

1-1-2013

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### Recommended Citation

POLATOĞLU, YAŞAR; ÖZKAN, HATİCE ESRA; and DUMAN, EMEL YAVUZ (2013) "Growth and distortion theorems for multivalent Janowski close-to-convex harmonic functions with shear construction method," *Turkish Journal of Mathematics*: Vol. 37: No. 3, Article 7. <https://doi.org/10.3906/mat-1012-543>  
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## Growth and distortion theorems for multivalent Janowski close-to-convex harmonic functions with shear construction method

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Received: 07.12.2010 • Accepted: 20.04.2012 • Published Online: 26.04.2013 • Printed: 27.05.2013

**Abstract:** In this paper we introduce the class of  $m$ -valent Janowski close to convex harmonic functions. Growth and distortion theorems are obtained for this class.

Our study is based on the harmonic shear methods for harmonic functions.

**Key words:** Multivalent harmonic functions, distortion theorem, growth theorem

### 1. Introduction

Let  $U$  be a simply connected domain in the complex plane. A harmonic function  $f$  has the representation  $f = h(z) + \overline{g(z)}$ , where  $h(z)$  and  $g(z)$  are analytic in  $U$  and are called the analytic and co-analytic part of  $f$ , respectively. Let  $h(z) = z^m + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \dots$ , and  $g(z) = b_m z^m + b_{m+1}z^{m+1} + b_{m+2}z^{m+2} + \dots$  be analytic functions in the open unit disc  $\mathbb{D}$ . The jacobian  $J_f$  of  $f = h(z) + \overline{g(z)}$  is defined by  $J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |h'(z)|^2 - |g'(z)|^2$ . If  $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ , then  $f = h(z) + \overline{g(z)}$  is called a sense-preserving multivalent harmonic function in  $\mathbb{D}$ . The class of all sense-preserving multivalent harmonic functions with  $|b_m| < 1$  is denoted by  $\mathcal{S}_H(m)$  and the class of all sense-preserving multivalent harmonic functions with  $b_m = 0$  is denoted by  $\mathcal{S}_H^0(m)$ . For convenience, we will investigate sense-preserving harmonic functions, that is functions for which  $J_f(z) > 0$ . If  $J_f(z) < 0$ , then  $\bar{f}$  is sense-preserving. The second analytic dilatation of a harmonic function is given by  $w(z) = g'(z)/h'(z)$ . We also note that if  $f$  is locally univalent and sense-preserving, then  $|w(z)| < 1$  for every  $z \in \mathbb{D}$ , and  $f$  is the solution of the differential equation  $f_z w(z) = \overline{f_{\bar{z}}}$  (see [3], [1] and [4]).

Let  $\Omega$  be the family of functions  $\varphi(z)$  which are regular and analytic in the open unit disc  $\mathbb{D}$  and satisfying the conditions  $\varphi(0) = 0$ ,  $|\varphi(z)| < 1$  for every  $z \in \mathbb{D}$ . For arbitrary fixed numbers  $A, B$ ,  $-1 < A \leq 1$ ,  $-1 \leq B < 1$ , denote by  $\mathcal{P}(A, B, m)$  the class of functions  $p(z) = m + \sum_{n=1}^{\infty} b_n z^n$  analytic in  $\mathbb{D}$  such that  $p(z) \in \mathcal{P}(A, B, m)$  if and only if

$$p(z) = m \frac{1 + A\varphi(z)}{1 + B\varphi(z)}, \varphi \in \Omega, z \in \mathbb{D}. \quad (1.1)$$

Moreover, let  $\mathcal{S}(A, B, m)$  denote the class of functions  $f(z) = z^m + \sum_{n=m+1}^{\infty} a_n z^n$  analytic in  $\mathbb{D}$  and satisfying the condition that  $f(z) \in \mathcal{S}(A, B, m)$  if and only if  $z \frac{f'(z)}{f(z)} = p(z)$  for some  $p(z) \in \mathcal{P}(A, B, m)$  and all  $z \in \mathbb{D}$ .

