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On modules which satisfy the radical formula

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Abstract: In this paper, the authors prove that every representable module over a commutative ring with identity satisfies the radical formula. With this result, they extend the class of modules satisfying the radical formula from that of Artinian modules to a larger one. They conclude their work by giving a description of the radical of a submodule of a representable module.

Key words: Prime submodule, prime radical, radical formula, secondary module, secondary representation, representable module

1. Introduction

Throughout this work R will denote a commutative ring with identity and every module will be unitary. Let M be an R-module. For submodules K and L of M, we use the notation $(K : L)$ to show the ideal $\{r \in R : rL \leq K\}$ of R. A proper submodule N of M is said to be *prime* submodule of M, if, for every $r \in R$ and $m \in M$, $rm \in N$ implies $m \in N$ or $r \in (N : M)$. It is not difficult to see that if N is a prime submodule of M and $P = (N : M)$ then P is a prime ideal of R and, in this case, we say that N is P-prime. It is easy to see that if $M = R$ prime ideals and prime submodules of R coincide. For any submodule N of M, the (prime) *radical* of N in M, denoted by $rad_M(N)$, is defined to be the intersection of all prime submodules of M containing N. (If there is no such prime submodule in M we put $rad_M(N) = M$). It is not easy to calculate the radical of a submodule, in general. Several authors tried to give simple descriptions for the radical in some particular cases.

In this note, we shall need the notion of the envelope of a submodule introduced by R. L. McCasland and M. E. Moore in [7]. For a submodule N of an R-module M, the *envelope* of N in M, denoted by $E_M(N)$, is defined to be the subset

$$\{rm : r \in R \text{ and } m \in M \text{ such that } r^k m \in N \text{ for some } k \in \mathbb{N} \text{ with } k \geq 1\}$$

of M. Note that, in general, $E_M(N)$ is not an R-module. It is clear that $N \subseteq E_M(N) \subseteq rad_M(N)$, where the equalities does not need to hold. With the help of envelopes, the notion of the radical formula is defined as follows: A submodule N of an R-module M is said to *satisfy the radical formula* in M, if $RE_M(N) = rad_M(N)$. Also, an R-module M is said to *satisfy the radical formula*, if every submodule of M satisfies the radical formula in M. The radical formula has been studied extensively by various authors (see [3], [5], [7], [9] and [10]). In

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[10], Sharif, Sharifi and Namazi prove that if R is an Artinian ring then every module over R satisfies the radical formula. More generally, they show that if R is a ring such that every prime ideal is maximal then every R-module satisfy the radical formula. Yılmaz and Smith give a generalization of this fact in [9]. They define the notion of a special module and prove that every special module satisfies the radical formula. It is also proved in [9] that every Artinian module and every module over a ring whose prime ideals are maximal are special.

To give one of our main results in this work it is appropriate for us to recall what the notion of a representable module means: An R-module S is said to be *secondary* if $S \neq 0$ and, for all $r \in R$, either $rS = S$ or there exists $n \in \mathbb{N}$ such that $r^n S = 0$. Note that if S is a secondary R-module then $P = \sqrt{Ann_R(S)}$ is a prime ideal of R, and, in this case, we say that S is P-secondary. Let M be an R-module. A *secondary representation* for M is an expression of the form

$$M = S_1 + \cdots + S_n,$$

where S_i is a secondary submodule of M for all $i = 1, \ldots, n$. Such a secondary representation is said to be *minimal,* if, whenever $P_i = \sqrt{Ann_R(S_i)}$ for $i = 1, \ldots, n$,

(i) $P_1, \ldots, P_n$ are all distinct, and

(ii) no term in the sum is redundant, that is $S_i \nsubseteq \sum_{j=1, j\neq i}^{n} S_j$ for all $i = 1, \ldots, n$.

