilbert C^*-modules

Hilbert C^*- modules form a category between the category of Banach spaces and the category of Hilbert spaces. The basic idea was to study modules over C^*- algebras instead of linear spaces and to allow an inner product to take its values in a more general C^*- algebra than that of the complex numbers $\mathbb{C}$. The structure was used by Kaplansky [9] in 1952 and was investigated in detail by Rieffel [13] and Paschke [12] in 1972–73.

We shall give only a brief introduction to the theory of Hilbert C^*- modules to make our explanations self-contained. For a comprehensive account, readers are referred to the books by Lance [10] and Wegge-Olsen [16].

Let A be a C^*-algebra and H be a (left) A-module. Suppose that the linear structures given on A and H are compatible, i.e., $\lambda(ax) = a(\lambda x)$ for every $\lambda \in \mathbb{C}$, $a \in A$ and $x \in H$. If there exists a mapping $\langle .,. \rangle : H \times H \longrightarrow A$ with the properties,

(i) $\langle x, x\rangle \geq 0$ for every $x \in H$,

(ii) $\langle x, x\rangle = 0$ if and only if $x = 0$,

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ for every $x, y \in H$,

(iv) $\langle ax, y\rangle = a\langle x, y\rangle$ for every $a \in A$ and $x, y \in H$,

(v) $\langle x+y, z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for every $x, y, z \in H$,

then the pair $\{H, \langle .,. \rangle\}$ is called a (left) pre-Hilbert A-module. The map $\langle .,. \rangle$ is called an A-valued inner product. If the pre-Hilbert A-module $\{H, \langle .,. \rangle\}$ is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$, then it is called a Hilbert C^*-module over A, or a Hilbert A-module. For example, the C^*-algebra A itself can be recognized to become a Hilbert A-module if the inner product is defined by $\langle a, b\rangle = ab^*$, for all $a, b \in A$.

Frames, orthogonal bases and Riesz bases for Hilbert C^*-modules were defined by Frank and Larson [6].

Let A be a unital C^*-algebra. A sequence $\{x_i\}_{i\in I}$ of elements in a Hilbert A-module H is called a frame for H if there exist two constants $C, D > 0$, such that

$$C\langle x, x\rangle \leq \sum_{i\in I} \langle x, x_i\rangle \langle x_i, x\rangle \leq D\langle x, x\rangle, \quad \forall x \in H,$$

where the sum converges weakly. The constants C and D are called the lower and upper frame bounds, respectively.

The frame $\{x_i\}_{i\in I}$ is called a tight frame if $C = D$, and is said to be a Parseval or normalizes tight frame if $C = D = 1$. Likewise, $\{x_i\}_{i\in I}$ is called a Bessel sequence for H with positive bound D if

$$\sum_{i\in I} \langle x, x_i\rangle \langle x_i, x\rangle \leq D\langle x, x\rangle, \quad \forall x \in H.$$

A sequence $\{x_i\}_{i\in I}$ in a Hilbert A-module H is called an orthogonal basis for H if it is a generating set (i.e., the A-linear hull of $\{x_i\}_{i\in I}$ is weak-dense in H) such that

i) $\langle x_i, x_j \rangle = 0$ for each $i \neq j$,

ii) $\|x_i\| = 1$ for each $i \in I$,

iii) the A-linear combinations $\sum_{i\in S} a_i x_i$ with coefficients $\{a_i : i \in S\} \subseteq A$ and $S \subseteq I$ are equal to zero if and only if every summand $a_i x_i$ is equal to zero, $i \in S$.

A sequence $\{x_i\}_{i\in I}$ in a separable Hilbert A-module H is called a Riesz basis for H if it is a frame and a generating set with the additional property that A-linear combinations $\sum_{i\in I} a_i x_i$ with coefficients $\{a_i : i \in S\} \subseteq A$ and $S \subseteq I$ are equal to zero if and only if every summand $a_i x_i$ is equal to zero, $i \in S$.

