

1-1-2013

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### Recommended Citation

WANG, YUPING; Yang, Chuan Fu; and HUANG, ZHENYOU (2013) "Half inverse problem for Sturm-Liouville operators with boundary conditions dependent on the spectral parameter," *Turkish Journal of Mathematics*: Vol. 37: No. 3, Article 8. <https://doi.org/10.3906/mat-1109-28>  
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## Half inverse problem for Sturm-Liouville operators with boundary conditions dependent on the spectral parameter

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Received: 14.09.2011 • Accepted: 05.03.2012 • Published Online: 26.04.2013 • Printed: 27.05.2013

**Abstract:** In this paper, we discuss the half inverse problem for the Sturm-Liouville operator with boundary conditions dependent on the spectral parameter and show that if  $q(x)$  is prescribed on  $[\frac{\pi}{2}, \pi]$ , then one spectrum is sufficient to determine the potential  $q(x)$  on the whole interval  $[0, \pi]$  and coefficient function  $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$  of the boundary condition.

**Key words:** Half inverse problem, Sturm-Liouville operator, boundary condition dependent on the spectral parameter

### 1. Introduction

Consider the following Sturm-Liouville operator  $L$  defined by

$$Ly = -y'' + q(x)y = \lambda y, (x \in [0, \pi]) \quad (1.1)$$

with boundary conditions dependent on the spectral parameter

$$y'(0, \lambda) - hy(0, \lambda) = 0 \quad (1.2)$$

or

$$(a_1\lambda + b_1)y(0, \lambda) - (c_1\lambda + d_1)y'(0, \lambda) = 0, \quad (1.2')$$

and

$$y'(\pi, \lambda) + Hy(\pi, \lambda) = 0 \quad (1.3)$$

or

$$(a_2\lambda + b_2)y(\pi, \lambda) - (c_2\lambda + d_2)y'(\pi, \lambda) = 0, \quad (1.3')$$

where  $h, H, a_k, b_k, c_k, d_k \in \mathbf{R}, c_1c_2 \neq 0$  such that

$$(-1)^k \delta_k = a_k d_k - b_k c_k > 0 (k = 1, 2)$$

and  $q$  is a real-valued function and  $q \in L^1[0, \pi]$ .

Let  $B(q, h, H)$ ,  $B(q, h)$  and  $B(q)$  be the Sturm-Liouville problem (1.1)–(1.3), the Sturm-Liouville problem (1.1), (1.2), (1.3') and the Sturm-Liouville problem (1.1), (1.2'), (1.3'), respectively.

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2000 AMS Mathematics Subject Classification: 34A55, 34B24, 47E05.

Sturm-Liouville operators with boundary conditions dependent on the spectral parameter were originated from engineering, physics and mathematics and have received substantial attention (see [1–3, 6, 8, 9, 11]). Fulton [8] discussed the Sturm-Liouville problem  $B(q, h)$  and obtained the eigenfunction expansion and asymptotic estimates of eigenvalues. Binding, Browne and Seddighi [2] considered the Sturm-Liouville operator  $l$  satisfying

$$ly = -(py')' + qy = \lambda ry$$

and boundary conditions (1.2') and (1.3'). They got oscillation and comparison results as well as the asymptotic estimates of eigenvalues, which can be considered as extension of Fulton's results. Browne and Sleeman [3] discussed the inverse nodal problem for the Sturm-Liouville problem  $B(q, h)$  and showed that a dense set of nodal points of eigenfunctions for the Sturm-Liouville problem  $B(q, h)$  is sufficient to determine the potential  $q$  and coefficient  $h$  of the boundary condition. Guliyev [11] considered the regularized trace problem for the Sturm-Liouville equation with spectral parameter in the boundary conditions and obtained the trace formula.

Inverse problem for Sturm-Liouville operators consists of reconstruction of the operator by its spectral data (see [3, 4, 10, 12–14, 17–23]). Half inverse problem for Sturm-Liouville operators is to determine a differential operator by one spectrum and half of its potential (see [4, 10, 13, 14, 17–19]). Hochstadt and Lieberman [13] firstly considered the half inverse problem for the Sturm-Liouville problem  $B(q, h, H)$  and showed that if  $q(x)$  is prescribed on  $[\frac{\pi}{2}, \pi]$ , then the potential  $q(x)$  on the interval  $[0, \pi]$  can be uniquely determined by one spectrum. Castillo [4] also discussed the half inverse problem for the Sturm-Liouville problem  $B(q, h, H)$ . By an example, Castillo showed that the necessity of the boundary condition (1.3) is given. Koyunbakan and Panakhov [14] considered the half inverse problem for diffusion operators on the finite interval  $[0, \pi]$  and showed that if  $p(x)$  is known on the  $[0, \pi]$ , and  $q(x)$  is prescribed on  $[\frac{\pi}{2}, \pi]$ , then only one spectrum is sufficient to determine the potential  $q(x)$  on the interval  $[0, \frac{\pi}{2})$  for the diffusion operator on the finite interval  $[0, \pi]$ . Hryniv and Mykytyuk [17] studied the half inverse spectral problems for Sturm-Liouville operators with singular potentials. Sakhnovich [18] proved the existence of the solution for the half inverse problem of Sturm-Liouville problems and gave a method of reconstructing this solution under some conditions. Using Weyl  $m$ -function techniques, Gesztesy and Simon [10] established a uniqueness theorem (see [10, Theorem 1.3]) by partial spectra and information on the potential, which is a generalization of Hochstadt and Lieberman's theorem [13]. In 2009, Wei and Xu [19] considered the half inverse problem missing one eigenvalue and obtained a uniqueness theorem on the potential  $q$ .