Note that any secondary representation for M can be modified to a minimal one. An R-module M is called *representable*, if M has a (minimal) secondary representation. It is known that if $M = S_1 + \cdots + S_n$ is a minimal secondary representation for M, with $\sqrt{Ann_R(S_i)} = P_i$ for $i = 1, \ldots, n$, then the set $\{P_1, \ldots, P_n\}$ of n prime ideals of R is independent of the choice of a minimal secondary representation for M (for details, see [4] or [6]). We denote this set by $Att_R(M)$. It is easy to see that if G is a proper submodule of a representable module M then M/G is again a representable module with $Att_R(M/G) \subseteq Att_R(M)$.

In this work, we generalize the fact that every Artinian module satisfies the radical formula (see [9]) by showing that every representable module satisfies the radical formula. In addition, we characterize the radical of a submodule of a representable module. Moreover, as a new technique in this subject, we use Lemma 6 which is actually *the lying over property* for prime submodules.

We shall require a number of preliminary results on secondary modules.

Lemma 1 *Let P be a prime ideal of R and let M be a P-secondary R-module. Then the following are satisfied:*

(i) $(N : M) \subseteq P$, *for every proper submodule N of M,*

(ii) $PM \subseteq RE_M(0)$, *and*

(iii) *for any proper submodule N of M, either $RE_M(N) = M$ or $(RE_M(N) : M) = P$.*

Proof (i) Let $rM \subseteq N$ for some $r \in R$. Since N is proper then so too is rM. Thus $r^k M = 0$ for some $k \in \mathbb{N}$. This gives that $r \in \sqrt{Ann_R(M)} = P$.

(ii) Let $p \in P$ and $m \in M$. Since M is P-secondary there exists $n \in \mathbb{N}$ such that $p^n m = 0$. Thus $pm \in RE_M(0)$. This gives that $PM \subseteq RE_M(0)$.

(iii) Suppose $RE_M(N) \neq M$. Applying part (i) above to $RE_M(N)$ instead of N we obtain $(RE_M(N) : M) \subseteq P$. On the other hand, by part (ii) above, we have $P \subseteq (RE_M(0) : M) \subseteq (RE_M(N) : M)$. □

A submodule N of an R-module M is called *semi-prime* if, whenever $r^k m \in N$ for some $r \in R$, $m \in M$ and $k \in \mathbb{N}$ then $rm \in N$. It is clear that N is a semi-prime submodule of M if and only if $RE_M(N) = N$. Note that an intersection of prime submodules of M is a semi-prime submodule of M.

Lemma 2 *[2, Theorem 2.3 (ii)] Let P be a prime ideal of R and let M be a P-secondary R-module. Then every semi-prime submodule of M is P-prime.*

Proposition 3 *Let P be a prime ideal of R and let M be a P-secondary R-module. Then for any submodule N of M,*

$$rad_M(N) = \{m \in M : cm \in N + PM \textit{ for some } c \in R\backslash P\}.$$

Proof Let $L = \{m \in M : cm \in N + PM$ for some $c \in R\backslash P\}$. By [8, Proposition 1.7], $rad_M(N) \subseteq L$. Suppose that $rad_M(N) \neq M$. Since $rad_M(N)$ is a semi-prime submodule of M , by Lemma 2, $rad_M(N)$ is a P-prime submodule of M. Thus the result follows from [8, Lemma 1.6]. □

