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Strong solution for a high order boundary value problem with integral condition

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Abstract: The present paper is devoted to a proof of the existence and uniqueness of strong solution for a high order boundary value problem with integral condition. The proof is based by a priori estimate and on the density of the range of the operator generated by the studied problem.

Key words: Integral condition, energy inequality, boundary value problem

1. Introduction

In the rectangular domain $Q = (0, 1) \times (0, T)$, with $T < \infty$, we consider the differential equation

$$\mathcal{L}u = \frac{\partial^4 u}{\partial t^4} + (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha} \left(a(x, t) \frac{\partial^\alpha u}{\partial x^\alpha} \right) = f(x, t), \quad (1.1)$$

where $a(x, t)$ satisfy the assumptions

$$0 < a_0 \leq a(x, t) \leq a_1, \quad (1.2)$$

$$c'_k \leq \frac{\partial^k a(x, t)}{\partial x^k} \leq c_k, \quad k = \overline{1, 4}, \quad \text{with } c'_1 \geq 0, \quad \forall (x, t) \in \overline{Q}, \quad (1.3)$$

subject to the initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad x \in (0, 1), \quad (1.4)$$

final conditions

$$\frac{\partial^2 u(x, T)}{\partial t^2} = 0, \quad \frac{\partial^3 u(x, T)}{\partial t^3} = 0, \quad x \in (0, 1), \quad (1.5)$$

boundary conditions

$$\frac{\partial^i u(0, t)}{\partial x^i} = 0, \quad \text{for } 0 \leq i \leq \alpha - 1, \quad t \in (0, T), \quad (1.6)$$

$$\frac{\partial^i u(1, t)}{\partial x^i} = 0, \quad \text{for } 0 \leq i \leq \alpha - 2, \quad t \in (0, T), \quad (1.7)$$

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and the integral (nonlocal) condition

$$\int_0^1 u(\xi, t) d\xi = 0, \quad t \in (0, T). \tag{1.8}$$

The importance of boundary value problems with integral boundary conditions has been pointed out by Samarski [21]. We remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics; see for example [7, 12, 22, 17]. Boundary value problems for parabolic equations with an integral boundary condition are investigated by Batten [1], Bouziani and Benouar [2], Cannon [4, 5], Cannon, et al. [6], Ionkin [15], Kamynin [16], Shi and Shillor [23], Shi [22], Marhoune and Bouziti [19], Denche and Marhoune [8, 9, 10, 11], Yurchuk [24], and many references therein. The problem with an integral one-space-variable condition is studied in Kartynnik [17], and Denche and Marhoune [11]

2. Preliminaries

In this paper, we prove the existence and uniqueness of a strong solution of the problem stated in equation (1.1) – (1.8). The demonstration is based on an a priori estimate and the density of the image of the operator generated by the problem (1.1) – (1.8). This problem can be written in the operator form

$$Lu = F, \tag{2.9}$$

where the operator L is considered from E to F . We consider the domain of definition $D(L)$ such that E is the Banach space consisting of all functions $u \in L^2(Q)$, satisfying equations (1.1) – (1.8), with the finite norm

$$\|u\|_E^2 = \int_Q \frac{(1-x)}{2} \left[\left| \frac{\partial^4 u}{\partial t^4} \right|^2 + \left| \frac{\partial^\alpha}{\partial x^\alpha} \left(a(x, t) \frac{\partial^\alpha u}{\partial x^\alpha} \right) \right|^2 + \left| \frac{\partial^\alpha u}{\partial x^\alpha} \right|^2 \right] dxdt, \tag{2.10}$$

and F is the Hilbert space with norm given by

$$\|f\|_F^2 = \int_Q (1-x)^\nu |f|^2 dxdt, \tag{2.11}$$

where ν is an arbitrary number such that $0 < \nu < 1$. Using the energy inequalities method proposed in [18], we establish an energy inequality

$$\|u\|_E^2 \leq C_1 \|Lu\|_F^2 \tag{2.12}$$

and we show that the operator L has the closure \overline{L} .

Definition 1 A solution of the operator equation $\overline{L}u = F$ is called a strong solution of the problem (1.1) – (1.8).