In this paper, we discuss the half inverse problem for the Sturm-Liouville problem  $B(q)$  and show that if  $q(x)$  is prescribed on  $[\frac{\pi}{2}, \pi]$ , then the potential  $q(x)$  on the whole interval  $[0, \pi]$  and coefficient function  $\frac{a_1\lambda + b_1}{c_1\lambda + d_1}$  of the boundary condition can be uniquely determined by one spectrum. Although the technique is based on the Hochstadt and Lieberman's methods, our model is different from that in the reference [13].

The following two Lemmas are important to prove our results.

**Lemma 1.1** ([2, Theorem 4.2]) *Let  $\sigma(L) = \{\lambda_n\}_{n=0}^\infty$  be the spectrum of the Sturm-Liouville problem  $B(q)$ , then  $\lambda_n (n = 0, 1, 2, \dots)$  is real and simple and satisfies the relations*

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots \rightarrow +\infty$$

and

$$\sqrt{\lambda_n} = n[1 + O(\frac{1}{n})]. \tag{1.4}$$

Suppose  $\varphi(x), \theta(x)$  are the two fundamental solutions of the equation (1.1) and satisfy

$$\varphi(0) = 1, \varphi'(0) = 0$$

and

$$\theta(0) = 0, \theta'(0) = 1,$$

respectively, then the solution of the equation (1.1) with boundary condition (1.2') is

$$y(x, \lambda) = (c_1\lambda + d_1)\varphi(x) + (a_1\lambda + b_1)\theta(x). \tag{1.5}$$

By the transformations (see [15, p. 144–145 or 7, p. 20–22]), we have

$$\begin{aligned} \varphi(x) &= \cos \sqrt{\lambda}x + \int_0^x A(x, t) \cos \sqrt{\lambda}tdt \\ \theta(x) &= \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x B(x, t) \sin \sqrt{\lambda}tdt, \end{aligned} \tag{1.6}$$

where the kernels  $A(x, t)$  and  $B(x, t)$  satisfy

$$\frac{\partial^2 K(x, t)}{\partial x^2} - q(x)K(x, t) = \frac{\partial^2 K(x, t)}{\partial t^2},$$

where  $q(x) = 2\frac{d}{dx}A(x, x) = 2\frac{d}{dx}B(x, x)$ ,  $\frac{\partial A(x, t)}{\partial t}|_{t=0} = B(x, 0) = 0$ .

By virtue of (1.5) and (1.6), this yields the following lemma.

**Lemma 1.2** ([7, 15]) *The solution of the equation (1.1) with boundary condition (1.2') can be written as*

$$\begin{aligned} y(x, \lambda) &= (c_1\lambda + d_1)[\cos \sqrt{\lambda}x + \int_0^x A(x, t) \cos \sqrt{\lambda}tdt] \\ &+ (a_1\lambda + b_1)[\frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x B(x, t) \sin \sqrt{\lambda}tdt]. \end{aligned} \tag{1.7}$$

## 2. Main results and Proofs

Consider another Sturm-Liouville operator  $\tilde{L}$  satisfying

$$\tilde{L}y = -y'' + \tilde{q}(x)y = \lambda y \tag{2.1}$$

with boundary conditions dependent on the spectral parameter

$$(\tilde{a}_1\lambda + \tilde{b}_1)y(0, \lambda) - (\tilde{c}_1\lambda + \tilde{d}_1)y'(0, \lambda) = 0 \tag{2.2}$$

and

$$(a_2\lambda + b_2)y(\pi, \lambda) - (c_2\lambda + d_2)y'(\pi, \lambda) = 0, \tag{2.3}$$

where  $\tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{d}_1, a_2, b_2, c_2, d_2 \in \mathbf{R}, \tilde{c}_1c_2 \neq 0$  such that

$$\tilde{\delta}_1 = \tilde{a}_1\tilde{d}_1 - \tilde{b}_1\tilde{c}_1 < 0, \delta_2 = a_2d_2 - b_2c_2 > 0$$

and  $\tilde{q}$  is a real-valued function and  $\tilde{q} \in L^1[0, \pi]$ .

Denote the Sturm-Liouville problem (2.1)-(2.3) by  $B(\tilde{q})$ . We have the following main results.

**Theorem 2.1** Let  $\{\lambda_n\}_{n=0}^\infty$  be real and simple spectrum of the Sturm-Liouville problem  $B(q)$  and Let  $\{\tilde{\lambda}_n\}_{n=0}^\infty$  be real and simple spectrum of the Sturm-Liouville problem  $B(\tilde{q})$ , respectively. If  $\lambda_n = \tilde{\lambda}_n (n = 0, 1, 2, \dots)$ ,  $q(x) = \tilde{q}(x)$  on  $[\frac{\pi}{2}, \pi]$  and  $q, \tilde{q} \in W_2^6[0, \pi]$ , then

$$q(x) = \tilde{q}(x) \text{ a.e. on } [0, \pi],$$

and

$$\frac{\tilde{a}_1\lambda + \tilde{b}_1}{\tilde{c}_1\lambda + \tilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}, \quad \forall \lambda \in \mathbf{C},$$

where  $q, \tilde{q}$  are real-valued functions.

