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On the maximal operators of Vilenkin-Fejér means

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Abstract: The main aim of this paper is to prove that the maximal operator $\widetilde{\sigma}^{*}f := \sup_{n\in P} \frac{|\sigma_n f|}{\log^2(n+1)}$ is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$, where $\sigma_n f$ are Fejér means of bounded Vilenkin-Fourier series.

Key words and phrases: Vilenkin system, Fejér means, martingale Hardy space

1. Introduction

In one-dimensional case the weak type inequality

$$\mu\left(\sigma^{*}f > \lambda\right) \leq \frac{c}{\lambda}\left\|f\right\|_1 \qquad (\lambda > 0)$$

can be found in Zygmund [19] for trigonometric series, in Schipp [11] for Walsh series and in Pál and Simon [10] for bounded Vilenkin series. Again in one-dimensional, Fujii [4] and Simon [13] verified that σ^{*} is bounded from H_1 to L_1. Weisz [16] generalized this result and proved the boundedness of σ^{*} from the martingale space H_p to the space L_p for $p > 1/2$. Simon [12] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ is due to Goginava [7], (see also [3]). In the endpoint case, $p = 1/2$, two positive results were showed. Weisz [18] proved that σ^{*} is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. For Walsh-Paley system in 2008 Goginava [6] proved that the maximal operator $\widetilde{\sigma}^{*}$ defined by

$$\widetilde{\sigma}^{*}f := \sup_{n\in P}\frac{|\sigma_n f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. He also proved that for any nondecreasing function $\varphi : P_{+} \to [1,\ \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{\log^2(n+1)}{\varphi(n)} = +\infty \tag{1}$$

the maximal operator

$$\sup_{n\in P}\frac{|\sigma_n f|}{\varphi(n)}$$

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is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

For a Walsh-Kaczmarz system an analogical theorem is proved in [9].

The main aim of this paper is to prove that the maximal operator $\widetilde{\sigma}^{*}f$ with respect to Vilenkin system is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$ (see Theorem 1).

We also prove that under the condition (1) the maximal operator

$$\sup_{n\in P}\frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. Actually, we prove stronger result (see Theorem 2) than the unboundedness of the maximal operator $\widetilde{\sigma}^{*}f$ from the Hardy space $H_{1/2}$ to the spaces $L_{1/2}$. In particular, we prove that

$$\sup_{n\in P}\left\|\frac{\sigma_n f}{\varphi(n)}\right\|_{L_{1/2}}=\infty.$$

2. Definitions and notation

Let P_{+} denote the set of the positive integers , $P:=P_{+}\cup\{0\}$.

Let $m:=(m_0,m_{1....})$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k}:=\{0,1,...m_k-1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}s.

The direct product μ of the measures

$$\mu_k(\{j\}):=1/m_k \qquad (j\in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m)=1$.

If $\sup_n m_n<\infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded then G_m is said to be an unbounded Vilenkin group. **In this paper we discuss bounded Vilenkin groups only.**

The elements of G_m are represented by sequences

$$x:=(x_0,x_{1,...,}x_{j,...}) \qquad (x_k\in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m

$$I_0(x):=G_m,$$

$$I_n(x):=\{y\in G_m \mid y_0=x_0,...y_{n-1}=x_{n-1}\}(x\in G_m,n\in P)$$

Denote $I_n:=I_n(0)$ for $n\in P$ and $\overline{I_n}:=G_m\ \backslash\ I_n$.

Let

$$e_n := (0, ..., 0, x_n = 1, 0, ...) \in G_m \qquad (n \in P) .$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0, ..., 0, x_k \neq 0, 0, ..., 0, x_l \neq 0, x_{l+1,...,}x_{N-1}), k < l < N \\ I_N(0, ..., 0, x_k \neq 0, 0, ..., , 0), l = N \end{cases}$$

and

$$I_N^{k,\alpha,l,\beta} := I_N(0, ..., 0, x_k = \alpha, 0, ..., 0, x_l = \beta, x_{l+1,...,}x_{N-1}), k < l < N.$$

It is evident

$$I_N^{k,l} = \bigcup_{\alpha=1}^{m_k-1} \bigcup_{\beta=1}^{m_l-1} I_N^{k,\alpha,l,\beta} \tag{2}$$

and

$$\bar{I_N} = \left(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \right) \cup \left(\bigcup_{k=0}^{N-1} I_N^{k,N} \right). \tag{3}$$

If we define the so-called generalized number system based on m as

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \qquad (k \in P),$$

then every $n \in P$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$ where $n_j \in Z_{m_j} \quad (j \in P)$ and only a finite number of n_js differ from zero. Let $|n| := \max \ \{j \in P; \ n_j \neq 0\}$.