Inequality (2.12) can be extended by

$$\|u\|_E^2 \leq C_1 \|\overline{L}u\|_F^2, \text{ for all } u \in D(\overline{L}). \tag{2.13}$$

From this inequality, we obtain the uniqueness of a strong solution if it exists, and the equality of sets $R(\overline{L})$ and $\overline{R(L)}$. Thus, to prove the existence of a strong solution of the problem in equations (1.1)–(1.8), it remains to prove that the set $R(L)$ is dense in F .

3. An energy inequality and its consequences

Theorem 1 For any function $u \in D(L)$ we have the a priori estimate

$$\|u\|_E^2 \leq k \|Lu\|_F^2, \tag{3.1}$$

where

$$k = \frac{\exp(cT) \max\left(\left(\frac{2\alpha}{(1-\nu)}\right)^2 + \frac{5}{4}\right)}{\min\left(\frac{1}{4}, \delta\right)} \tag{3.2}$$

and

$$\delta = c'_4 - 4cc_3 + 6c^2c'_2 - 4c^3c_1 + c^4a_1 > 0, \tag{3.3}$$

with the constant c satisfying the region

$$\left\{ \begin{array}{l} \sup \left[\frac{1}{a} \frac{\partial a}{\partial t} - \sqrt{\left(\frac{\partial a}{\partial t}\right)^2 - \frac{1}{a} \frac{\partial a}{\partial t}} \right] < c < \inf \left[1 + \frac{1}{a} \frac{\partial a}{\partial t} - \sqrt{\left(\frac{\partial a}{\partial t}\right)^2 - \frac{1}{a} \frac{\partial a}{\partial t} + \frac{1}{2}} \right], \\ a_0c^3 - c_1c(3c+2) + c'_2(3c+1) - c_3 \geq 0, \\ \delta = c'_4 - 4cc_3 + 6c^2c'_2 - 4c^3c_1 + c^4a_1 > 0 \end{array} \right. \tag{3.4}$$

Proof Denote

$$Mu = (1-x) \frac{\partial^4 u}{\partial t^4} + \alpha J \frac{\partial^4 u}{\partial t^4},$$

where

$$Ju = \int_0^x u(\xi, t) d\xi.$$

We consider the quadratic formula

$$\operatorname{Re} \int_0^\tau \int_0^1 \exp(-ct) \mathcal{L}u \overline{Mu} dx dt, \tag{3.5}$$

with the constant c satisfying condition (3.4); obtained by multiplying equation (1.1) by $\exp(-ct) \mathcal{L}u \overline{Mu}$; and integrating over Q^τ , where $Q^\tau = (0, 1) \times (0, \tau)$, with $0 \leq \tau \leq T$, and by taking the real part. Integrating by parts α times in formula (3.5) with the use of boundary conditions in equations (1.6), (1.7), and (1.8), we obtain

$$\operatorname{Re} \int_0^\tau \int_0^1 \exp(-ct) \mathcal{L}u \overline{Mu} dx dt = \tag{3.6}$$

$$\begin{aligned}
 & \int_0^\tau \int_0^1 \exp(-ct) (1-x) \left| \frac{\partial^4 u}{\partial t^4} \right|^2 dx dt + \\
 & 2\operatorname{Re} \int_0^1 \exp(-c\tau) \left(\frac{\partial^2 a(x, \tau)}{\partial t^2} - 2c \frac{\partial a(x, \tau)}{\partial t} + c^2 a(x, \tau) \right) \frac{(1-x)}{2} \frac{\partial}{\partial t} \left(\frac{\partial^\alpha \overline{u(x, \tau)}}{\partial x^\alpha} \right) \frac{\partial^\alpha u(x, \tau)}{\partial x^\alpha} dx - \\
 & 4 \int_0^\tau \int_0^1 \exp(-ct) \left(\frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a \right) \frac{(1-x)}{2} \left| \frac{\partial}{\partial t} \left(\frac{\partial^\alpha u}{\partial x^\alpha} \right) \right| dx dt - \\
 & \int_0^1 \exp(-c\tau) \left(\frac{\partial^3 a(x, \tau)}{\partial t^3} - 3c \frac{\partial^2 a(x, \tau)}{\partial t^2} + 3c^2 \frac{\partial a(x, \tau)}{\partial t} - c^3 a(x, \tau) \right) \frac{(1-x)}{2} \left| \frac{\partial^\alpha u(x, \tau)}{\partial x^\alpha} \right| dx + \\
 & 2 \int_0^1 \exp(-c\tau) \left(\frac{\partial a(x, \tau)}{\partial t} - ca(x, \tau) \right) \frac{(1-x)}{2} \left| \frac{\partial}{\partial t} \left(\frac{\partial^\alpha u(x, \tau)}{\partial x^\alpha} \right) \right| dx + \\
 & \int_0^\tau \int_0^1 \exp(-ct) \left(\frac{\partial^4 a}{\partial t^4} - 4c \frac{\partial^3 a}{\partial t^3} + 6c^2 \frac{\partial^2 a}{\partial t^2} - 4c^3 \frac{\partial a}{\partial t} + c^4 a \right) \frac{(1-x)}{2} \left| \frac{\partial^\alpha u}{\partial x^\alpha} \right| dx dt + \\
 & 2 \int_0^\tau \int_0^1 a \exp(-ct) \frac{(1-x)}{2} \left| \frac{\partial^2}{\partial t^2} \left(\frac{\partial^\alpha u}{\partial x^\alpha} \right) \right| dx dt.
 \end{aligned}$$