**Remark 2.2** We use the condition  $q, \tilde{q} \in W_2^6[0, \pi]$  in the proof of Theorem 2.1 (see below).

**Proof** According to Lemma 1.2, the solutions of the equation (1.1) with boundary condition (1.2') and the equation (2.1) with boundary condition (2.2) can be expressed in the integral forms

$$y(x, \lambda) = (c_1\lambda + d_1)[\cos \sqrt{\lambda}x + \int_0^x A(x, t) \cos \sqrt{\lambda}tdt] + (a_1\lambda + b_1)[\frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x B(x, t) \sin \sqrt{\lambda}tdt] \tag{2.4}$$

and

$$\tilde{y}(x, \lambda) = (\tilde{c}_1\lambda + \tilde{d}_1)[\cos \sqrt{\lambda}x + \int_0^x \tilde{A}(x, t) \cos \sqrt{\lambda}tdt] + (\tilde{a}_1\lambda + \tilde{b}_1)[\frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x \tilde{B}(x, t) \sin \sqrt{\lambda}tdt], \tag{2.5}$$

respectively. Let  $\lambda = s^2$ , from (2.4) and (2.5), we have

$$y\tilde{y} = (c_1s^2 + d_1)(\tilde{c}_1s^2 + \tilde{d}_1)(\cos sx + \int_0^x A(x, t) \cos stdt) \cdot (\cos sx + \int_0^x \tilde{A}(x, t) \cos stdt) + \frac{(a_1s^2 + b_1)(\tilde{a}_1s^2 + \tilde{b}_1)}{s^2} \cdot (\sin sx + \int_0^x B(x, t) \sin stdt)(\sin sx + \int_0^x \tilde{B}(x, t) \sin stdt) + \frac{1}{s}(c_1s^2 + d_1)(\tilde{a}_1s^2 + \tilde{b}_1) \cdot (\cos sx + \int_0^x A(x, t) \cos stdt) \cdot (\sin sx + \int_0^x \tilde{B}(x, t) \sin stdt) + \frac{1}{s}(\tilde{c}_1s^2 + \tilde{d}_1)(a_1s^2 + b_1) \cdot (\cos sx + \int_0^x \tilde{A}(x, t) \cos stdt)(\sin sx + \int_0^x B(x, t) \sin stdt). \tag{2.6}$$

Using the trigonometric addition formulae, extending the range of  $A(x, t), \tilde{A}(x, t), B(x, t)$  and  $\tilde{B}(x, t)$  with respect to the second argument, respectively. i.e.,  $A(x, -t) = A(x, t), \tilde{A}(x, -t) = \tilde{A}(x, t), B(x, -t) = -B(x, t)$  and  $\tilde{B}(x, -t) = -\tilde{B}(x, t)$ , we can find that

$$(\cos sx + \int_0^x A(x, t) \cos stdt)(\cos sx + \int_0^x \tilde{A}(x, t) \cos stdt) = \frac{1}{2}[1 + \cos 2sx + \int_0^x k(x, \tau) \cos 2s\tau d\tau], \tag{2.7}$$

$$(\sin sx + \int_0^x B(x, t) \sin stdt)(\sin sx + \int_0^x \tilde{B}(x, t) \sin stdt) = \frac{1}{2}[1 - \cos 2sx + \int_0^x h(x, \tau) \cos 2s\tau d\tau], \tag{2.8}$$

$$(\cos sx + \int_0^x A(x, t) \cos stdt)(\sin sx + \int_0^x \tilde{B}(x, t) \sin stdt) = \frac{1}{2}[\sin 2sx + \int_0^x l(x, \tau) \sin 2s\tau d\tau] \tag{2.9}$$

and

$$(\cos sx + \int_0^x \tilde{A}(x, t) \cos stdt)(\sin sx + \int_0^x B(x, t) \sin stdt) = \frac{1}{2}[\sin 2sx + \int_0^x m(x, \tau) \sin 2s\tau d\tau], \tag{2.10}$$

where

$$\begin{aligned}
 k(x, \tau) &= A(x, x - 2\tau) + \tilde{A}(x, x - 2\tau) + A(x, x + 2\tau) + \tilde{A}(x, x + 2\tau) \\
 &\quad + 2[\int_{-x+2\tau}^x A(x, v)\tilde{A}(x, v - 2\tau)ds + \int_{-x}^{x-2\tau} A(x, v)\tilde{A}(x, v + 2\tau)dv], \\
 h(x, \tau) &= B(x, x + 2\tau) + \tilde{B}(x, x + 2\tau) - B(x, x - 2\tau) - \tilde{B}(x, x - 2\tau) \\
 &\quad + 2[\int_{-x+2\tau}^x B(x, v)\tilde{B}(x, v - 2\tau)dv + \int_{-x}^{x-2\tau} B(x, v)\tilde{B}(x, v + 2\tau)dv], \\
 l(x, \tau) &= 2[A(x, x - 2\tau) + \tilde{B}(x, x - 2\tau)] \\
 &\quad + 2[\int_{-x+2\tau}^x A(x, v)\tilde{B}(x, v - 2\tau)ds + \int_{-x}^{x-2\tau} A(x, v)\tilde{B}(x, v + 2\tau)dv]
 \end{aligned}$$

and

$$\begin{aligned}
 m(x, \tau) &= 2[B(x, x - 2\tau) + \tilde{A}(x, x - 2\tau)] \\
 &\quad + 2[\int_{-x+2\tau}^x B(x, v)\tilde{A}(x, v - 2\tau)ds + \int_{-x}^{x-2\tau} B(x, v)\tilde{A}(x, v + 2\tau)dv].
 \end{aligned}$$

