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Structure theorems for rings under certain coactions of a Hopf algebra

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Abstract: Let $\{D_1, \dots, D_n\}$ be a system of derivations of a k -algebra A , k a field of characteristic $p > 0$, defined by a coaction δ of the Hopf algebra $H_c = k[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p)$, $c \in \{0, 1\}$, the Lie Hopf algebra of the additive group and the multiplicative group on A , respectively. If there exist $x_1, \dots, x_n \in A$, with the Jacobian matrix $(D_i(x_j))$ invertible, $[D_i, D_j] = 0$, $D_i^p = cD_i$, $c \in \{0, 1\}$, $1 \leq i, j \leq n$, we obtain elements $y_1, \dots, y_n \in A$, such that $D_i(y_j) = \delta_{ij}(1 + cy_i)$, using properties of H_c -Galois extensions. A concrete structure theorem for a commutative k -algebra A , as a free module on the subring A^δ of A consisting of the coinvariant elements with respect to δ , is proved in the additive case.

Key words: Hopf algebras, derivations, Jacobian criterion

1. Introduction

A series of articles in commutative algebra ([5], [6], [7], [8] have focused on the following problem:

(P): Let $\{D_1, \dots, D_n\}$ be a system of derivations of a k -algebra A , k field of characteristic $p > 0$, such that there exist $x_1, \dots, x_n \in A$, with the Jacobian matrix $(D_i(x_j))$ invertible, $[D_i, D_j] = 0$, $D_i^p = c_i^{p-1}D_i$, $c_i \in k$, $1 \leq i, j \leq n$. Do elements $y_1, \dots, y_n \in A$ exist such that $D_i(y_j) = (1 + c_j y_j)\delta_{ij}$?

If a positive answer is given, structure theorems for A follow in terms of the subring of constants of A with respect to the derivations D_1, \dots, D_n , the main one of which is contained in [5]. We recall that a finite dimensional Hopf algebra over k is a k -algebra, with comultiplication $\Delta : H \rightarrow H \otimes_k H$, antipode $S : H \rightarrow H$ and counity $\varepsilon : H \rightarrow k$ and a coaction of H on a k -algebra A (or an H -comodule algebra structure on A) is a morphism of algebras $\delta : A \rightarrow A \otimes H$ such that $(1 \otimes \varepsilon)\delta \cong 1$ and $(1 \otimes \Delta)\delta = (\delta \otimes 1)\delta$. Given such a coaction, the subalgebra $\{a \in A : \delta(a) = a \otimes 1\}$ of A is called the algebra of coinvariant elements of δ and it is denoted by $A^\delta = A^{\text{co}H}$.

In [6], surprisingly, for a local commutative algebra A , the authors prove that the jacobian condition (which states that there are elements $y_1, \dots, y_n \in A$ such that for all $1 \leq m \leq n$ the $m \times m$ matrix $(D_i(y_j))_{1 \leq i, j \leq m}$ over A is invertible) is equivalent to the property for A to be an H -Galois extension over the subring A^δ of the coinvariant elements of A with respect to a coaction $\delta : A \rightarrow A \otimes H$, where H is a (co)commutative Hopf algebra with underlying algebra

$$H = k[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}), \quad n \geq 1, \quad s_1 \geq \dots \geq s_n \geq 1.$$

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For the Lie Hopf algebra H of the additive group, from the strong jacobian condition (which states that there are elements $y_1, \dots, y_n \in A$ such that $D_i(y_j)_{1 \leq i, j \leq n} = \delta_{ij}$) an important structure theorem follows for A (not necessarily commutative), precisely A has an A^δ -basis as a left A^δ -module, consisting of the monomials $y_1^{\alpha_1} \dots y_n^{\alpha_n}$, $\alpha_i \in \mathbb{N}$, $0 \leq \alpha_i < p^{s_i}$, $1 \leq i \leq n$, ([6], Theorem 3.1).

In this paper we consider Hopf algebras that “live” on the truncated algebra

$H_{\underline{s}} = k[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}})$ $\underline{s} = (s_1, \dots, s_n)$. According to ([11], 14.4), the assumption is not too restrictive because any finite-dimensional, commutative and local algebra over a perfect field has this structure. Using the notion just mentioned, we formulate a more general theorem where we postulate the existence of the elements $y_1, \dots, y_n \in A$ with the strong jacobian condition in the Lie algebra case of the additive group for $H = H_0 = k[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p)$, with $c_i = 0$ in (\mathbf{P}) , $i = 1, \dots, n$. The same result is given in the Lie algebra case of the multiplicative group for $H = H_1$ with $c_i = 1$ in (\mathbf{P}) , $i = 1, \dots, n$, under the hypotheses A local and $A = A^\delta + m$, where m is the maximal ideal of A . More precisely, the main result of section 1 concerns a positive answer to the previous question that can be deduced from the following theorem.