By substituting the expression of Mu in formula (3.5), using elementary inequalities and the inequality

$$\int_0^1 \frac{\left| J \frac{\partial^4 u}{\partial t^4} \right|^2}{(1-x)^\nu} dx \leq \frac{4}{(1-x)^\nu} \int_0^1 (1-x) \left| \frac{\partial^4 u}{\partial t^4} \right|^2 dx, \quad \text{where } 0 < \nu < 1, \tag{3.7}$$

yields

$$\begin{aligned}
 \operatorname{Re} \int_0^\tau \int_0^1 \exp(-ct) \mathcal{L}u \overline{Mu} dx dt & \leq \left(\frac{4\alpha^2}{(1-x)^\nu} + 1 \right) \int_0^\tau \int_0^1 \exp(-ct) (1-x)^\nu |\mathcal{L}u|^2 dx dt \\
 & + \frac{1}{2} \int_0^\tau \int_0^1 \exp(-ct) (1-x) \left| \frac{\partial^4 u}{\partial t^4} \right|^2 dx dt.
 \end{aligned} \tag{3.8}$$

From equation (1.1), we have

$$\begin{aligned}
 \frac{1}{4} \int_0^\tau \int_0^1 \exp(-ct) \frac{(1-x)}{2} \left| \frac{\partial^\alpha}{\partial x^\alpha} \left(a \frac{\partial^\alpha u}{\partial x^\alpha} \right) \right|^2 dx dt & \leq \frac{1}{4} \int_0^\tau \int_0^1 \exp(-ct) (1-x) |\mathcal{L}u|^2 dx dt + \\
 & \frac{1}{2} \int_0^\tau \int_0^1 \exp(-ct) \frac{(1-x)}{2} \left| \frac{\partial^4 u}{\partial t^4} \right|^2 dx dt.
 \end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
 & \int_Q \frac{(1-x)}{2} \left[\left| \frac{\partial^4 u}{\partial t^4} \right|^2 + \left| \frac{\partial^\alpha}{\partial x^\alpha} \left(a(x, t) \frac{\partial^\alpha u}{\partial x^\alpha} \right) \right|^2 + \left| \frac{\partial^\alpha u}{\partial x^\alpha} \right|^2 \right] dx dt \\
 & \leq \frac{\exp(cT) \max \left(\left(\frac{2\alpha}{(1-\nu)} \right)^2 + \frac{5}{4} \right)}{\min \left(\frac{1}{4}, \delta \right)} \int_Q (1-x)^\nu |f|^2 dx dt.
 \end{aligned} \tag{3.9}$$

□

Lemma 1 *The operator L from E to F admits a closure.*

Proof Suppose that $(u_n) \in D(L)$ is a sequence such that

$$u_n \longrightarrow 0 \text{ in } E, \tag{3.10}$$

and

$$Lu_n \longrightarrow f \text{ in } F, \tag{3.11}$$

We must show that $f = 0$.