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesgue space.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system.

At first define the complex valued function $r_k(x) : G_m \to C$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \qquad \left(i^2 = -1, x \in G_m, k \in P\right).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in P)$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in P) .$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 14].

Now we introduce analogues of the usual definitions in Fourier-analysis.

If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system ψ in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \ , \ (k \in P) ,$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \ , (\ n \in P_+, S_0 f := 0) ,$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f \, (n \in P_+) \, ,$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, (n \in P_+) \, ,$$

$$K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k \ \, , \, (n \in P_+) \, .$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n \\ 0 & \text{if } \ x \notin I_n. \end{cases} \tag{4}$$

It is well known that

$$\sup_n \int_{G_m} |K_n(x)| \, d\mu(x) \leq c < \infty, \tag{5}$$

$$n \, |K_n(x)| \leq c \sum_{A=0}^{|n|} M_A \, |K_{M_A}(x)| \, . \tag{6}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_{L_p} := \left(\int_{G_m} |f(x)|^p \, d\mu(x)\right)^{1/p} \qquad (0 < p < \infty) \, .$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n $(n \in P)$. Denote by $f = \left(f^{(n)}, n \in P\right)$ a martingale with respect to F_n $(n \in P)$ (for details, see e.g. [15]). The maximal function of a martingale f is defended by

$$f^* = \sup_{n \in P} \left| f^{(n)} \right| .$$

In case $f \in L_1$, the maximal functions are also given by

$$f^*(x) = \sup_{n \in P} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \, \mu(u) \right| .$$

For $0 < p < \infty$ the Hardy martingale spaces H_p (G_m) consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$

If $f \in L_1$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in P)$ is a martingale. If $f = \left(f^{(n)}, n \in P\right)$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x) \, \overline{\Psi}_i(x) \, d\mu(x) \, .$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in P)$ obtained from f .

For the martingale f we consider maximal operators

$$\sigma^* f = \sup_{n \in P} |\sigma_n f| ,$$

$$\widetilde{\sigma}^* f := \sup_{n \in P} \frac{|\sigma_n f|}{\log^2 (n+1)}.$$

A bounded measurable function a is p-atom, if there exist a dyadic interval I, such that

$$\begin{cases} a) & \int_I a d\mu = 0, \\ b) & \|a\|_\infty \leq \mu(I)^{-1/p}, \\ c) & \operatorname{supp}(a) \subset I. \end{cases}$$

3. Formulation of main results

Theorem 1 *The maximal operator*

$$\overset{\sim}{\sigma}^* f := \sup_{n \in P} \frac{|\sigma_n f|}{\log^2 (n+1)}$$

is bounded from the Hardy space $H_{1/2}(G_m)$ to the space $L_{1/2}(G_m)$.

Theorem 2 *Let* $\varphi : P_+ \to [1, \infty)$ *be a nondecreasing function satisfying the condition*

$$\overline{\lim_{n \to \infty}} \frac{\log^2 (n+1)}{\varphi(n)} = +\infty. \tag{7}$$

Then there exists a martingale $f \in H_{1/2}$, such that

$$\sup_{n \in P} \left\| \frac{\sigma_n f}{\varphi(n)} \right\|_{L_{1/2}} = \infty.$$

Corollary 1 *The maximal operator*

$$\sup_{n \in P} \frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

4. Auxiliary propositions

Lemma 1 *[17] Suppose that an operator T is sublinear and for some $0<p\leq 1$*

$$\int_{\bar{I}} |Ta|^p \, d\mu \leq c_p < \infty$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_{L_p(G_m)} \leq c_p \|f\|_{H_p(G_m)}.$$