From (2.7), (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned}
 y\tilde{y} &= \frac{(c_1s^2+d_1)(\tilde{c}_1s^2+\tilde{d}_1)}{2}(1 + \cos 2sx + \int_0^x k(x, \tau) \cos 2s\tau d\tau) \\
 &\quad + \frac{(a_1s^2+b_1)(\tilde{a}_1s^2+\tilde{b}_1)}{2s^2}(1 - \cos 2sx + \int_0^x h(x, \tau) \cos 2s\tau d\tau) \\
 &\quad + \frac{1}{2s}(c_1s^2 + d_1)(\tilde{a}_1s^2 + \tilde{b}_1)(\sin 2sx + \int_0^x l(x, \tau) \sin 2s\tau d\tau) \\
 &\quad + \frac{1}{2s}(a_1s^2 + b_1)(\tilde{c}_1s^2 + \tilde{d}_1)(\sin 2sx + \int_0^x m(x, \tau) \sin 2s\tau d\tau),
 \end{aligned} \tag{2.11}$$

Define the function  $\omega(\lambda)$  by

$$\omega(\lambda) = (a_2\lambda + b_2)y(\pi, \lambda) - (c_2\lambda + d_2)y'(\pi, \lambda).$$

From (2.4), we have the following asymptotic forms

$$y(\pi, \lambda) = (c_1\lambda + d_1) \cos \sqrt{\lambda}\pi + O(|\sqrt{\lambda}|e^{Im\sqrt{\lambda}|\pi})]$$

and

$$y'(\pi, \lambda) = -(c_1\lambda + d_1)\sqrt{\lambda} \sin \sqrt{\lambda}\pi + O(|\lambda|e^{Im\sqrt{\lambda}|\pi}).$$

Hence,

$$\omega(\lambda) = (c_1\lambda + d_1)(c_2\lambda + d_2)\sqrt{\lambda} \sin \sqrt{\lambda}\pi + O(|\lambda|^2e^{Im\sqrt{\lambda}|\pi}). \tag{2.12}$$

Zeros of  $\omega(\lambda)$  are the eigenvalues of the Sturm-Liouville problem  $B(q)$ .  $\omega(\lambda)$  is an entire function of order  $\frac{1}{2}$  of  $\lambda$ . Multiplying (2.1) by  $y$ , (1.1) by  $\tilde{y}$  and subtracting and integrating from 0 to  $\pi$ , we obtain

$$(\tilde{y}y' - y\tilde{y}')|_0^\pi + \int_0^\pi (\tilde{q} - q)y\tilde{y}dx = 0.$$

Using  $y(0, \lambda) = c_1\lambda + d_1$ ,  $\tilde{y}(0, \lambda) = \tilde{c}_1\lambda + \tilde{d}_1$ ,  $y'(0, \lambda) = a_1\lambda + b_1$ ,  $\tilde{y}'(0, \lambda) = \tilde{a}_1\lambda + \tilde{b}_1$  and  $\tilde{q}(x) = q(x)$ ,  $x \in [\frac{\pi}{2}, \pi]$ , we get

$$\begin{aligned}
 &[\tilde{y}(\pi, \lambda)y'(\pi, \lambda) - y(\pi, \lambda)\tilde{y}'(\pi, \lambda)] + (a_1\lambda + b_1)(\tilde{c}_1\lambda + \tilde{d}_1) \\
 &- (c_1\lambda + d_1)(\tilde{a}_1\lambda + \tilde{b}_1) + \int_0^{\frac{\pi}{2}} (\tilde{q}(x) - q(x))y\tilde{y}dx = 0.
 \end{aligned} \tag{2.13}$$

Let

$$Q(x) = \tilde{q}(x) - q(x)$$

and

$$K(\lambda) = (a_1\tilde{c}_1 - \tilde{a}_1c_1)\lambda^2 + (a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1d_1 - \tilde{b}_1c_1)\lambda + (b_1\tilde{d}_1 - \tilde{b}_1d_1) + \int_0^{\frac{\pi}{2}} Q(x)y(x, \lambda)\tilde{y}(x, \lambda)dx. \tag{2.14}$$

Obviously, the function  $K(\lambda)$  is an entire function. Since the first term of equation (2.13) for  $\lambda = \lambda_n$  is zero, then

$$K(\lambda_n) = 0.$$

In addition, using (2.4) and (2.5) for  $0 \leq x \leq \frac{\pi}{2}$ , we have

$$|K(\lambda)| \leq M|\lambda|^2 e^{Im\sqrt{\lambda}\pi}, \tag{2.15}$$

where  $M$  is constant. Let

$$\psi(\lambda) = \frac{K(\lambda)}{\omega(\lambda)}. \tag{2.16}$$

Then,  $\psi(\lambda)$  is an entire function. Using (2.12) and (2.16), we obtain

$$|\psi(\lambda)| = O\left(\frac{1}{\sqrt{|\lambda|}}\right).$$

By the Liouville theorem, we get

$$\psi(\lambda) = 0, \forall \lambda \in \mathbf{C}.$$

Hence

$$K(\lambda) = 0, \forall \lambda \in \mathbf{C}.$$

i.e.,

$$(a_1\tilde{c}_1 - \tilde{a}_1c_1)\lambda^2 + (a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1d_1 - \tilde{b}_1c_1)\lambda + (b_1\tilde{d}_1 - \tilde{b}_1d_1) + \int_0^{\frac{\pi}{2}} Q(x)y(x, \lambda)\tilde{y}(x, \lambda)dx = 0. \tag{2.17}$$