Lemma 4 *Any secondary R-module satisfies the radical formula.*

Proof Let P be a prime ideal of R and let N be a proper submodule of a P-secondary R-module M. If $RE_M(N) = M$ then clearly $RE_M(N) = rad_M(N)$. Hence we assume that $RE_M(N) \neq M$. Then by Lemma 1 *(iii)*, $(RE_M(N) : M) = P$. Let $r \in R\backslash P$ and $m \in M$ such that $rm \in RE_M(N)$. So, there exist $k \in \mathbb{N}$, $r_i \in R$ and $m_i \in M$ $(1 \leq i \leq k)$ such that $rm = r_1 m_1 + \cdots r_k m_k$ and $r_i^n m_i \in N$ $(1 \leq i \leq k)$ for some $n \in \mathbb{N}$. Since $r \in R\backslash P$, $rM = M$. Then there exist $m'_1, \ldots, m'_k \in M$ such that $rm'_i = m_i$. Set $x = m - r_1 m'_1 - \cdots - r_k m'_k$. Now, we have $rx = 0$. We may also write $x = rx'$ for some $x' \in M$. This gives that $r^2 x' = 0$, and so $x = rx' \in RE_M(0) \subseteq RE_M(N)$. Moreover, $m'_i = rm''_i$ for some $m''_i \in M$. Thus $r_i^n r^2 m''_i = r_i^n rm'_i = r_i^n m_i \in N$, and so $r_i m'_i = r_i rm''_i \in RE_M(N)$. Therefore $m = r_1 m'_1 + \cdots + r_k m'_k + x \in RE_M(N)$. This shows that $RE_M(N)$ is a P-prime submodule of M. Thus we have $rad_M(N) \subseteq RE_M(N)$. This completes the proof. □

In [9], Pusat-Yılmaz and Smith introduce the notion of special modules as follows: An R-module M is called *special* if for each $m \in M$ and each element a of any maximal ideal $\mathfrak{M}$, there exist $n \in \mathbb{N}$ and $c \in R\backslash\mathfrak{M}$ such that $ca^n m = 0$. In their work, they prove that every special module satisfies the radical formula. The following example shows that the class of secondary modules differs from the class of special modules. Note that, in [11], it is shown that, for any injective module E over a commutative ring R and a primary ideal Q of R, $\operatorname{ann}_E(Q)$ is a secondary R-module.

Example 5 *Let R be a ring with a non-maximal prime ideal P of R and let E be the injective envelope of the R-module R/P. Then $M = ann_E(P)$ is a secondary module over R. But M is not special (and hence not Artinian). To see this, take a maximal ideal $\mathfrak{M}$ of R containing P and choose $r \in \mathfrak{M}\backslash P$. Since $R/P \subseteq M$, $1 + P \in M$. If there exist $c \in R\backslash\mathfrak{M}$ and $n \in \mathbb{N}$ such that $cr^n(1+P) = 0$ then $cr^n \in P$, which is impossible.*

Lemma 6 *Let M be an R-module and N a submodule of M. If K is a P-prime submodule of N and $(N : M) \nsubseteq P$, then there exists a P-prime submodule K' of M such that $K' \cap N = K$.*

Proof Define $K' = \{m \in M : cm \in K$ for some $c \in R\backslash P\}$. Then clearly K' is a submodule of M. We first show that $K' \cap N = K$. Suppose $x \in K' \cap N$. Then there exists $c \in R\backslash P$ such that $cx \in K$. Because K is a prime submodule of N we have $x \in K$. Thus $K = K' \cap N$. If $K' = M$, then $K = K' \cap N = N$, a contradiction since K is prime in N. Now let $c \in R\backslash P$ such that $cM \subseteq N$. Then $cPM \subseteq PN \subseteq K$, which shows that $PM \subseteq K'$. Now suppose that $rm \in K'$ for some $r \in R$ and $m \in M$. Let $r \in R\backslash P$. By definition of K', there exists $d \in R\backslash P$ such that $drm \in K$. Since $dr \in R\backslash P$, we have $m \in K'$, and so K' is a prime submodule of M. □

Now, we state the following lemma whose proof is straightforward.

Lemma 7 *Let N and L be submodules of an R-module M with $N \leq L$. Then* $rad_{M/N}(L/N) = rad_M(L)/N$.