Lemma 2 *[2, 8] Let $2<A\in P_+$, $k\leq s<A$ and $q_A=M_{2A}+M_{2A-2}+...+M_2+M_0$. Then*

$$q_{A-1}\left|K_{q_{A-1}}(x)\right| \geq \frac{M_{2k}M_{2s}}{4}$$

for

$$x\in I_{2A}\left(0,...,x_{2k}\neq 0,0,...,0,x_{2s}\neq 0,x_{2s+1},...x_{2A-1}\right);$$

$$k=0,1,...,A-3. \qquad s=k+2,k+3,...,A-1.$$

Lemma 3 *[5] Let $A>t$, $t,A\in P$, $z\in I_t\backslash\ I_{t+1}$. Then*

$$K_{M_A}(z)=\begin{cases} 0 & \text{if } z-z_te_t\notin I_A, \\ \frac{M_t}{1-r_t(z)} & \text{if } z-z_te_t\in I_A. \end{cases}$$

Lemma 4 *Let $x\in I_N^{k,l}$, $\qquad k=0,...,N-1$, $l=k+1,...,N$. Then*

$$\int_{I_N} |K_n(x-t)| \, d\mu(t) \leq \frac{cM_lM_k}{M_N^2} \quad \text{when } n\geq M_N.$$

Proof. Let $x\in I_N^{k,\alpha,l,\beta}$. Then applying lemma 3 we have

$$K_{M_A}(x)=0 \text{ when } A>l.$$

Hence we can suppose that $A\leq l$. Let $k<A\leq l$. Then we have

$$|K_{M_A}(x)| = \frac{M_k}{|1-r_k(x)|} \leq \frac{m_kM_k}{2\pi\alpha}. \tag{8}$$

Let $A\leq k<l$. Then it is easy to show that

$$|K_{M_A}(x)| \leq cM_k. \tag{9}$$

Combining (8) and (9), from (2) we have

$$|K_{M_A}(x)| \leq cM_k, \text{ when } x \in I_N^{k,l}$$

and if we apply (6) we conclude that

$$n|K_n(x)| \leq c\sum_{A=0}^{l-1} M_A M_k \leq cM_k M_l. \tag{10}$$

Let $x \in I_N^{k,l}$, for some $0 \leq k < l \leq N-1$. Since $x - t \in I_N^{k,l}$ and $n \geq M_N$ from (10) we obtain

$$\int_{I_N} |K_n(x-t)|\, d\mu(t) \leq \frac{cM_k M_l}{M_N^2}. \tag{11}$$

Let $x \in I_N^{k,N}$, then applying (6) we have

$$\int_{I_N} n|K_n(x-t)|\, d\mu(t) \leq \sum_{A=0}^{|n|} M_A \int_{I_N} |K_{M_A}(x-t)|\, d\mu(t). \tag{12}$$

Let

$$\begin{cases} x = \left(0, ..., 0, x_k \neq 0, 0, ...0, x_N, x_{N+1}, x_q, ..., x_{|n|-1}, , ...\right) \\ t = \left(0, ..., 0, x_N, ...x_{q-1}, t_q \neq x_q, t_{q+1}, ..., t_{|n|-1}, ...\right), \ q = N, ..., |n|-1. \end{cases}$$

Then using Lemma 3 in (12) it is easy to show that

$$\begin{aligned} \int_{I_N} |K_n(x-t)|\, d\mu(t) &\leq \frac{c}{n}\sum_{A=0}^{q-1} M_A \int_{I_N} M_k d\mu(t) \\ &\leq \frac{cM_k M_q}{nM_N} \leq \frac{cM_k}{M_N}. \end{aligned} \tag{13}$$

Let

$$\begin{cases} x = \left(0, ..., 0, x_m \neq 0, 0, ..., 0, x_N, x_{N+1}, x_q, ..., x_{|n|-1}, ...\right), \\ t = \left(0, 0, ..., x_N, ..., x_{|n|-1}, t_{|n|}, ...\right). \end{cases}$$

If we apply Lemma 3 in (12), we obtain

$$\begin{aligned} &\int_{I_N} |K_n(x-t)|\, d\mu(t) \\ \leq\ &\frac{c}{n}\sum_{A=0}^{|n|-1} M_A \int_{I_N} M_k d\mu(t) \leq \frac{cM_k}{M_N}. \end{aligned} \tag{14}$$

Combining (11), (13) and (14) we complete the proof of lemma 4.

5. Proof of the theorems

Proof of Theorem 1. By Lemma 1, the proof of Theorem 1 will be complete, if we show that

$$\int_{\overline{I}_N} \left(\sup_{n\in P} \frac{|\sigma_n a|}{\log^2 (n+1)} \right)^{1/2} d\mu \le c < \infty$$

for every 1/2-atom a, where I denotes the support of the atom. The boundedness of maximal operator $\sup\limits_{n\in P} \frac{\sigma_n f}{log^2(n+1)}$ from L_∞ to L_∞ follows from (5).