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2000 AMS Mathematics Subject Classification: 30C45, 30C55.

Next, denote by  $\mathcal{P}(m)$  (with  $m$  being a positive integer) the family of functions  $p(z) = m + p_1z + p_2z^2 + \dots$  which are regular in  $\mathbb{D}$  and satisfying the conditions  $p(0) = m$ ,  $\operatorname{Re} p(z) > 0$  for all  $z \in \mathbb{D}$ , and such that  $p(z) \in \mathcal{P}(m)$  if and only if for some function  $\phi(z) \in \Omega$  and every  $z \in \mathbb{D}$  ([2], [6]).

Let  $\mathcal{C}(A, B, m)$  denote the class of functions  $f(z) = z^m + \sum_{n=m+1}^{\infty} c_n z^n$  regular in  $\mathbb{D}$  and satisfies the condition

$$1 + z \frac{f''(z)}{f'(z)} = p(z), \quad (1.2)$$

for some  $p(z) \in \mathcal{P}(A, B, m)$  and every  $z \in \mathbb{D}$ . Finally, a function  $f(z) = z^m + \sum_{n=m+1}^{\infty} d_n z^n$  is in the class of  $\mathcal{K}(A, B, m)$  if there is a function  $\phi(z)$  in  $\mathcal{C}(A, B, m)$  such that

$$z \frac{f'(z)}{f(z)} = p(z), \quad (1.3)$$

where  $p(z) \in \mathcal{P}(A, B, m)$  and every  $z \in \mathbb{D}$ .

Let  $F(z) = z + a_2z^2 + \dots$  and  $G(z) = z + b_2z^2 + \dots$  be analytic functions in  $\mathbb{D}$ . If there exists a function  $\phi(z) \in \Omega$  such that  $F(z) = G(\phi(z))$  for all  $z \in \mathbb{D}$ , then we say that  $F(z)$  subordinate to  $G(z)$  and we write  $F(z) \prec G(z)$ . We also note that if  $F(z) \prec G(z)$  then  $F(\mathbb{D}) \subset G(\mathbb{D})$  ([5]).

Denote by  $\mathcal{S}_H\mathcal{K}(A, B, m)$  the class of all  $m$ -valent close to convex harmonic functions in the open unit disc  $\mathbb{D}$ .

## 2. Main results

**Lemma 2.1** *Let  $\phi(z) = z^m + c_{m+1}z^{m+1} + c_{m+2}z^{m+2} + \dots$  be analytic  $m$ -valent Janowski convex function in  $\mathbb{D}$ . Then the inequalities*

$$\frac{r^{m-1}}{(1+Br)^{\frac{m(B-A)}{B}}} \leq |\phi'(z)| \leq \frac{r^{m-1}}{(1-Br)^{\frac{m(B-A)}{B}}}, \quad B \neq 0 \quad (2.4)$$

$$r^{m-1}e^{-mAr} \leq |\phi'(z)| \leq r^{m-1}e^{mAr}, \quad B = 0$$

are realized.

**Proof** Since  $\phi(z) \in \mathcal{C}(A, B, m)$  and by using the subordination principle, we have

$$\left| \left( 1 + z \frac{\phi''(z)}{\phi'(z)} \right) - m \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{m(A-B)r}{1 - B^2 r^2}, \quad B \neq 0 \quad (2.5)$$

$$\left| \left( 1 + z \frac{\phi''(z)}{\phi'(z)} \right) - m \right| \leq mAr, \quad B = 0$$

for every  $|z| = r < 1$ . Therefore we have

$$\left\{ \begin{array}{l} \frac{(m-1) - m(A-B)r - (mAB - B^2)r^2}{1 - B^2r^2} \leq \operatorname{Re} \left( z \frac{\phi''(z)}{\phi'(z)} \right) \\ \leq \frac{(m-1) + p(A-B)r - (mAB - B^2)r^2}{1 - B^2r^2}, B \neq 0 \\ m-1 - mAr \leq \operatorname{Re} \left( z \frac{\phi''(z)}{\phi'(z)} \right) \leq m-1 + mAr, B = 0 \end{array} \right. \quad (2.6)$$

for all  $|z| = r < 1$ . On the other hand, we know that

$$\operatorname{Re} \left( z \frac{\phi''(z)}{\phi'(z)} \right) = r \frac{\partial}{\partial r} \log |\phi'(z)|. \quad (2.7)$$