Proposition 8 *Let $M = M_1 + M_2$ be a sum of submodules M_1 which satisfies the radical formula and M_2 which is a P-secondary R-module for a prime ideal P of R. If $Ann_R(M_1) \nsubseteq P$ then M satisfies the radical formula.*

Proof Since M/M_2 is isomorphic to a quotient of M_1, M/M_2 satisfies the radical formula. Let $N \leq M$ and let $m \in rad_M(N)$. Then

$$m + M_2 \in rad_{M/M_2}\left(\frac{N+M_2}{M_2}\right) = RE_{M/M_2}\left(\frac{N+M_2}{M_2}\right).$$

Thus there exist $k \in \mathbb{N}$, $r_1, \ldots, r_k \in R$, $m_1, \ldots, m_k \in M$ such that $m + M_2 = r_1(m_1 + M_2) + \cdots + r_k(m_k + M_2)$ and $r_i^t(m_i + M_2) \in (N + M_2)/M_2$ for some $t \in \mathbb{N}$. Write $r_i^t m_i = n_i + u_i$, where $n_i \in N$ and $u_i \in M_2$ $(1 \leq i \leq k)$. Suppose $r_j \in P$ for some $1 \leq j \leq k$. Since M_2 is P-secondary, there exists $s \in \mathbb{N}$ such that $r_j^s M_2 = 0$. Then we have $r_j^{t+s} m_j = r_j^s n_i \in N$, and so $r_j m_j \in RE_M(N)$. On the other hand, if $r_j \in R\backslash P$, then $r_j M_2 = M_2$. Hence there exists $u_j' \in M_2$ such that $r_j^t u_j' = u_j$. This gives $r_j^t(m_j - u_j') \in N$, and hence $r_j(m_j - u_j') \in RE_M(N)$. Set $J = \{j : 1 \leq j \leq k \text{ and } r_j \in P\}$ and $I = \{1, \ldots, k\}\backslash J$. Then we can write $m = \sum_{j \in J} r_j m_j + \sum_{i \in I} r_i(m_i - u_i') + x$ for some $x \in M_2$. Since $e = \sum_{j \in J} r_j m_j + \sum_{i \in I} r_i(m_i - u_i') \in RE_M(N)$, it is enough to show that $x \in RE_M(N)$. Observe that $x = m - e \in rad_M(N) \cap M_2$. Suppose $RE_{M_2}(N \cap M_2) \neq M_2$. Then, by the proof of Lemma 4, $RE_{M_2}(N \cap M_2)$ is a P-prime submodule of M_2. Since $Ann_R(M_1) \nsubseteq P$, clearly, $(M_2 : M) \nsubseteq P$, and so, by Lemma 6, there exists a P-prime submodule K of M such that $K \cap M_2 = RE_{M_2}(N \cap M_2)$. Then $K \supseteq N \cap M_2 \supseteq (M_2 : M)N$, and so $K \supseteq N$. Since $x \in rad_M(N)$ we have $x \in K \cap M_2 = RE_{M_2}(N \cap M_2)$. In any case, $x \in RE_{M_2}(N \cap M_2) \subseteq RE_M(N \cap M_2) \subseteq RE_M(N)$. This completes the proof. □

In [1], Brodmann and Sharp say that the class of representable modules is, in general, larger than the class of Artinian modules. And also they give the following examples:

Let E be an injective R-module.

1. Suppose that Q is an R-module whose zero submodule is primary. Then $Hom_R(Q, M)$ is a secondary R-module.
2. If M is a finitely generated R-module, then $Hom_R(M, E)$ is representable.

Thus, if E is taken to be an infinite direct product of some injective R-modules, then representable modules in (1) and (2) become non-Artinian. Now, as stated in the introductory section, we give the following theorem which generalizes the fact that every Artinian module s.t.r.f. On the other hand, sum of two modules which s.t.r.f. need not s.t.r.f. (see, for example, [5]). The following theorem also provides examples to the contrary.