Substituting (2.11) into (2.17), we have

$$\begin{aligned} & 2(a_1\tilde{c}_1 - \tilde{a}_1c_1)s^6 + 2(a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1d_1 - \tilde{b}_1c_1)s^4 + 2(b_1\tilde{d}_1 - \tilde{b}_1d_1)s^2 \\ & + \int_0^{\frac{\pi}{2}} Q(x)\{[c_1\tilde{c}_1s^6 + (c_1\tilde{d}_1 + \tilde{c}_1d_1)s^4 + d_1\tilde{d}_1s^2] \\ & \cdot (1 + \cos 2sx + \int_0^x k(x, \tau) \cos 2s\tau d\tau) \\ & + [a_1\tilde{a}_1s^4 + (a_1\tilde{b}_1 + \tilde{a}_1b_1)s^2 + b_1\tilde{b}_1](1 - \cos 2sx + \int_0^x h(x, \tau) \cos 2s\tau d\tau) \\ & + s^5[(\tilde{a}_1c_1 + a_1\tilde{c}_1) \sin 2sx + c_1\tilde{c}_1 \int_0^x k_5(x, \tau) \sin 2s\tau d\tau] \\ & + s^3[(\tilde{a}_1d_1 + a_1\tilde{d}_1 + \tilde{b}_1c_1 + b_1\tilde{c}_1) \sin 2sx + c_1\tilde{c}_1 \int_0^x k_3(x, \tau) \sin 2s\tau d\tau] \\ & + s[(\tilde{b}_1d_1 + b_1\tilde{d}_1) \sin 2sx + c_1\tilde{c}_1 \int_0^x k_1(x, \tau) \sin 2s\tau d\tau]\}dx = 0, \end{aligned} \tag{2.18}$$

where

$$k_1(x, \tau) = \frac{1}{c_1\tilde{c}_1}(\tilde{b}_1d_1l(x, \tau) + b_1\tilde{d}_1m(x, \tau)),$$

$$k_5(x, \tau) = \frac{1}{c_1\tilde{c}_1}(\tilde{a}_1c_1l(x, \tau) + a_1\tilde{c}_1m(x, \tau))$$

and

$$k_3(x, \tau) = \frac{1}{c_1\tilde{c}_1}(\tilde{a}_1d_1l(x, \tau) + a_1\tilde{d}_1m(x, \tau) + \tilde{b}_1c_1l(x, \tau) + b_1\tilde{c}_1m(x, \tau)).$$

Changing the integral variables, it can be written as

$$\begin{aligned}
 & 2(a_1\tilde{c}_1 - \tilde{a}_1c_1)s^6 + 2(a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1d_1 - \tilde{b}_1c_1)s^4 + 2(b_1\tilde{d}_1 - \tilde{b}_1d_1)s^2 \\
 & + [c_1\tilde{c}_1s^6 + (c_1\tilde{d}_1 + \tilde{c}_1d_1)s^4 + d_1\tilde{d}_1s^2] \int_0^{\frac{\pi}{2}} Q(\tau) \{ (1 + \cos 2s\tau \\
 & + \cos 2s\tau \int_{\tau}^{\frac{\pi}{2}} Q(x)k(x, \tau)dx) + [a_1\tilde{a}_1s^4 + (a_1\tilde{b}_1 + \tilde{a}_1b_1)s^2 + b_1\tilde{b}_1] \\
 & \cdot [(1 - \cos 2s\tau + \cos 2s\tau \int_{\tau}^{\frac{\pi}{2}} Q(x)h(x, \tau)dx] \\
 & + s^5 [(\tilde{a}_1c_1 + a_1\tilde{c}_1) \sin 2s\tau + c_1\tilde{c}_1 \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x)k_5(x, \tau)dx] \\
 & + s^3 [(\tilde{a}_1d_1 + a_1\tilde{d}_1 + \tilde{b}_1c_1 + b_1\tilde{c}_1) \sin 2s\tau + c_1\tilde{c}_1 \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x)k_3(x, \tau)dx] \\
 & + s [(\tilde{b}_1d_1 + b_1\tilde{d}_1) \sin 2s\tau + c_1\tilde{c}_1 \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x)k_1(x, \tau)dx] d\tau = 0.
 \end{aligned} \tag{2.19}$$

Divided by  $s^6$  in (2.19) and letting  $s = \sqrt{\lambda} \rightarrow +\infty (\lambda \in \mathbf{R})$ , from the Riemann-Lebesgue lemma, we obtain

$$2(a_1\tilde{c}_1 - \tilde{a}_1c_1) + c_1\tilde{c}_1 \int_0^{\frac{\pi}{2}} Q(x)dx = 0, \tag{2.20}$$