Thus, by using equality (2.7) in the inequalities (2.6) we obtain that

$$\left\{ \begin{array}{l} \frac{(m-1) - m(A-B)r - (mAB - B^2)r^2}{r(1 - B^2r^2)} \leq \frac{\partial}{\partial r} \log |\phi'(z)| \\ \leq \frac{(m-1) + m(A-B)r - (mAB - B^2)r^2}{r(1 - B^2r^2)}, B \neq 0 \\ \frac{m-1-mAr}{r} \leq \frac{\partial}{\partial r} \log |\phi'(z)| \leq \frac{m-1+mAr}{r}, B = 0 \end{array} \right. \quad (2.8)$$

where  $|z| = r < 1$ . Integrating from 0 to  $r$  of the above inequalities we can get (2.4).  $\square$

**Lemma 2.2** Let  $w(z)$  be the second analytic dilatation of the class  $\mathcal{S}_H\mathcal{K}(A, B, m)$ , i.e.,  $w(z) = \frac{g'(z)}{h'(z)}$ . Then

$$\frac{|b_m| - r}{1 - |b_m|r} \leq |w(z)| \leq \frac{|b_m| + r}{1 + |b_m|r}, \quad (2.9)$$

$$\frac{(1 + |b_m|)(1 - r)}{1 - |b_m|r} \leq 1 + |w(z)| \leq \frac{(1 + |b_m|)(1 + r)}{1 + |b_m|r}, \quad (2.10)$$

and

$$\frac{(1 - |b_m|)(1 - r)}{1 + |b_m|r} \leq 1 - |w(z)| \leq \frac{(1 - |b_m|)(1 + r)}{1 - |b_m|r}. \quad (2.11)$$

**Proof** Since  $w(z) = \frac{g'(z)}{h'(z)} = \frac{mb_m z^{m-1} + (m+1)b_{m+1} z^m + \dots}{mz^{m-1} + (m+1)a_{m+1} z^m + \dots}$  we have  $w(0) = b_m$ . Define the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)} = \frac{w(z) - b_m}{1 - \overline{b_m}w(z)}.$$

This function satisfies the conditions of Schwarz lemma. Therefore we have

$$w(z) = \frac{b_m + \phi(z)}{1 + \overline{b_m}\phi(z)},$$

which shows that the second dilatation  $w(z)$  is subordinate to  $\left(\frac{z+b_m}{1+b_m z}\right)$ . On the other hand, the transformation  $\left(\frac{z+b_m}{1+b_m z}\right)$  maps  $|z| = r$  onto the disc with the center  $C(r) = \left(\frac{\alpha_1(1-r^2)}{1-|b_m|^2 r^2}, \frac{\alpha_2(1-r^2)}{1-|b_m|^2 r^2}\right)$ , and radius  $\rho(r) = \frac{(1-|b_m|^2)r}{1-|b_m|^2 r^2}$ . Using the subordination principle, we can write

$$\left|w(z) - \frac{b_m(1-r^2)}{1-|b_m|^2 r^2}\right| \leq \frac{(1-|b_m|^2)r}{1-|b_m|^2 r^2}. \quad (2.12)$$

After straightforward calculations from the last inequality, we get (2.9), (2.10) and (2.11).  $\square$

**Theorem 2.3** Let  $f(z)$  be a  $m$ -valent Janowski close to convex function and  $\phi(z)$  be a  $m$ -valent convex function in  $\mathbb{D}$ . Thus we obtain those inequalities

$$\frac{m(1-Ar)}{1-Br} \leq \left|\frac{f'(z)}{\phi'(z)}\right| \leq \frac{m(1+Ar)}{1+Br}, \quad B \neq 0 \quad (2.13)$$

$$m(1-Ar) \leq \left|\frac{f'(z)}{\phi'(z)}\right| \leq m(1+Ar), \quad B = 0,$$

where  $|z| = r < 1$ .