2.2. Positive operators in $B(H,K)$

In the rest of this paper, let H and K be separable Hilbert spaces and let $B(H,K)$ be the set of all bounded linear operators from H into K. $B(H,K)$ is a Hilbert $B(K)$-module with a $B(K)$-valued inner product $\langle S,T \rangle = ST^*$ for all $S,T \in B(H,K)$, and with a linear operation of $B(K)$ on $B(H,K)$ by the composition of operators. On the other hand, $B(H,K)$ is also a Banach space with respect to the operator norm $\|T\|_o = \sup\{\|Tx\| : \|x\| \leq 1, x \in H\}$, for all $T \in B(H,K)$. The norm in $B(H,K)$ considered as a Hilbert $B(K)$-module is defined by

$$\|T\|_{c^*} = \| \langle T,T \rangle \|_o^{\frac{1}{2}} = \|TT^*\|_o^{\frac{1}{2}} = \|T\|_o.$$

Therefore, the norms in $B(H,K)$ considered as a Hilbert A-module and as a Banach space are the same. However, $B(H,K)$ is not a Hilbert space, and some facts that are true for Hilbert spaces may not hold for $B(H,K)$.

In the study of frame operators on $B(H,K)$, we need to know some facts about operators and, especially, positive operators on $B(H,K)$.

Proposition 2.1 *Let S be an operator on $B(H,K)$, then $\langle SU,U \rangle = 0$ for all $U \in B(H,K)$ if and only if $S = 0$.*

Proof Clearly if $S = 0$, then $\langle SU,U \rangle = 0$ for all $U \in B(H,K)$. On the other hand, we have

$$\langle S(U+V), U+V \rangle = 0, \quad \forall U,V \in B(H,K),$$

so that $\langle SV,U \rangle + \langle SU,V \rangle = 0$. If V changes with iV, we have

$$i \langle SV,U \rangle - i \langle SU,V \rangle = 0,$$

that implies $2i \langle SU,V \rangle = 0$ or $\langle SU,V \rangle = 0$. By setting $V = SU$, we conclude that $SU = 0$ for all $U \in B(H,K)$ and so $S = 0$. □

A map S on $B(H,K)$ is said to be adjointable if there exists a map S^* on $B(H,K)$ such that

$$\langle SU,V \rangle = \langle U,S^*V \rangle, \quad \forall U,V \in B(H,K).$$

Such a map S^* is called the adjoint of S. It follows that S and S^* are bounded linear $B(K)$-module maps. By $B(B(H,K))$ we denote the set of all adjointable linear $B(K)$-module maps on $B(H,K)$, and $B_b(B(H,K))$

denotes the set of all bounded linear $B(K)$-module maps on $B(H,K)$. An adjointable map S on $B(H,K)$ is said to be self-adjoint if $S = S^*$ [16].

Proposition 2.2 *Let S be a adjointable linear $B(K)$-module map on $B(H,K)$. Then*

i) S is self adjoint if and only if $\langle SU, U\rangle$ is self adjoint for all $U \in B(H,K)$,

ii) S is self adjoint if and only if for all $U \in B(H;K)$, $\langle SU, U\rangle$ is normal and the spectrum of $\langle SU, U\rangle$ is a subset of the real line.

Proof i) If S is self adjoint, then

$$\langle SU, U\rangle = \langle U, SU\rangle = \langle SU, U\rangle^* \qquad \forall U \in B(H,K).$$

Conversely, if $\langle SU, U\rangle = \langle SU, U\rangle^*$ for all $U \in B(H,K)$, then

$$\langle SU, U\rangle = \langle U, SU\rangle = \langle S^*U, U\rangle$$

or

$$\langle SU, U\rangle = \langle S^*U, U\rangle \qquad \forall U \in B(H,K)$$

and this means $S = S^*$.
ii) If $S = S^*$, then

$$\langle SU, U\rangle = \langle U, S^*U\rangle = \langle U, SU\rangle = \langle SU, U\rangle^*$$

is a self adjoint operator on $B(K)$ for each $U \in B(H,K)$, and by [14] its spectrum is a subset of the real line. Conversely, let the spectrum of $\langle SU, U\rangle$ be a subset of the real line for all $U \in B(H,K)$. For $\alpha \in \mathbb{C}$ and $U, V \in B(H,K)$ we have

$$\langle S(U+\alpha V), U+\alpha V\rangle = \langle SU, U\rangle + \bar{\alpha}\langle SV, U\rangle + \alpha\langle SU, V\rangle + |\alpha|^2\langle SV, V\rangle.$$