Theorem 9 *Let $M = M_1 + M_2$ be a sum of submodules M_1 which satisfies the radical formula and M_2 which is a representable R-module. If $Ann_R(M_1)$ is not contained in any element of $Att_R(M_2)$ then M satisfies the radical formula. In particular, every representable module satisfies the radical formula.*

Proof Let

$$M_2 = L_1 + \cdots + L_k \qquad (L_i \text{ is } P_i\text{-secondary})$$

be a minimal secondary representation for M_2. We use induction on k. If $k = 1$, by Proposition 8, we are done. Now suppose the assertion is valid for every number smaller than k. Without losing generality we may assume that P_k is a minimal element of $Att_R(M_2)$. Let $Ann_R(M_1 + L_1 + \cdots + L_{k-1}) \subseteq P_k$. Then $Ann_R(M_1) \cap Ann_R(L_1) \cap \ldots \cap Ann_R(L_{k-1}) \subseteq P_k$, and so $\sqrt{Ann_R(M_1)} \cap P_1 \cap \ldots \cap P_{k-1} \subseteq P_k$. Since $P_j \nsubseteq P_k$, for every $1 \le j \le k-1$, $Ann_R(M_1) \subseteq \sqrt{Ann_R(M_1)} \subseteq P_k$, a contradiction. Then $Ann_R(M_1 + L_1 + \cdots + L_{k-1}) \nsubseteq P_k$. By the induction hypothesis, $M_1 + L_1 \cdots + L_{k-1}$ satisfies the radical formula, and so, by Proposition 8, M satisfies the radical formula. □

We know that, over a domain, every injective module s.t.r.f. Here is a generalization of this fact: Combining the above theorem with [11, Theorem 2.3], one can easily show that if the zero ideal of a ring R has a primary decomposition then every injective R-module satisfies the radical formula. Moreover, every injective module whose zero submodule has a primary decomposition satisfies the radical formula. With this theorem, we also generalize the fact that every finitely generated torsion module over a one-dimensional Noetherian domain s.t.r.f. proved in [9, Theorem 4.12]. Indeed, if M is a finitely generated torsion module over a one-dimensional Noetherian domain R, then there exist positive integers $t_1, \ldots, t_n$ and maximal ideals $P_1, \ldots, P_n$ of R such that $P_1^{t_1} \ldots P_n^{t_n} M = 0$. In this case M is isomorphic to $\left(M/P_1^{t_1}M\right) \oplus \cdots \oplus \left(M/P_n^{t_n}M\right)$, which is clearly representable.

In the remaining part of this note, we describe the radical of a submodule of a representable module.

Proposition 10 *Let M be an R-module and let N be a submodule of M. If $\{P_1, \ldots, P_n\}$ is a set of pairwise non-comparable prime ideals of R then the subset*

$$L = \{m \in M : cm \in N + (P_1 \cap \ldots \cap P_n)M \text{ for some } c \in R \backslash (P_1 \cup \ldots \cup P_n)\}$$

of M is either M or a radical submodule of M (i.e., an intersection of prime submodules of M).

Proof Let $K = N + (P_1 \cap \ldots \cap P_n)M$ and $S = R \backslash (P_1 \cup \ldots \cup P_n)$. Then S is a multiplicatively closed subset of R. Let R_S and M_S denote the localizations of R and M at S, respectively. Since

$$P_1 R_S \cap \ldots \cap P_n R_S \subseteq (K : M) R_S \subseteq (K_S :_{R_S} M_S),$$

if $K_S \neq M_S$, then M_S / K_S becomes a semisimple R_S-module. In this case K_S is a radical submodule of M_S, being an intersection of maximal submodules of M_S, and so $L = K_S \cap M$ is a radical submodule of M since primeness and intersections are invariant under taking contractions. □

With the following theorem which is a generalization of Proposition 3, we give a characterization of the radical of a submodule of a representable module in a particular case. Note that this theorem will also help us consider the general case.