**Proof** Since  $f(z) \in \mathcal{K}(A, B, m)$  and  $\phi(z) \in \mathcal{C}(A, B, m)$  then we know that

$$\frac{f'(z)}{\phi'(z)} \prec m \frac{1+Az}{1+Bz},$$

from the last subordination we can write the inequalities

$$\left|\frac{f'(z)}{\phi'(z)} - \frac{m(1-ABr^2)}{1-B^2r^2}\right| \leq \frac{m(A-B)r}{1-B^2r^2}, \quad B \neq 0 \quad (2.14)$$

$$\left|\frac{f'(z)}{\phi'(z)} - m\right| \leq mA, \quad B = 0.$$

By using the triangle inequality in the inequalities (2.14) we get (2.13).  $\square$

**Theorem 2.4** If  $f(z)$  is a  $m$ -valent Janowski close to convex function and  $\phi(z)$  is a  $m$ -valent Janowski convex function in  $\mathbb{D}$ , then the following inequalities

$$\frac{m(1-Ar)r^{m-1}}{(1-Br)(1+Br)^{\frac{m(B-A)}{B}}} \leq |f'(z)| \leq \frac{m(1+Ar)r^{m-1}}{(1+Br)(1-Br)^{\frac{m(B-A)}{B}}}, \quad B \neq 0 \quad (2.15)$$

$$mr^{m-1}e^{-mAr}(1-Ar) \leq |f'(z)| \leq mr^{m-1}e^{mAr}(1+Ar), \quad B = 0$$

are realized.

**Proof** Using lemma 2.1 in theorem 2.4, we obtain the result.  $\square$

**Theorem 2.5** Let  $f = h(z) + \overline{g(z)}$  be an element of  $\mathcal{S}_H\mathcal{K}(A, B, m)$ . Then

$$\left\{ \begin{array}{l} \frac{m(1-Ar)r^{m-1}}{(1-Br)(1+Br)^{\frac{m(B-A)}{B}}} \cdot \frac{(1+|b_m|r)}{(1+|b_m|)(1+r)} \leq |f_z| \\ \leq \frac{m(1+Ar)r^{m-1}}{(1+Br)(1-Br)^{\frac{m(B-A)}{B}}} \cdot \frac{(1+|b_m|r)}{(1-|b_m|)(1-r)}, B \neq 0, \\ mr^{m-1}e^{-mAr}(1-Ar) \cdot \frac{(1+|b_m|r)}{(1+|b_m|)(1+r)} \leq |f_z| \leq mr^{m-1}e^{mAr}(1+Ar) \cdot \frac{(1+|b_m|r)}{(1-|b_m|)(1-r)}, B = 0, \end{array} \right. \quad (2.16)$$

and

$$\left\{ \begin{array}{l} \frac{m(1-Ar)r^{m-1}}{(1-Br)(1+Br)^{\frac{m(B-A)}{B}}} \cdot \frac{(|b_m|-r)(1+|b_m|r)}{(1-|b_m|r)(1+|b_m|)(1+r)} \leq |f_{\bar{z}}| \\ \leq \frac{m(1+Ar)r^{m-1}}{(1+Br)(1-Br)^{\frac{m(B-A)}{B}}} \cdot \frac{(|b_m|+r)}{(1-|b_m|)(1-r)}, B \neq 0, \\ mr^{m-1}e^{-mAr}(1-Ar) \cdot \frac{(|b_m|-r)(1+|b_m|r)}{(1-|b_m|r)(1+|b_m|)(1+r)} \leq |f_{\bar{z}}| \leq mr^{m-1}e^{mAr}(1+Ar) \cdot \frac{(|b_m|+r)}{(1-|b_m|)(1-r)}, B = 0. \end{array} \right. \quad (2.17)$$