Introducing the operator

$$\mathcal{L}_0 v = \frac{\partial^4 v}{\partial t^4} + (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha} \left(a(x, t) \frac{\partial^\alpha v}{\partial x^\alpha} \right), \tag{3.12}$$

defined on the domain $D(\mathcal{L}_0)$ of function $v \in L^2(Q)$ verifying

$$\begin{aligned} v(x, 0) &= \frac{\partial v(x, 0)}{\partial t} = \frac{\partial^2 v(x, T)}{\partial t} = \frac{\partial^3 v(x, T)}{\partial t} = 0, \\ \frac{\partial^i v(0, t)}{\partial x^i} &= 0, \text{ for } 0 \leq i \leq \alpha - 1, \\ \frac{\partial^i v(1, t)}{\partial x^i} &= 0, \text{ for } 0 \leq i \leq \alpha - 2, \end{aligned} \tag{3.13}$$

we note that $D(\mathcal{L}_0)$ is dense in the Hilbert space obtained from the completion of $L^2(Q)$ with respect to the norm

$$\|f\|_F^2 = \int_Q (1-x)^\nu |f|^2 dx dt. \tag{3.14}$$

Additionally, since

$$\int_Q (1-x)^\nu f \bar{v} dx dt = \lim_{n \rightarrow \infty} \int_Q \mathcal{L} u_n [(1-x)^\nu \bar{v}] dx dt = \lim_{n \rightarrow \infty} \int_Q u_n \mathcal{L}_0 [(1-x)^\nu \bar{v}] dx dt = 0, \tag{3.15}$$

this holds for every function $v \in D(\mathcal{L}_0)$, and yields $f = 0$.

Theorem 2 *The priori estimate in Theorem 1 can be extended to include all functions u , i.e.*

$$\|u\|_E^2 \leq k \|\bar{\mathcal{L}}u\|_F^2, \forall u \in D(\bar{\mathcal{L}}), \tag{3.16}$$

Hence we obtain the following corollary.

Corollary 1 *A strong solution of the problem in equations (1.1)–(1.8) is unique if it exists, and depends continuously on f .*

Corollary 2 *The range $R(L)$ of the operator $\bar{\mathcal{L}}$ is closed in F , and $R(\bar{\mathcal{L}}) = \overline{R(L)}$.*

□

4. Solvability of the problem

To prove the solvability of problem in equations (1.1)–(1.8), it is sufficient to show that $R(L)$ is dense in F . The proof is based on the following lemma.

Lemma 2 For all $\omega \in L^2(Q)$,

$$\int_Q (1-x) \mathcal{L}u \cdot \bar{\omega} dx dt = 0, \tag{4.1}$$

then $\omega = 0$.

Proof Equality (4.1) can be written as

$$-\int_Q \frac{\partial^4 u}{\partial t^4} (1-x) \bar{\omega} dx dt = (-1)^\alpha \int_Q \frac{\partial^\alpha}{\partial x^\alpha} \left(a(x,t) \frac{\partial^\alpha u}{\partial x^\alpha} \right) (1-x) \bar{\omega} dx dt \tag{4.2}$$

If we introduce the smoothing operators with respect to t [24, 20, 14, 3], $J_\xi^{-1} = (I + \xi \frac{\partial}{\partial t})^{-1}$ and $(J_\xi^{-1})^*$, then these operators provide the solutions of the respective problems

$$\begin{aligned} \xi \frac{dg_\xi(t)}{dt} + g_\xi(t) &= g(t), \\ g(t)|_{t=0} &= 0, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} -\xi \frac{dg_\xi^*(t)}{dt} + g_\xi^*(t) &= g(t), \\ g(t)|_{t=T} &= 0. \end{aligned} \tag{4.4}$$

The operators also have the following properties: for any $g \in L_2(0, T)$, the function $g_\xi = (J_\xi^{-1})g$ and $g_\xi^* = (J_\xi^{-1})^*g$ are in $W_2^1(0, T)$ such that $g_\xi|_{t=0} = 0$. and $g_\xi^*|_{t=T} = 0$. Moreover, J_ξ^{-1} commutes with $\frac{\partial}{\partial t}$, so $\int_0^T |g_\xi - g|^2 dt \rightarrow 0$ and $\int_0^T |g_\xi^* - g|^2 dt \rightarrow 0$ for $\xi \rightarrow 0$.