and

$$\begin{aligned}
 & \frac{1}{c_1\tilde{c}_1} \{ [2(a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1d_1 - \tilde{b}_1c_1) + a_{04} \int_0^{\frac{\pi}{2}} Q(x)dx] s^4 \\
 & + [2(b_1\tilde{d}_1 - \tilde{b}_1d_1) + a_{02} \int_0^{\frac{\pi}{2}} Q(x)dx] s^2 \} \\
 & + \int_0^{\frac{\pi}{2}} \{ s^6 [Q(\tau) \cos 2s\tau + \cos 2s\tau \int_{\tau}^{\frac{\pi}{2}} Q(x)k(x, \tau)dx] d\tau \\
 & + s^5 [a_{15}Q(\tau) \sin 2s\tau + \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x)k_5(x, \tau)dx] \\
 & + s^4 [a_{14}Q(\tau) \cos 2s\tau + \cos 2s\tau \int_{\tau}^{\frac{\pi}{2}} Q(x)k_4(x, \tau)dx] \\
 & + s^3 [a_{13}Q(\tau) \sin 2s\tau + \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x)k_3(x, \tau)dx] \\
 & + s^2 [a_{12}Q(\tau) \cos 2s\tau + \cos 2s\tau \int_{\tau}^{\frac{\pi}{2}} Q(x)k_2(x, \tau)dx] \\
 & + s [a_{11}Q(\tau) \sin 2s\tau + \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x)k_1(x, \tau)dx] \\
 & + [-a_{10}Q(\tau) \cos 2s\tau + a_{10} \cos 2s\tau \int_{\tau}^{\frac{\pi}{2}} Q(x)h(x, \tau)dx] \} d\tau = 0,
 \end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
 a_{10} &= \frac{1}{c_1\tilde{c}_1} b_1\tilde{b}_1, & a_{11} &= \frac{1}{c_1\tilde{c}_1} (\tilde{b}_1d_1 + b_1\tilde{d}_1), \\
 a_{02} &= \frac{1}{c_1\tilde{c}_1} (d_1\tilde{d}_1 + a_1\tilde{b}_1 + \tilde{a}_1d_1), & a_{12} &= \frac{1}{c_1\tilde{c}_1} (d_1\tilde{d}_1 - a_1\tilde{b}_1 - \tilde{a}_1d_1), \\
 a_{13} &= \frac{1}{c_1\tilde{c}_1} (\tilde{a}_1d_1 + a_1\tilde{d}_1 + \tilde{b}_1c_1 + b_1\tilde{c}_1), \\
 a_{04} &= \frac{1}{c_1\tilde{c}_1} (c_1\tilde{d}_1 + \tilde{c}_1d_1 + a_1\tilde{a}_1), & a_{14} &= \frac{1}{c_1\tilde{c}_1} (c_1\tilde{d}_1 + \tilde{c}_1d_1 - a_1\tilde{a}_1), \\
 a_{15} &= \frac{1}{c_1\tilde{c}_1} (\tilde{a}_1c_1 + a_1\tilde{c}_1), \\
 k_2(x, \tau) &= \frac{1}{c_1\tilde{c}_1} [d_1\tilde{d}_1k(x, \tau) + (a_1\tilde{b}_1 + \tilde{a}_1b_1)h(x, \tau)], \\
 k_4(x, \tau) &= \frac{1}{c_1\tilde{c}_1} [(c_1\tilde{d}_1 + \tilde{c}_1d_1)k(x, \tau) + a_1\tilde{a}_1h(x, \tau)].
 \end{aligned}$$

By integration by parts, letting  $s = \sqrt{\lambda} \rightarrow +\infty$ , from the Riemann-Lebesgue lemma we get

$$Q\left(\frac{\pi}{2}\right) = 0$$



and

$$\begin{aligned}
 & \frac{1}{c_1 \tilde{c}_1} \{ [2(a_1 \tilde{d}_1 + b_1 \tilde{c}_1 - \tilde{a}_1 d_1 - \tilde{b}_1 c_1) + a_{04} \int_0^{\frac{\pi}{2}} Q(x) dx] s^4 \\
 & + [2(b_1 \tilde{d}_1 - \tilde{b}_1 d_1) + a_{02} \int_0^{\frac{\pi}{2}} Q(x) dx] s^2 \} + \int_0^{\frac{\pi}{2}} \{ -\frac{1}{2} s^5 \\
 & \cdot [(Q'(\tau) - Q(\tau)k(\tau, \tau)) \sin 2s\tau + \sin 2s\tau \int_{\tau}^{\frac{\pi}{2}} Q(x) \frac{\partial k(x, \tau)}{\partial \tau} dx] \\
 & + s^5 [a_{15} Q(\tau) \sin 2s\tau + \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x) k_5(x, \tau) dx] \\
 & + s^4 [a_{14} Q(\tau) \cos 2s\tau + \cos 2s\tau \int_{\frac{\tau}{2}}^{\frac{\pi}{2}} Q(x) k_4(x, \tau) dx] \\
 & + s^3 [a_{13} Q(\tau) \sin 2s\tau + \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x) k_3(x, \tau) dx] \\
 & + s^2 [a_{12} Q(\tau) \cos 2s\tau + \cos 2s\tau \int_{\frac{\tau}{2}}^{\frac{\pi}{2}} Q(x) k_2(x, \tau) dx] \\
 & + s [a_{11} Q(\tau) \sin 2s\tau + \sin 2s\tau \int_0^{\frac{\pi}{2}} Q(x) k_1(x, \tau) dx] \\
 & + [-a_{10} Q(\tau) \cos 2s\tau + a_{10} \cos 2s\tau \int_{\tau}^{\frac{\pi}{2}} Q(x) h(x, \tau) dx] \} d\tau = 0.
 \end{aligned} \tag{2.22}$$

Repeating the above arguments five times, we obtain

$$2(a_1 \tilde{d}_1 + b_1 \tilde{c}_1 - \tilde{a}_1 d_1 - \tilde{b}_1 c_1) + a_{04} \int_0^{\frac{\pi}{2}} Q(x) dx = 0, \tag{2.23}$$

$$2(b_1 \tilde{d}_1 - \tilde{b}_1 d_1) + a_{02} \int_0^{\frac{\pi}{2}} Q(x) dx = 0, \tag{2.24}$$

$$Q^{(i)}\left(\frac{\pi}{2}\right) = 0, (i = 0, 1, 2, 3, 4, 5) \tag{2.25}$$

and

$$\begin{aligned}
 & \int_0^{\frac{\pi}{2}} \cos 2s\tau \{ Q^{(6)}(\tau) - k_{77}(\tau)Q^{(5)}(\tau) - k_{76}(\tau)Q^{(4)}(\tau) - k_{75}(\tau)Q'''(\tau) \\
 & - k_{74}(\tau)Q''(\tau) - k_{73}(\tau)Q'(\tau) - k_{72}(\tau)Q(\tau) + \int_{\tau}^{\frac{\pi}{2}} Q(x)k_{71}(x, \tau) dx \} d\tau = 0,
 \end{aligned} \tag{2.26}$$