**Proof** If we take  $\psi(z) = h(z) - g(z)$ , then we have

$$h'(z) = \frac{\psi'(z)}{1-w(z)}, \quad g'(z) = \frac{w(z)\psi'(z)}{1-w(z)}, \quad |w(z)| < 1.$$

Therefore we have

$$\frac{|\psi'(z)|}{1+|w(z)|} \leq |f_z| \leq \frac{|\psi'(z)|}{1-|w(z)|}, \quad (2.18)$$

$$\frac{|w(z)||\psi'(z)|}{1+|w(z)|} \leq |\overline{f_z}| \leq \frac{|w(z)||\psi'(z)|}{1-|w(z)|}. \quad (2.19)$$

Using lemma 2.1 and lemma 2.2 in the inequalities (2.18) and (2.19) we get (2.16) and (2.17), respectively. Since

$$\phi(z) = \frac{w(z) - b_m}{1 - \overline{b_m}w(z)},$$

we have

$$h'(z) = f_z = \frac{\psi'(z)}{1-w(z)},$$

and so

$$h(z) = \int_0^z \frac{\psi'(\xi)}{1-w(\xi)} d\xi.$$

Also, since

$$g'(z) = \overline{f'_z} = \int_0^z \frac{\psi'(\xi)w(\xi)}{1-w(\xi)} d\xi,$$

it follows that

$$g(z) = \int_0^z \frac{\psi'(\xi)w(\xi)}{1-w(\xi)} d\xi.$$

(The solution  $h(z)$  and  $g(z)$  must be found under the conditions  $h(0) = g(0) = 0$ .) Thus

$$\begin{aligned} f(z) &= h(z) + \overline{g(z)} = \int_0^z \frac{\psi'(\xi)}{1-w(\xi)} d\xi + \overline{\int_0^z \frac{\psi'(\xi)w(\xi)}{1-w(\xi)} d\xi} = \\ &= \int_0^z \frac{\psi'(\xi)}{1-w(\xi)} d\xi + \overline{\int_0^z \frac{\psi'(\xi)}{1-w(\xi)} d\xi} - \int_0^z \psi'(\xi) d\xi = \operatorname{Re} \left( \int_0^z \frac{2\psi'(\xi)}{1-w(\xi)} \right) - \overline{\psi(z)}. \end{aligned}$$

□

**Corollary 2.6** *If we choose the following values for theorem 2.5, we get the accompanying inequalities:*

- $A = 1, B = -1$  :

$$\frac{m(1-r)r^{m-1}}{(1+r)^2(1-r)^{2m}} \cdot \frac{(1+|b_m|r)}{(1+|b_m|)} \leq |f_z| \leq \frac{m(1+r)r^{m-1}}{(1-r)^2(1+r)^{2m}} \cdot \frac{(1+|b_m|r)}{(1-|b_m|)}$$

$$\frac{m(1-r)r^{m-1}}{(1+r)^2(1-r)^{2m}} \cdot \frac{(1+|b_m|r)(|b_m|-r)}{(1+|b_m|)(1-|b_m|r)} \leq |f_{\bar{z}}| \leq \frac{m(1+r)r^{m-1}}{(1-r)^2(1+r)^{2m}} \cdot \frac{(|b_m|+r)}{(1-|b_m|)}$$

- $A = 1 - 2\alpha, B = -1, 0 \leq \alpha < 1$  :

$$\frac{m(1-r+2\alpha r)r^{m-1}}{(1+r)^2(1-r)^{2m(1-\alpha)}} \cdot \frac{(1+|b_m|r)}{(1+|b_m|)} \leq |f_z| \leq \frac{m(1+r-2\alpha r)r^{m-1}}{(1-r)^2(1+r)^{2m(1-\alpha)}} \cdot \frac{(1+|b_m|r)}{(1-|b_m|)}$$

$$\frac{m(1-r+2\alpha r)r^{m-1}}{(1+r)^2(1-r)^{2m(1-\alpha)}} \cdot \frac{(1+|b_m|r)(|b_m|-r)}{(1+|b_m|)(1-|b_m|r)} \leq |f_{\bar{z}}| \leq \frac{m(1+r-2\alpha r)r^{m-1}}{(1-r)^2(1+r)^{2m(1-\alpha)}} \cdot \frac{(|b_m|+r)}{(1-|b_m|)}$$