Since the spectrum of $\langle S(U+\alpha V), U+\alpha V\rangle$ is a subset of the real line,

$$\langle S(U+\alpha V), U+\alpha V\rangle = \langle S(U+\alpha V), U+\alpha V\rangle^*,$$

and

$$\begin{aligned}\alpha\langle SV, U\rangle + \bar{\alpha}\langle SU, V\rangle &= \bar{\alpha}\langle SV, U\rangle^* + \alpha\langle SU, V\rangle^* \\ &= \bar{\alpha}\langle U, SV\rangle + \alpha\langle V, SU\rangle \\ &= \bar{\alpha}\langle S^*U, V\rangle + \alpha\langle S^*V, U\rangle.\end{aligned}$$

By setting $\alpha = 1$ and $\alpha = i$,

$$\langle SV, U\rangle + \langle SU, V\rangle = \langle S^*U, V\rangle + \langle S^*V, U\rangle,$$

$$i\langle SV, U\rangle - i\langle SU, V\rangle = -i\langle S^*U, V\rangle + i\langle S^*V, U\rangle.$$

Now by product i in the second equality, we obtain $\langle SU, V\rangle = \langle S^*U, V\rangle$. Therefore, $S = S^*$. □

Remark 2.3 *An element $S \in B(B(H,K))$ is said to be positive if $S = S^*$ and the spectrum of S is contained in the positive real line [11]. Wegge-Olsen [16] has shown that $S \geq 0$ if and only if the spectrum of $\langle ST, T\rangle$ is a subset of [0,∞) for all $T \in B(H,K)$.*

Proposition 2.4 *Let S be a positive operator in $B(B(H,K))$. Then*

$$\|S\| = \sup_{\|T\|\leq 1} \|\langle ST, T\rangle\|.$$

Proof Since $B(B(H,K))$ is a C^*-algebra [16] and S is positive, $S^{\frac{1}{2}}$ exists. Then, we have

$$\begin{aligned}
\sup_{\|T\|\leq 1} \|\langle ST, T\rangle\| &= \sup_{\|T\|\leq 1} \left\|\left\langle S^{\frac{1}{2}} S^{\frac{1}{2}} T, T\right\rangle\right\| \\
&= \sup_{\|T\|\leq 1} \left\|\left\langle S^{\frac{1}{2}} T, S^{\frac{1}{2}} T\right\rangle\right\| \\
&= \sup_{\|T\|\leq 1} \left\|S^{\frac{1}{2}} T\right\|^2 \\
&= \left\|S^{\frac{1}{2}}\right\|^2 = \left\|S^{\frac{1}{2}} (S^{\frac{1}{2}})^*\right\| \\
&= \left\|S^{\frac{1}{2}} S^{\frac{1}{2}}\right\| = \|S\|.
\end{aligned}$$

□

Lemma 2.5 *Let $\Lambda \in B(H)$. Then Λ is positive if and only if $T\Lambda T^* \in B(K)$ is positive for all $T \in B(H,K)$.*

Proof Let Λ be positive. Since $B(H)$ is a C^*- algebra, there is $\Gamma \in B(H)$ such that $\Lambda = \Gamma\Gamma^*$, and so

$$T\Lambda T^* = T\Gamma\Gamma^* T^* = T\Gamma(T\Gamma)^*.$$

On the other hand, for all $f \in H$, we have

$$\langle T\Gamma(T\Gamma)^* f, f\rangle = \langle (T\Gamma)^* f, (T\Gamma)^* f\rangle = \|(T\Gamma)^* f\|^2 \geq 0.$$

Hence $T\Lambda T^*$ is positive.