Theorem 11 *Let N be a submodule of an R-module M. Suppose that M/N is a representable R-module and every element of $Att_R(M/N)$ is minimal (with respect to the inclusion). Then*

$$rad_M(N) = \left\{ m \in M : cm \in N + \sqrt{(N : M)}M \text{ for some } c \in R \backslash \bigcup Att_R(M/N) \right\}.$$

Proof In view of Lemma 7, we may assume that $N = 0$. Let $M = S_1 + \cdots + S_k$ be a minimal secondary representation for M, with $P_i = \sqrt{Ann_R(S_i)}$. Suppose that every element of $Att_R(M)$ is minimal. Clearly, we have $\sqrt{Ann_R(M)} = \bigcap_{i=1}^k P_i$. Define $L = \{m \in M : cm \in (\bigcap_{i=1}^k P_i)M$ for some $c \in R\backslash \bigcup_{i=1}^k P_i\}$. Let $m \in L$. Then there exists $c \in R\backslash \bigcup_{i=1}^k P_i$ such that $cm \in (\bigcap_{i=1}^k P_i)M = \sqrt{Ann_R(M)}M$. On the other hand, $M = cM$. (Use the secondary representation given for M above). Now, write $m = cm'$ for a suitable $m' \in M$. Then $cm = c^2m' \in \sqrt{Ann_R(M)}M \subseteq rad_M(0)$. Since $rad_M(0)$ is a semi-prime submodule of M we have $m = cm' \in rad_M(0)$. Therefore $L \subseteq rad_M(0)$. If $L = M$, then we are thorough. Thus we assume $L \neq M$. Now, by Proposition 10, L is a radical submodule of M, and so $rad_M(L) = L$. This completes the proof. □

Lemma 12 *Let M be a representable R-module and let $M = S_1 + \cdots + S_n$ be a minimal secondary representation for M, with $\sqrt{Ann_R(S_i)} = P_i$, for $i = 1, \ldots, n$. Without losing generality, assume that $\{P_1, \ldots, P_k\}$ is the set of all minimal elements of $Att_R(M)$, where $k \leq n$ and set $G = S_1 + \cdots + S_k$. Then $rad_G(0) = rad_M(0) \cap G$.*

Proof Clearly, $Att_R(G) = \{P_1, \ldots, P_k\}$. If $G = M$, then there is nothing to prove. Thus we assume $G \neq M$. In this case, $k < n$. By [3, Lemma 4], $rad_G(0) \subseteq rad_M(0) \cap G$. Let $x \in rad_M(0) \cap G$, and let H be a P-prime submodule of G. We shall first show that $(G : M) \nsubseteq P$. Assume contrarily that $(G : M) \subseteq P$. Since

$$\bigcap_{j=k+1}^{n} P_j = Ann_R(S_{k+1} + \cdots + S_n) \subseteq (S_1 + \cdots + S_k : S_{k+1} + \cdots + S_n) = (G : M) \subseteq P$$

we have $P_j \subseteq P$ for some $j = k + 1, \ldots, n$. Now, let $r \in P$. Then $rG \subseteq H$. If $r \in R\backslash \bigcup_{i=1}^k P_i$, then $rS_i = S_i$ for all $i = 1, \ldots, k$, and hence $G = rG \subseteq H$, which contradicts with the choice of H. Thus $r \in \bigcup_{i=1}^k P_i$. This gives that $P \subseteq \bigcup_{i=1}^k P_i$, and, by the Prime Avoidance Theorem (see [12]), $P \subseteq P_i$ for some $i = 1, \ldots, k$. But, in this case, $P_j \subseteq P \subseteq P_i$, which contradicts the fact that P_i is a minimal element of $Att_R(M)$ but P_j is not. It follows that $(G : M) \nsubseteq P$. By Lemma 6, there exists a prime submodule K of M such that $H = K \cap G$. Since $x \in rad_M(0) \subseteq K$, we obtain that $x \in K \cap G = H$. Since H is an arbitrary prime submodule of G, we have $x \in rad_G(0)$. □