- $A = 1, B = \frac{1}{M} - 1, M > \frac{1}{2}$  :

$$\left\{ \begin{array}{l} \frac{m(1-r)r^{m-1}}{(1+r-\frac{r}{M})(1-r+\frac{r}{M})^{m\frac{1-2M}{1-M}}} \cdot \frac{(1+|b_m|r)}{(1+|b_m|)(1+r)} \leq |f_z| \\ \leq \frac{m(1+r)r^{m-1}}{(1-r+\frac{r}{M})(1+r-\frac{r}{M})^{m\frac{1-2M}{1-M}}} \cdot \frac{(1+|b_m|r)}{(1-|b_m|)(1-r)} \\ \frac{m(1-r)r^{m-1}}{(1+r-\frac{r}{M})(1-r+\frac{r}{M})^{m\frac{1-2M}{1-M}}} \cdot \frac{(1+|b_m|r)(|b_m|-r)}{(1+|b_m|)(1-|b_m|r)(1+r)} \leq |f_{\bar{z}}| \leq \frac{m(1+r)r^{m-1}}{(1-r+\frac{r}{M})(1+r-\frac{r}{M})^{m\frac{1-2M}{1-M}}} \cdot \frac{(|b_m|+r)}{(1-|b_m|)(1-r)} \end{array} \right.$$

- $A = \beta, B = -\beta, 0 < \beta \leq 1$  :

$$\left\{ \begin{array}{l} \frac{mr^{m-1}(1+|b_m|r)}{(1+\beta r)(1-\beta r)^{2m-1}(1+|b_m|)(1+r)} \leq |fz| \leq \frac{mr^{m-1}(1+|b_m|r)}{(1-\beta r)(1+\beta r)^{2m-1}(1-|b_m|)(1-r)} \\ \frac{mr^{m-1}(|b_m|-r)(1+|b_m|r)}{(1+\beta r)(1-\beta r)^{2m-1}(1-|b_m|r)(1+|b_m|)(1+r)} \leq |f\bar{z}| \\ \leq \frac{mr^{m-1}(|b_m|+r)}{(1-\beta r)(1+\beta r)^{2m-1}(1-|b_m|)(1-r)} \end{array} \right.$$

**Corollary 2.7** Let  $f = h(z) + \overline{g(z)}$  be an element of  $S_H\mathcal{K}(A, B, m)$ . Then

$$\left\{ \begin{array}{l} \frac{m^2(1-Ar)^2r^{2(m-1)}}{(1-Br)^2(1+Br)^{\frac{2m(B-A)}{B}}} \cdot \frac{(1-|b_m|^2)(1-r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2(1+r)^2} \leq J_f(z), \\ \leq \frac{m^2(1+Ar)^2r^{2(m-1)}}{(1+Br)^2(1-Br)^{\frac{2m(B-A)}{B}}} \cdot \frac{(1-|b_m|^2)(1+r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2(1-r)^2}, B \neq 0, \\ m^2r^{2(m-1)}e^{-2mAr}(1-Ar)^2 \cdot \frac{(1-|b_m|^2)(1-r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2(1+r)^2} \leq J_f(z), \\ \leq m^2r^{2(m-1)}e^{2mAr}(1+Ar)^2 \cdot \frac{(1-|b_m|^2)(1+r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2(1-r)^2}, B = 0. \end{array} \right. \tag{2.20}$$

**Proof** Since  $J_f(z) = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2(1 - |w(z)|^2)$ , then using theorem 2.5 and lemma 2.2, we get (2.20). □

**Corollary 2.8** For the last results, if we take the following values, we get the accompanying inequalities:

- $A = 1, B = -1$  :

$$\begin{aligned} \frac{m^2(1-r)^4r^{2(m-1)}}{(1+r)^4(1-r)^{4m}} \cdot \frac{(1-|b_m|^2)(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2} &\leq J_f(z) \\ &\leq \frac{m^2(1+r)^2r^{2(m-1)}}{(1-r)^4(1+r)^{4m}} \cdot \frac{(1-|b_m|^2)(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2} \end{aligned}$$