Now, for given $\omega(x, t)$, we introduce the function

$$v(x, t) = -\alpha (1-x)^{\alpha-1} \int_0^x \frac{\omega}{(1-\xi)^\alpha} d\xi + \omega(x, t).$$

Integrating by parts, we obtain

$$(1-x)v + \alpha Jv = (1-x)\omega, \text{ and } \int_0^x v(x, t) dx = 0. \tag{4.5}$$

Then from equality (4.2), we have

$$-\int_Q \frac{\partial^4 u}{\partial t^4} N \bar{v} dx dt = \int_Q A(t) u \bar{v} dx dt, \tag{4.6}$$

where $Nv = (1 - x)v + \alpha Jv$, and $A(t)u = (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha} (a(x, t) \frac{\partial^\alpha u}{\partial x^\alpha})$.

Putting $u = \int_0^t \int_0^\eta \int_\delta^h \int_\xi^T \exp(c\tau) v_\xi^*(\tau) d\tau d\xi d\eta dh$ in (4.6), and using (4.4), we obtain

$$-\int_Q \exp(ct) v_\xi^* \overline{Nv} dxdt = \int_Q A(u) uv_\xi^* dxdt - \xi \int_Q A(t) u \frac{\partial^4 v_\xi^*}{\partial t^4} dxdt. \tag{4.7}$$

Integrating by parts each term in the right-hand side of (4.7) and taking the real parts, we have

$$\operatorname{Re} \left(\int_Q A(u) uv_\xi^* dxdt \right) \geq 0, \tag{4.8}$$

$$\operatorname{Re} \left(-\xi \int_Q A(t) u \frac{\partial^4 v_\xi^*}{\partial t^4} dxdt \right) \geq -\xi M, \tag{4.9}$$

where

$$\begin{aligned} M = & 16 \int_Q \frac{(1-x)}{2} \left| \frac{\partial^4 v_\xi^*}{\partial t^4} \right|^2 dxdt + \int_Q \frac{(1-x)}{2} \left(\frac{\partial^4 a}{\partial t^4} \right)^2 \left| \frac{\partial^\alpha u}{\partial t^\alpha} \right|^2 dxdt + \\ & 4 \int_Q \frac{(1-x)}{2} \left(\frac{\partial^3 a}{\partial t^3} \right)^2 \left| \frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}} \right|^2 dxdt + \\ & 6 \int_Q \frac{(1-x)}{2} \left(\frac{\partial^2 a}{\partial t^2} \right)^2 \left| \frac{\partial^{\alpha+2} u}{\partial t^{\alpha+2}} \right|^2 dxdt + \\ & 4 \int_Q \frac{(1-x)}{2} \left(\frac{\partial a}{\partial t} \right)^2 \left| \frac{\partial^{\alpha+3} u}{\partial t^{\alpha+3}} \right|^2 dxdt + \int_Q \frac{(1-x)}{2} a^2 \left| \frac{\partial^{\alpha+4} u}{\partial t^{\alpha+4}} \right|^2 dxdt. \end{aligned} \tag{4.10}$$

Now, using inequalities (4.8) and (4.9) in equation (4.7), we have

$$\operatorname{Re} \left(\int_Q \exp(ct) v_\xi^* \overline{Nv} dxdt \right) \leq 0, \tag{4.11}$$

then for $\xi \rightarrow 0$, we obtain

$$\operatorname{Re} \left(\int_Q \exp(ct) v \overline{Nv} dxdt \right) \leq 0. \tag{4.12}$$

We conclude that $v = 0$, hence, $\omega = 0$, which ends the proof of the lemma.

Theorem 3 *The range $R(\overline{L})$ of \overline{L} coincides with F .*

□

Proof Since F is a Hilbert space, we have $R(\overline{L}) = F$ if and only if the relation

$$\int_Q (1-x)^\nu \mathcal{L}u \cdot \overline{f} dxdt = 0 \tag{4.13}$$

for arbitrary function $u \in E$ and $f \in F$, implies that $f = 0$.

Putting $u \in D(L)$ in relation (4.13), taking $\omega = \frac{f}{(1-x)^{\nu-1}}$, and using lemma 7, we obtain $\omega = \frac{f}{(1-x)^{\nu-1}} = 0$, then $f = 0$. \square

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