Conversely, let $f \in H$ be arbitrary. We can find $g \in K$ and $T \in B(H,K)$ such that $T^* g = f$. Then by the positivity of $T\Lambda T^*$,

$$\langle \Lambda f, f\rangle = \langle T\Lambda T^* g, g\rangle \geq 0.$$

Therefore, Λ is positive. □

3. Operator sequences, g-sequences and their relations

3.1. Frames

A sequence $\{T_i \in B(H,K) : i \in I\}$ is said to be a frame for B(H,K) if there exist $0 < A, B < \infty$ such that

$$A \langle T, T\rangle \leq \sum_{i\in I} \langle T, T_i\rangle \langle T_i, T\rangle \leq B \langle T, T\rangle, \quad \forall T \in B(H,K), \qquad (3.1)$$

where the series converges in the strong operator topology. The frame operator on B(H,K) is defined by

$$S\colon B(H,K) \longrightarrow B(H,K),$$

$$ST = \sum_{i\in I} \langle T, T_i\rangle T_i = \sum_{i\in I} TT_i^* T_i.$$

Proposition 2.2, Remark 2.3 and (3.1) assert that S is a positive, self adjoint and invertible operator, and

$$\langle ST, T\rangle = \sum_{i\in I} TT_i^* T_i T^* = \sum_{i\in I} \langle T, T_i\rangle \langle T_i, T\rangle .$$

Therefore, we have

$$A\langle T, T\rangle \leq \langle ST, T\rangle \leq B\langle T, T\rangle .$$

Convergence in the definition of frames, Bessel sequences, orthogonal and Riesz bases in B(H,K) as a Hilbert $B(K)$-module is in the strong operator topology.

Various generalizations of frames have been studied by many authors. Sun [15] introduced a type of frames called g-frames, and showed that most generalizations of frames can be regarded as special cases of g-frames. Here we point out that g-frames can be regarded as frames in $B(H,K)$ with the same bounds.

A sequence $\{\Lambda_i \in B(H,K_i) : i \in I\}$ is called a generalized frame, or simply a g-frame for H with respect to a sequence of Hilbert spaces $\{K_i\}_{i\in I}$ if there exist two positive constants A and B such that

$$B\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq A\|f\|^2, \quad \forall f \in H.$$

A and B are called the lower and upper g-frame bounds, respectively. A g-frame is called tight if $A = B$ and Parseval g-frame if $A = 1$. In simple terms, $\{\Lambda_i\}_{i\in I}$ is called a g-frame for H whenever the space sequence $\{K_i : i \in I\}$ is clear, and also a g-frame for H with respect to K whenever $K_i = K$ for each $i \in I$. A sequence $\Lambda_i \in B(H,K_i) : i \in I$ is called a g-Bessel sequence with bound B if we have only an upper bound in the definition of g-frames. The space $(\sum_{i\in I} \oplus K_i)_{l_2}$ is defined by

$$\left(\sum_{i\in I} \oplus K_i\right)_{l_2} = \left\{\{f_i\}_{i\in I} : f_i \in K_i, \quad i \in I \quad \text{and} \quad \sum_{i\in I} \|f_i\|^2 \leq \infty \right\}$$

and has the inner product,

$$\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle .$$

It is clear that $(\sum_{i\in I} \oplus K_i)_{l_2}$ is a Hilbert space and contains K_i as a subspace, $i \in I$.

Remark 3.1 *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{K_i\}_{i\in I}$ and let $K = (\sum_{i\in I} \oplus K_i)_{l_2}$. For $i \in I$, define $\Lambda_i' : H \longmapsto K$ by*

$$\Lambda_i' f = (..., 0, 0, 0, \Lambda_i f, 0, 0, 0, ...), \quad \forall f \in H.$$

Then

$$\|\Lambda_i{}' f\| = \|\Lambda_i f\|, \quad \forall i \in I, \forall f \in H.$$

Hence $\{\Lambda_i\}_{i\in I}$ *is a g-frame for* H *with respect to* $\{K_i\}_{i\in I}$ *if and only if* $\{\Lambda_i'\}_{i\in I}$ *is a g-frame for* H *with respect to* K. *Therefore, without loss of generality, we may deal with g-frames for* H *with respect to* K.

Now we shall show that a g-frame for H *with respect to* K *is a frame for* $B(H,K)$, *and vice versa.*

Theorem 3.2 *Let* $\{\Lambda_i \in I\}_{i\in I}$ *be a sequence in B(H,K). Then it is a frame for* $B(H,K)$ *considered as a Hilbert* C^**-module if and only if it is a g-frame for H with respect to* K.