Let a be an arbitrary 1/2-atom with support I and $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $\sigma_n(a) = 0$ when $n \le M_N$. Therefore we can suppose that $n > M_N$.

Since $\|a\|_\infty \le cM_N^2$, we can write

$$\begin{aligned} & \frac{|\sigma_n(a)|}{\log^2(n+1)} \\ \le \quad & \frac{1}{\log^2(n+1)} \int_{I_N} |a(t)| \, |K_n(x-t)| \, d\mu(t) \\ \le \quad & \frac{\|a\|_\infty}{\log^2(n+1)} \int_{I_N} |K_n(x-t)| \, d\mu(t) \\ \le \quad & \frac{cM_N^2}{\log^2(n+1)} \int_{I_N} |K_n(x-t)| \, d\mu(t). \end{aligned}$$

Let $x \in I_N^{k,l}$, $0 \le k < l \le N$. Then from Lemma 4 we get

$$\frac{|\sigma_n(a)|}{\log^2(n+1)} \le \frac{cM_N^2}{N^2} \frac{M_l M_k}{M_N^2} = \frac{cM_l M_k}{N^2} \tag{15}$$

Combining (3) and (15) we obtain

$$\begin{aligned} & \int_{\overline{I_N}} |\widetilde{\sigma}^* a(x)|^{1/2} \, d\mu(x) \\ = \quad & \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j\in\{l+1,\dots,N-1\}}^{m_j-1} \int_{I_N^{k,l}} \left|\widetilde{\sigma}^* a(x)\right|^{1/2} d\mu(x) \\ & + \sum_{k=0}^{N-1} \int_{I_N^{k,N}} |\widetilde{\sigma}^* a(x)|^{1/2} \, d\mu(x) \\ \le \quad & c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \dots m_{N-1}}{M_N} \frac{\sqrt{M_l M_k}}{N} \\ & + c \sum_{k=0}^{N-1} \frac{1}{M_N} \frac{\sqrt{M_N M_k}}{N} \\ \le \quad & c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{\sqrt{M_k}}{N\sqrt{M_l}} + c \sum_{k=0}^{N-1} \frac{1}{\sqrt{M_N}} \frac{\sqrt{M_k}}{N} \le c < \infty. \end{aligned}$$

Which completes the proof of Theorem 1. □

Proof of Theorem 2. Let $\{\lambda_k ; k \in P_+\}$ be an increasing sequence of the positive integers such that

$$\lim_{k\to\infty} \frac{\log^2(\lambda_k)}{\varphi(\lambda_k)} = \infty.$$

It is evident that for every λ_k there exists positive integers $m_k^{'}$ such that $q_{m_k^{'}} \leq \lambda_k < q_{m_k^{'}+1} < M^5 \ q_{m_k^{'}}$, $M := \sup_k m_k$. Since $\varphi(n)$ is a nondecreasing function we have

$$\overline{\lim_{k\to\infty}} \frac{\left(m_k^{'}\right)^2}{\varphi\left(q_{m_k^{'}}\right)} \geq c \lim_{k\to\infty} \frac{\log^2(\lambda_k)}{\varphi(\lambda_k)} = \infty; \tag{16}$$

let $\{n_k ; k \in P_+\} \subset \{m_k^{\prime} ; k \in P_+\}$ such that

$$\lim_{k\to\infty} \frac{n_k^2}{\varphi(q_{n_k})} = \infty$$

and

$$f_{n_k}(x) = D_{M_{2n_k+1}}(x) - D_{M_{2n_k}}(x).$$

It is evident

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, \ \text{if } i = M_{2n_k}, ..., M_{2n_k+1} - 1; \\ 0, \text{otherwise.} \end{cases}$$

Then we can write

$$S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{M_{2n_k}}(x), \ \text{if } i = M_{2n_k}, ..., M_{2n_k+1} - 1 \\ f_{n_k}(x), \text{if } i \geq M_{2n_k+1} \\ 0, \qquad \text{otherwise} \ . \end{cases} \tag{17}$$