where

$$\begin{aligned}
 k_{77}(\tau) &= k(\tau, \tau) + 2a_{15}, \\
 k_{76}(\tau) &= \left( 5 \frac{dk(\tau, \tau)}{d\tau} + \frac{\partial k(x, \tau)}{\partial \tau} \Big|_{x=\tau} \right) - 2k_5(\tau, \tau) + 4a_{14}, \\
 k_{75}(\tau) &= \left( 10 \frac{d^2 k(\tau, \tau)}{d\tau^2} + 4 \frac{d\left(\frac{\partial k(x, \tau)}{\partial \tau}\right)}{d\tau} \Big|_{x=\tau} + \frac{\partial^2 k(x, \tau)}{\partial \tau^2} \Big|_{x=\tau} \right) \\
 & \quad - 2 \left( 4 \frac{dk_5(\tau, \tau)}{d\tau} + \frac{\partial k_5(x, \tau)}{\partial \tau} \Big|_{x=\tau} \right) - 4k_4(\tau, \tau) - 8a_{13}, \\
 k_{74}(\tau) &= \left( 10 \frac{d^3 k(\tau, \tau)}{d\tau^3} + 6 \frac{d^2\left(\frac{\partial k(x, \tau)}{\partial \tau}\right)}{d\tau^2} \Big|_{x=\tau} + 3 \frac{d\left(\frac{\partial^2 k(x, \tau)}{\partial \tau^2}\right)}{d\tau} \Big|_{x=\tau} + \frac{\partial^3 k(x, \tau)}{\partial \tau^3} \Big|_{x=\tau} \right) \\
 & \quad - 2 \left( 6 \frac{d^2 k_5(\tau, \tau)}{d\tau^2} + 3 \frac{d\left(\frac{\partial k_5(x, \tau)}{\partial \tau}\right)}{d\tau} \Big|_{x=\tau} + \frac{\partial^2 k_5(x, \tau)}{\partial \tau^2} \Big|_{x=\tau} \right) \\
 & \quad - 4 \left( 3 \frac{dk_4(\tau, \tau)}{d\tau} + \frac{\partial k_4(x, \tau)}{\partial \tau} \Big|_{x=\tau} \right) + 8k_3(\tau, \tau) - 16a_{12}, \\
 k_{73}(\tau) &= \left( 5 \frac{d^4 k(\tau, \tau)}{d\tau^4} + 4 \frac{d^3\left(\frac{\partial k(x, \tau)}{\partial \tau}\right)}{d\tau^3} \Big|_{x=\tau} + 3 \frac{d^2\left(\frac{\partial^2 k(x, \tau)}{\partial \tau^2}\right)}{d\tau^2} \Big|_{x=\tau} + 2 \frac{d\left(\frac{\partial^3 k(x, \tau)}{\partial \tau^3}\right)}{d\tau} \Big|_{x=\tau} \right) \\
 & \quad + \frac{\partial^4 k(x, \tau)}{\partial \tau^4} \Big|_{x=\tau} - 2 \left( 4 \frac{d^3 k_5(\tau, \tau)}{d\tau^3} + 3 \frac{d^2\left(\frac{\partial k_5(x, \tau)}{\partial \tau}\right)}{d\tau^2} \Big|_{x=\tau} + 2 \frac{d\left(\frac{\partial^2 k_5(x, \tau)}{\partial \tau^2}\right)}{d\tau} \Big|_{x=\tau} \right) \\
 & \quad + \frac{\partial^3 k_5(x, \tau)}{\partial \tau^3} \Big|_{x=\tau} - 4 \left( 3 \frac{d^2 k_3(\tau, \tau)}{d\tau^2} + 2 \frac{d\left(\frac{\partial k_3(x, \tau)}{\partial \tau}\right)}{d\tau} \Big|_{x=\tau} + \frac{\partial^2 k_3(x, \tau)}{\partial \tau^2} \Big|_{x=\tau} \right) \\
 & \quad + 8 \left( 2 \frac{dk_2(\tau, \tau)}{d\tau} + \frac{\partial k_2(x, \tau)}{\partial \tau} \Big|_{x=\tau} \right) + 16k_1(\tau, \tau) + 32a_{11}, \\
 k_{72}(\tau) &= \left( \frac{d^5 k(\tau, \tau)}{d\tau^5} + \frac{d^4\left(\frac{\partial k(x, \tau)}{\partial \tau}\right)}{d\tau^4} \Big|_{x=\tau} + \frac{d^3\left(\frac{\partial^2 k(x, \tau)}{\partial \tau^2}\right)}{d\tau^3} \Big|_{x=\tau} + \frac{d^2\left(\frac{\partial^3 k(x, \tau)}{\partial \tau^3}\right)}{d\tau^2} \Big|_{x=\tau} \right) \\
 & \quad + \frac{d\left(\frac{\partial^4 k(x, \tau)}{\partial \tau^4}\right)}{d\tau} \Big|_{x=\tau} + \frac{\partial^5 k(x, \tau)}{\partial \tau^5} \Big|_{x=\tau} - 2 \left( \frac{d^4 k_5(\tau, \tau)}{d\tau^4} + \frac{d^3\left(\frac{\partial k_5(x, \tau)}{\partial \tau}\right)}{d\tau^3} \Big|_{x=\tau} \right) \\
 & \quad + \frac{d^2\left(\frac{\partial^2 k_5(x, \tau)}{\partial \tau^2}\right)}{d\tau^2} \Big|_{x=\tau} + \frac{d\left(\frac{\partial^3 k_5(x, \tau)}{\partial \tau^3}\right)}{d\tau} \Big|_{x=\tau} + \frac{\partial^4 k_5(x, \tau)}{\partial \tau^4} \Big|_{x=\tau} \\
 & \quad - 4 \left( \frac{d^3 k_4(\tau, \tau)}{d\tau^3} + \frac{d^2\left(\frac{\partial k_4(x, \tau)}{\partial \tau}\right)}{d\tau^2} \Big|_{x=\tau} + \frac{d\left(\frac{\partial^2 k_4(x, \tau)}{\partial \tau^2}\right)}{d\tau} \Big|_{x=\tau} + \frac{\partial^3 k_4(x, \tau)}{\partial \tau^3} \Big|_{x=\tau} \right) \\
 & \quad + 8 \left( \frac{d^2 k_3(\tau, \tau)}{d\tau^2} + \frac{d\left(\frac{\partial k_3(x, \tau)}{\partial \tau}\right)}{d\tau} \Big|_{x=\tau} + \frac{\partial^2 k_3(x, \tau)}{\partial \tau^2} \Big|_{x=\tau} \right) \\
 & \quad + 16 \left( \frac{dk_2(\tau, \tau)}{d\tau} + \frac{\partial k_2(x, \tau)}{\partial \tau} \Big|_{x=\tau} \right) - 32k_1(\tau, \tau) - 64a_{10},
 \end{aligned}$$