- $A = 1 - 2\alpha, B = -1, 0 \leq \alpha < 1$  :

$$\begin{aligned} \frac{m^2(1-r+2\alpha r)^2r^{2(m-1)}}{(1+r)^4(1-r)^{4m(1-\alpha)}} \cdot \frac{(1-|b_m|^2)(1-r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2} &\leq J_f(z) \\ &\leq \frac{m^2(1+r-2\alpha r)^2r^{2(m-1)}}{(1-r)^4(1+r)^{4m(1-\alpha)}} \cdot \frac{(1-|b_m|^2)(1+r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2}. \end{aligned}$$

- $A = 1, B = \frac{1}{M} - 1, M > \frac{1}{2}$  :



$$\frac{m^2(1-r)^4r^{2(m-1)}}{(1+r-\frac{r}{M})^2(1-r+\frac{r}{M})^{2m\frac{1-2M}{1-M}}} \cdot \frac{(1-|b_m|^2)(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2(1+r)^2} \leq J_f(z)$$

$$\leq \frac{m^2(1+r)^4r^{2(m-1)}}{(1-r+\frac{r}{M})^2(1+r-\frac{r}{M})^{2m\frac{1-2M}{1-M}}} \cdot \frac{(1-|b_m|^2)(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2(1-r)^2}.$$

- $A = \beta, B = -\beta$  :

$$\frac{m^2r^{2(m-1)}(1-|b_m|^2)(1-r)^2(1+|b_m|r)^2}{(1+\beta r)^2(1-\beta r)^{4m-2}(1-|b_m|^2r^2)(1+|b_m|)^2(1+r)^2} \leq J_f(z)$$

$$\leq \frac{m^2r^{2(m-1)}(1-|b_m|^2)(1+r)^2(1+|b_m|r)^2}{(1-\beta r)^2(1+\beta r)^{4m-2}(1-|b_m|^2r^2)(1-|b_m|)^2(1-r)^2}$$

**Corollary 2.9** If  $f = h(z) + \overline{g(z)} \in \mathcal{S}_H(A, B, m)$ , then

$$\left\{ \begin{array}{l} m \int_0^r \rho^{m-1} \left[ \frac{(1-A\rho)(1+|b_m|\rho)}{(1-B\rho)(1+B\rho)^{\frac{m(B-A)}{B}}} - \frac{(1+A\rho)(|b_m|+\rho)}{(1+B\rho)(1-B\rho)^{\frac{m(B-A)}{B}}(1-|b_m|)(1-\rho)} \right] d\rho \leq \\ |f| \leq m \left( \frac{1+|b_m|}{1-|b_m|} \right) \int_0^r \rho^{m-1} \frac{(1+A\rho)(1+\rho)}{(1+B\rho)(1-B\rho)^{\frac{m(B-A)}{B}}(1-\rho)} d\rho, B \neq 0, \\ \\ m \int_0^r \rho^{m-1} \left[ \frac{(1-A\rho)(1+|b_m|\rho)}{(1+|b_m|)(1+\rho)} - \frac{(1+A\rho)(|b_m|+\rho)}{(1-|b_m|)(1-\rho)} \right] e^{-A\rho} d\rho \leq \\ |f| \leq m \left( \frac{1+|b_m|}{1-|b_m|} \right) \int_0^r \rho^{m-1} \frac{(1+A\rho)(1+\rho)}{(1-\rho)} e^{A\rho} d\rho, B = 0. \end{array} \right. \quad (2.21)$$

**Proof** Since  $(|f_z| - |f_{\bar{z}}|)|dz| \leq |df| \leq (|f_z| + |f_{\bar{z}}|)|dz|$ , it follows that  $(|h'(z)| - |g'(z)|)|dz| \leq |df| \leq (|h'(z)| + |g'(z)|)|dz|$ , and using theorem 2.5 we can write the result.  $\square$

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