Now, we conclude our work with a generalization of Theorem 11, as follows:

Theorem 13 *With the notation of Lemma 12,*

$$rad_M(0) = rad_M(G) \cap \{m \in M : cm \in (\bigcap_{i=1}^{k} P_i)M \text{ for some } c \in R\backslash \bigcup_{i=1}^{k} P_i\}.$$

Proof If $G = M$, then we are thorough, by Theorem 11. So, assume that $G \neq M$. It follows easily from Proposition 10 that

$$rad_M(0) \subseteq rad_M(G) \cap \{m \in M : cm \in \left(\bigcap_{i=1}^{k} P_i\right) M \text{ for some } c \in R\backslash \bigcup_{i=1}^{k} P_i\}.$$

Now let $m \in rad_M(G)$ and let there exist $c \in R\backslash \bigcup_{i=1}^k P_i$ such that $cm \in \left(\bigcap_{i=1}^k P_i\right) M$. Let K be a P-prime submodule of M. If $G \subseteq K$, then $m \in rad_M(G) \subseteq K$. Now, assume that $G \nsubseteq K$. It is not difficult to see that $\sqrt{Ann_R(M)} = \bigcap_{i=1}^k P_i$. Thus $P = (K : M) \supseteq \sqrt{(0 : M)} = \bigcap_{i=1}^k P_i$, and so $cm \in K$. Now, suppose that $c \in P$. Then $cG \subseteq K$. But, since $G \nsubseteq K$ we have $cG \neq G$. This gives that $c \in \bigcup_{i=1}^k P_i$, a contradiction.

Therefore $c \notin P$, and hence $m \in K$ because K is P-prime. It follows from the fact that K is arbitrarily chosen $m \in rad_M(0)$. This completes the proof. □

In Theorem 13, we could calculate $rad_M(0)$ inductively: Note that

$$M/G = (G + S_{k+1})/G + \cdots + (G + S_n)/G$$

is again a minimal secondary representation, and $Att_R(M/G) = \{P_{k+1}, \ldots, P_n\}$. In view of the equality $rad_M(G)/G = rad_{M/G}(0_{M/G})$, apply Theorem 13 to $rad_{M/G}(0_{M/G})$ to obtain $rad_M(G)$, and continue in this way, if possible. More precisely, if $M = S_1 + \cdots + S_n$ is a minimal secondary representation of M we can form submodules $G_1, \ldots, G_t$ of M each of which is a sum of some S_i's such that $M = G_1 + \cdots + G_t$ and $Att_R(G_{i+1})$ consists of all the minimal elements of $Att_R(M) \backslash Att_R(\sum_{j=0}^{i} G_j)$ for each $i = 0, \ldots, t-1$ where $G_0 = 0$ and $Att_R(G_0) = \varnothing$. Since $\bigcap Att_R(G_{i+1}) \nsubseteq \bigcup Att_R(G_1 + \cdots + G_i)$, we must have

$$\left(\bigcap Att_R(G_{i+1})\right) \frac{M}{G_1 + \cdots + G_i} = \frac{(\bigcap Att_R(G_{i+1}))\, M}{G_1 + \cdots + G_i}.$$

It is now easy to see that

$$rad_M(0) = \bigcap_{i=1}^{t} \left\{ m \in M : cm \in \left(\bigcap Att_R(G_i)\right) M \text{ for some } c \in R \backslash \left(\bigcup Att_R(G_i)\right) \right\}.$$

If, in particular, $Att_R(M) = \{P_1, \ldots, P_n\}$ where $P_1 \subset \ldots \subset P_n$, then $rad_M(0) = \bigcap_{i=1}^{n} \{m \in M : cm \in P_i M$ for some $c \in R \backslash P_i\}$.

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