$$k_{71}(x, \tau) = \frac{\partial^6 k(x, \tau)}{\partial \tau^6} - 2 \frac{\partial^5 k_5(x, \tau)}{\partial \tau^5} - 4 \frac{\partial^4 k_4(x, \tau)}{\partial \tau^4} + 8 \frac{\partial^3 k_3(x, \tau)}{\partial \tau^3} + 16 \frac{\partial^2 k_2(x, \tau)}{\partial \tau^2} + 32 \frac{\partial k_1(x, \tau)}{\partial \tau} - 64 a_{10} h(x, \tau).$$

On the other hand, by the completeness of the functions  $\{\cos 2s\tau\} (s = \sqrt{\lambda} \in \mathbf{R})$ , we see that  $Q(x)$  satisfies

$$Q^{(6)}(\tau) - k_{77}(\tau)Q^{(5)}(\tau) - k_{76}(\tau)Q^{(4)}(\tau) - k_{75}(\tau)Q'''(\tau) - k_{74}(\tau)Q''(\tau) - k_{73}(\tau)Q'(\tau) - k_{72}(\tau)Q(\tau) + \int_{\tau}^{\frac{\pi}{2}} Q(x)k_{71}(x, \tau)dx = 0, \forall \tau \in [0, \frac{\pi}{2}]. \tag{2.27}$$

From (2.25), we have the identities

$$Q^{(i)}(\tau) + \int_{\tau}^{\frac{\pi}{2}} Q^{(i+1)}(x)dx = 0, (i = 0, 1, 2, 3, 4, 5). \tag{2.28}$$

Using (2.28), (2.27) can be written as

$$Q^{(6)}(\tau) + \int_{\tau}^{\frac{\pi}{2}} [k_{77}(\tau)Q^{(6)}(x) + k_{76}(\tau)Q^{(5)}(x) + k_{75}(\tau)Q^{(4)}(x) + k_{74}(\tau)Q'''(x) + k_{73}(\tau)Q''(x) + k_{72}(\tau)Q'(x) + k_{71}(x, \tau)Q(x)]dx = 0, \forall \tau \in [0, \frac{\pi}{2}]. \tag{2.29}$$

Let

$$\tilde{Q}(x) = (Q(x), Q'(x), \dots, Q^{(6)}(x))^T.$$

From (2.28) and (2.29), we get a vectorial integral equation

$$\tilde{Q}(\tau) + \int_{\tau}^{\frac{\pi}{2}} \tilde{K}(x, \tau)\tilde{Q}(x)dx = 0, \forall \tau \in [0, \frac{\pi}{2}], \tag{2.30}$$

where  $\tilde{K}(x, \tau) = (k_{ij}(x, \tau))_{7 \times 7}$  is a  $7 \times 7$  matrix-value function, where  $k_{ij}(x, \tau) = 1 (i = 1, 2, \dots, 6, j = i + 1), k_{ij}(x, \tau) = 0 (i = 1, 2, \dots, 6, j \neq i + 1), k_{7j}(x, \tau) (j = 1, 2, \dots, 7)$  are described above.

Since the equation (2.30) is a vectorial Volterra integral equation, it has only trivial solution (see [5, Theorem, p.1281, 16, Theorem 1.1.1, p.1]). Hence

$$\tilde{Q}(x) = (Q(x), Q'(x), \dots, Q^{(6)}(x))^T = 0 \text{ a.e. on } [0, \pi].$$

i.e.,

$$Q(x) = \tilde{q}(x) - q(x) = 0 \text{ a.e. on } [0, \pi]. \tag{2.31}$$

From (2.20), (2.23) and (2.24), we have

$$a_1 \tilde{c}_1 - \tilde{a}_1 c_1 = 0, \tag{2.32}$$

$$a_1 \tilde{d}_1 + b_1 \tilde{c}_1 - \tilde{a}_1 d_1 - \tilde{b}_1 c_1 = 0 \tag{2.33}$$

and

$$b_1 \tilde{d}_1 - \tilde{b}_1 d_1 = 0. \tag{2.34}$$

By virtue of (2.32)-(2.34), this yields

$$\frac{\tilde{a}_1 \lambda + \tilde{b}_1}{\tilde{c}_1 \lambda + \tilde{d}_1} = \frac{a_1 \lambda + b_1}{c_1 \lambda + d_1}, \forall \lambda \in \mathbf{C}.$$

Therefore, this completes the proof of Theorem 2.1. □

## Acknowledgements

The authors acknowledge helpful comments and suggestions from the referees, which obviously improved the manuscript.

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