Proof Let $\{\Lambda_i \in B(H,K) : i \in I\}$ be a g-frame for H with respect to K. Then there are positive constants A and B, such that

$$B\langle f,f\rangle \leq \sum_{i\in I} \langle \Lambda_i \Lambda_i{}^* f, f\rangle \leq A\langle f,f\rangle, \quad \forall f \in H.$$

Hence

$$BI_H \leq \sum_{i\in I} \Lambda_i^* \Lambda_i \leq AI_H.$$

Lemma 2.5 asserts that the inequality

$$BTT^* \leq \sum_{i\in I} T\Lambda_i^*\Lambda_i T^* \leq ATT^*,$$

satisfies for all $T \in B(H,K)$. Thus,

$$B\langle T,T\rangle \leq \sum_{i\in I} \langle T,\Lambda_i\rangle \langle \Lambda_i,T\rangle \leq A\langle T,T\rangle, \quad \forall T \in B(H,K),$$

and $\{\Lambda_i\}_{i\in I}$ is a frame for $B(H,K)$. Conversely, let $\{\Lambda_i \in B(H,K) : i \in I\}$ be a frame for $B(H,K)$ and $f \in H$. We can choose T in B(H,K) and g in K such that $T^*g = f$. Therefore,

$$\begin{aligned}
\left\langle \sum_{i\in I} \langle T,\Lambda_i\rangle \langle \Lambda_i,T\rangle g, g \right\rangle &= \left\langle \sum_{i\in I} T\Lambda_i^*\Lambda_i T^* g, g \right\rangle \\
&= \sum_{i\in I} \langle T\Lambda_i^*\Lambda_i T^* g, g\rangle \\
&= \sum_{i\in I} \langle \Lambda_i T^* g, \Lambda_i T^* g\rangle \\
&= \sum_{i\in I} \langle \Lambda_i f, \Lambda_i f\rangle \\
&= \sum_{i\in I} \|\Lambda_i f\|^2.
\end{aligned}$$

Also we have

$$\langle B\langle T,T\rangle g, g\rangle = \langle BTT^* g, g\rangle = B\langle T^*g, T^*g\rangle = B\langle f,f\rangle = B\|f\|^2.$$

Thus

$$B\langle T,T\rangle \le \sum_{i\in I} \langle T,\Lambda_i\rangle \langle \Lambda_i,T\rangle \le A\langle T,T\rangle,$$

implies that

$$B\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le A\|f\|^2, \quad \forall f\in H,$$

as desired. □

The following are immediate consequences.

Corollary 3.3 *The sequence* $\{\Lambda_i \in B(H,K) : i\in I\}$ *is a tight frame for* $B(H,K)$ *if and only if it is a g-tight frame for* H *with respect to* K*.*

Corollary 3.4 *The sequence* $\{\Lambda_i \in B(H,K) : i\in I\}$ *is a Bessel sequence for* $B(H,K)$ *if and only if it is a g-Bessel sequence for* H *with respect to* K*.*

Remark 3.5 *Let* $\{\Lambda_i \in B(H,K) : i\in I\}$ *be a g-frame for* H*. The g-frame operator of* $\{\Lambda_i\}_{i\in I}$ *is defined by*

$$S_g : H \longrightarrow H, \qquad f \longmapsto \sum_{i\in I} \Lambda_i^* \Lambda_i f.$$

Also, the frame operator of the frame $\{\Lambda_i\}_{i\in I}$ *is defined by* $ST = \sum_{i\in I} T\Lambda_i^*\Lambda_i$*. Therefore,* $ST = TS_g$*, and from this equation, for any* $T\in B(H,K)$ *a reconstruction formula is derived by* $T = S^{-1}TS_g$*.*

3.2. Orthonormal bases and Riesz bases

Now, we study the relations between g-orthonormal bases and g-Riesz bases for H with respect to K with orthogonal bases and Riesz bases for $B(H,K)$ considered as a Hilbert C^*-module.