From (4) we get

$$\begin{aligned} \|f_{n_k}\|_{H_{1/2}} &= \left\|\sup_{n\in P} S_{M_n} f_{n_k}\right\|_{L_{1/2}} \qquad (18) \\ &= \left\|D_{M_{2n_k+1}} - D_{M_{2n_k}}\right\|_{L_{1/2}} \\ &= \left(\int_{I_{2n_k} \setminus I_{2n_k+1}} M_{2n_k}^{1/2} d\mu(x) + \int_{I_{2n_k+1}} \left(M_{2n_k+1} - M_{2n_k}\right)^{1/2} d\mu(x)\right)^2 \\ &= \left(\frac{m_{2n_k} - 1}{M_{2n_k+1}} M_{2n_k}^{1/2} + \frac{\left(m_{2n_k} + 1\right)^{1/2}}{M_{2n_k+1}} M_{2n_k}^{1/2}\right)^2 \\ &\leq \frac{c}{M_{2n_k}}. \end{aligned}$$

By (17) we can write:

$$\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)}=\frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}}\left|\sum_{j=0}^{q_{n_k}-1} S_j f_{n_k}(x)\right|$$

$$=\frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}}\left|\sum_{j=M_{2n_k}}^{q_{n_k}-1} S_j f_{n_k}(x)\right|$$

$$=\frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}}\left|\sum_{j=M_{2n_k}}^{q_{n_k}-1}\Big(D_j\left(x\right)-D_{M_{2n_k}}\left(x\right)\Big)\right|$$

$$=\frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}}\left|\sum_{j=0}^{q_{n_k-1}-1}\Big(D_{j+M_{2n_k}}\left(x\right)-D_{M_{2n_k}}\left(x\right)\Big)\right|$$

Since

$$D_{j+M_{2n_k}}\left(x\right)-D_{M_{2n_k}}\left(x\right)=\psi_{M_{2n_k}} D_j,\ j=1,2,..,M_{2n_k}-1,$$

we obtain

$$\begin{aligned}\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} &= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}}\left|\sum_{j=0}^{q_{n_k-1}-1} D_j\left(x\right)\right| \\ &= \frac{1}{\varphi\left(q_{n_k}\right)}\frac{q_{n_k-1}}{q_{n_k}}\left|K_{q_{n_k-1}}\left(x\right)\right|.\end{aligned}$$

Let $x\in I_{2n_k}^{2s,2l}$. Then from Lemma 2 we have

$$\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)}\geq\frac{cM_{2s}M_{2l}}{M_{2n_k}\varphi\left(q_{n_k}\right)}.$$

Hence we can write:

$$\int_{G_m}\left(\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)}\right)^{1/2} d\mu\left(x\right)$$

$$\geq\sum_{s=0}^{n_k-3}\sum_{l=s+1}^{n_k-1}\sum_{x_{2l+1}=0}^{m_{2l+1}}\ldots\sum_{x_{2n_k-1}=0}^{m_{2n_k-1}}\int_{I_{2n_k}^{2s,2l}}\left(\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)}\right)^{1/2} d\mu\left(x\right)$$

$$\geq c\sum_{s=0}^{n_k-3}\sum_{l=s+1}^{n_k-1}\frac{m_{2l+1}\ldots m_{2n_k-1}}{M_{2n_k}}\frac{\sqrt{M_{2s}M_{2l}}}{\sqrt{\varphi\left(q_{n_k}\right)M_{2n_k}}}\geq$$

$$c\sum_{s=0}^{n_k-3}\sum_{l=s+1}^{n_k-1}\frac{\sqrt{M_{2s}}}{\sqrt{M_{2l}M_{2n_k}\varphi\left(q_{n_k}\right)}}\geq\frac{cn_k}{\sqrt{M_{2n_k}\varphi\left(q_{n_k}\right)}}$$

Then from (18) we have

$$\frac{\left(\int_{G_m}\left(\frac{|\sigma_{q_{n_k}} f_{n_k}(x)|}{\varphi(n_k)}\right)^{1/2} d\mu\left(x\right)\right)^2}{\left\|f_{n_k(x)}\right\|_{H_{1/2}}} \geq \frac{cn_k^2}{M_{2n_k}\varphi\left(q_{n_k}\right)} M_{2n_k}$$

$$\geq \frac{cn_k^2}{\varphi\left(q_{n_k}\right)} \to \infty \qquad \text{when} \qquad k \to \infty.$$

Theorem 2 is proved. □

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