A sequence $\{\Lambda_i : i\in I\}$ is called a g-orthonormal basis for H with respect to K if it satisfies the following:

1) $\left\langle \Lambda_i^* f, \Lambda_j^* g\right\rangle = \delta_{i,j}\langle f,g\rangle, \quad \forall i,j\in I$ and $f,g\in H$,

2) $\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2, \quad \forall f\in H$.

If $\{\Lambda_i : i\in I\}$ is a sequence in $B(H,K)$ and $\{f : \Lambda_i f = 0, i\in I\} = \{0\}$, then $\{\Lambda_i : i\in I\}$ is called g-complete.

A sequence $\{\Lambda_i \in B(H,K) : i\in I\}$ is called a g-Riesz basis for H with respect to K if it is g-complete and there are positive constants A and B such that

$$A\sum_{i\in I_1} \|g_i\|^2 \le \|\sum_{i\in I_1} \Lambda_i^* g_i\|^2 \le B\sum_{i\in I_1} \|g_i\|^2,$$

for any finite subset I_1 of I and $\{g_i\}_{i\in I_1} \subseteq K$ [15].

The following theorem provide that every g-orthonormal basis for H is an orthogonal basis for $B(H,K)$. We will give an example to show that the converse of the theorem is not correct.

Theorem 3.6 *If* $\{\Lambda_i \in B(H,K) : i \in I\}$ *is a g-orthonormal basis for* H *with respect to* K*, then it is an orthogonal basis for* $B(H,K)$ *considered as a Hilbert* C^**-module.*

Proof Since $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for H, for $i \neq j$, we have $\langle \Lambda_i, \Lambda_j\rangle = \Lambda_i\Lambda_j^* = 0$ and $\|\Lambda_i\|^2 = \|\langle \Lambda_i, \Lambda_i\rangle\| = \|\Lambda_i\Lambda_i^*\| = \|I_K\| = 1$, where I_K is the identity operator on K. Now suppose that $\sum_{i\in I} T_i\Lambda_i = 0$ where $T_i \in B(K), i \in I$. We have

$$\begin{aligned} 0 = \left\langle \sum_{i\in I} T_i\Lambda_i, \Lambda_j \right\rangle &= \sum_{i\in I} \langle T_i\Lambda_i, \Lambda_j\rangle \\ &= \sum_{i\in I} T_i \langle \Lambda_i, \Lambda_j\rangle \\ &= T_j \langle \Lambda_j, \Lambda_j\rangle \\ &= T_j I_H = T_j. \end{aligned}$$

Therefore, $T_j = 0$ and $T_j\Lambda_j = 0$, for each $j \in I$. It remains to show that every $T \in B(H,K)$ can be generated by $\{\Lambda_i\}_{i\in I}$. The second condition of g-orthonormal basis, $\sum_{i\in I}\|\Lambda_i f\|^2 = \|f\|^2$ for all $f \in H$, implies that $\sum_{i\in I}\Lambda_i^*\Lambda_i = I_H$. Then, for every $T \in B(H,K)$ we have

$$\begin{aligned} T = TI_H &= T\sum_{i\in I}\Lambda_i^*\Lambda_i \\ &= \sum_{i\in I} T\Lambda_i^*\Lambda_i \\ &= \sum_{i\in I}\langle T, \Lambda_i\rangle \Lambda_i \\ &= \sum_{i\in I} U_i\Lambda_i \end{aligned}$$

where $U_i = \langle T, \Lambda_i\rangle$ belongs to $B(K)$, for each $i \in I$. This completes the proof of the theorem. □

The relation between Riesz bases for $B(H,K)$ and g-Riesz bases for H with respect to K is similar to the above theorem.

Theorem 3.7 *If* $\{\Lambda_i \in B(H,K) : i \in I\}$ *is a g-Riesz basis for* H *with respect to* K*, then it is a Riesz basis for* $B(H,K)$ *considered as a Hilbert* C^**-module.*

Proof Let $\{\Lambda_i \in B(H,K) : i \in I\}$ be a g-Riesz basis for H with respect to K. By ([15], Corollary 3.3) $\{\Lambda_i\}_{i\in I}$ is a g-frame and by Theorem 3.2 it is a frame for $B(H,K)$. It is clear that $\Lambda_i \neq 0$ for each $i \in I$. Now let $\sum_{i\in I} T_i\Lambda_i = 0$, where $T_i \in B(K)$. We have $\sum_{i\in I}\Lambda_i^*T_i^* = 0$, therefore, $\sum_{i\in I}\Lambda_i^*T_i^*g = 0$, for each $g \in K$. By the definition of g-Riesz basis, $\sum_{i\in I}\|T_i^*g\|^2 = 0$, then $\|T_i^*g\|^2 = 0$, for each $i \in I$ and $g \in K$. Therefore, $T_i = 0$ and hence $T_i\Lambda_i = 0$ for each $i \in I$. The invertibility of the frame operator S implies that

$$T = \sum_{i\in I}\left\langle S^{-1}T, \Lambda_i\right\rangle \Lambda_i, \quad \forall T \in B(H,K).$$

Thus, $\{\Lambda_i\}_{i\in I}$ is a generating set for $B(H,K)$ and the proof is complete. □

By an example we show that the converse of Theorem 3.5 and Theorem 3.6 is not true.

Example 3.8 *Let H be a Hilbert space and $\{\varphi_i\}_{i\in I}$ be an orthonormal basis for H. For $i \in I$, define Λ_i and Λ_i^* by*

$$\Lambda_i : H \mapsto \mathbb{C}^2, \qquad f \longmapsto (\langle f, \varphi_i \rangle, 0),$$

$$\Lambda_i^* : \mathbb{C}^2 \mapsto H, \qquad (c_1, c_2) \longmapsto c_1 \varphi_1.$$

Since

$$\sum_{i\in I} T\Lambda_i^* \Lambda_i T^* = TT^*, \qquad for \quad all \quad T \in B(H, \mathbb{C}^2),$$

the sequence $\{\Lambda_i\}_{i\in I}$ is a Parseval frame for $B(H,\mathbb{C}^2)$. Now let $\{T_i\}_{i\in I}$ be a sequence in $B(\mathbb{C}^2)$ and $\sum_{i\in I} T_i\Lambda_i = 0$. Then, for each $f \in H$ we have

$$0 = \sum_{i\in I} T_i \Lambda_i f = \sum_{i\in I} T_i(\langle \varphi_i, f \rangle, 0) = \sum_{i\in I} \langle \varphi_i, f \rangle T_i(1, 0).$$

By the orthonormality of $\{\varphi_i\}_{i\in I}$, $T_i(1,0) = 0$, hence, $T_i\Lambda_i = 0$ for all $i \in I$. Also, $\sum_{i\in I} \Lambda_i^*\Lambda_i = I_H$ implies that

$$T = \sum_{i\in I} T\Lambda_i^*\Lambda_i = \sum_{i\in I} \langle T, \Lambda_i \rangle \Lambda_i.$$

This shows that $\{\Lambda_i\}_{i\in I}$ generates $B(H,\mathbb{C}^2)$ as $B(\mathbb{C}^2)$-module. Therefore, all conditions of a Riesz basis are satisfied and $\{\Lambda_i\}_{i\in I}$ is a Riesz basis for $B(H,\mathbb{C}^2)$. But $\{\Lambda_i\}_{i\in I}$ is not a g-Riesz basis since $\Lambda_i^*(0,1) = 0$, which implies that $A = 0$ in the definition of a g-Riesz basis.

However, $\langle \Lambda_i, \Lambda_i \rangle (0,1) = \Lambda_i\Lambda_i^* = (0,0)$, $\Lambda_i\Lambda_i^* \neq I_{\mathbb{C}^2}$, $\|\Lambda_i\| = 1$ and $\langle \Lambda_i, \Lambda_j \rangle = 0$ for $i \neq j$. Therefore, $\{\Lambda_i\}_{i\in I}$ is an orthogonal basis for $B(H,\mathbb{C}^2)$. On the other hand, $\langle \Lambda_i^*(0,1), \Lambda_i^*(0,1) \rangle = 0$ and $\delta_{ii} \langle (0,1), (0,1) \rangle = 1$, imply that $\{\Lambda_i\}_{i\in I}$ is not a g-orthonormal basis for H with respect to K.

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