Theorem *Let H_c be the Hopf algebra defined as before, $c \in \{0, 1\}$, A a right H_c -comodule algebra with structure map $\delta : A \rightarrow A \otimes H_c$. If there are $y_1, \dots, y_n \in A$ with $\delta(y_i) = y_i \otimes 1 + (1 + cy_i) \otimes x_i$, for all $1 \leq i \leq n$, then the map*

$$\gamma : A^\delta \otimes H_c \rightarrow A, r \otimes x^\alpha \mapsto ry^\alpha, r \in A^\delta, \alpha \in \mathbb{A}, x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, y^\alpha = y_1^{\alpha_1} \dots y_n^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{A}$, \mathbb{A} the set of all multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$, with $0 \leq \alpha_i < p$, $1 \leq i \leq n$, is a left A^δ -linear and right H_c -colinear isomorphism. In particular, the elements y^α , $\alpha \in \mathbb{A}$, form an A^δ -basis of A as a left A^δ -module.

By using the previous theorem we are able to prove Theorem 2.5, where the property of H_c -Galois extension permits, starting from the strong jacobian condition on $n - 1$ elements y_1, \dots, y_{n-1} of A , to have the strong jacobian condition on n elements of A , assuming there exists $y \in A$ such that $D_n(y) = 1 + cy$, $c \in \{0, 1\}$. In section 2 we use Theorem 2.5 in the additive case and for a commutative k -algebra A , to give “explicitly” $y_1, \dots, y_n \in A$, the special elements that verify the strong condition $D_i(y_j) = \delta_{ij}$ of derivability, $1 \leq i, j \leq n$. Some consequences are discussed in section 3, where we consider the structure of A as an $A^\delta = A^{\{D_1, \dots, D_n\}}$ -algebra, $A^{\{D_1, \dots, D_n\}}$ the constant subring of A with respect to the derivations D_1, \dots, D_n .

2. Coactions of a Hopf algebra H and H -Galois type extensions

Throughout the paper, k is an arbitrary field of characteristic $p > 0$. All vector spaces, algebras, coalgebras are over k and maps between them are at least k -linear. We refer to the books by Montgomery [4] and Sweedler [10] for general Hopf algebra theory and to the book by Schauenburg and Schneider [9] for Galois type extensions of Hopf algebras. In this section we recall some definitions and theorems and we establish a structure theorem for the Hopf algebra of the multiplicative group. For $H = H_0$ the result is known [6]. Let H be a Hopf algebra over the field k , with comultiplication $\Delta : H \rightarrow H \otimes H$, counit $\varepsilon : H \rightarrow k$, antipode $S : H \rightarrow H$. The augmentation ideal of H will be denoted by $H^+ = \ker \varepsilon$. If A is a right H -comodule algebra, with structure map $\delta : A \rightarrow A \otimes H$, then

$$A^{\text{coH}} = A^\delta := \{a \in A \mid \delta(a) = a \otimes 1\}$$

is the algebra of H -coinvariant elements of A . We are interested in algebra extensions $B \subseteq A$ in a Hopf algebraic context. Precisely, $A^{\text{co}H} \subseteq A$. In fact, by definition, the sequence

$$A^{\text{co}H} \xrightarrow{\subseteq} A \begin{array}{c} \xrightarrow{\delta} \\ \xrightarrow{i_1} \end{array} A \otimes H$$

is exact, that is $A^{\text{co}H} \subseteq A$ is the difference kernel of the maps δ and $i_1 : A \rightarrow A \otimes H, a \mapsto a \otimes 1$.

Definition 2.1 [2] Let A be a right H -comodule algebra with structure map $\delta : A \rightarrow A \otimes H$. Then the extension $A^{\text{co}H} \subseteq A$ is a *right H -Galois extension* if the canonical map $\text{can} : A \otimes_{A^{\text{co}H}} A \rightarrow A \otimes_k H$ given by $a \otimes b \mapsto (a \otimes 1)\delta(b) = ab_{(0)} \otimes b_{(1)}$ is bijective.

In the following we will consider commutative Hopf algebras with underlying algebra:

$$H = k[X_1, \dots, X_n] / (X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}), \quad n \geq 1, \quad s_1 \geq \dots \geq s_n \geq 1.$$

We denote by \mathbb{A} the set of all multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $0 \leq \alpha_i < p^{s_i}, 1 \leq i \leq n$. For $\beta = (\beta_1, \dots, \beta_n), \gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ we define

$$\beta + \gamma = (\beta_1 + \gamma_1, \dots, \beta_n + \gamma_n), \quad \text{and } |\beta| = \beta_1 + \dots + \beta_n.$$

If we denote by x_i the residue class of X_i in H , for all i , then the elements $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}, \alpha \in \mathbb{A}$ form a k -basis of H . Let A be an algebra, $\delta : A \rightarrow A \otimes H$ be an algebra map and a right H -comodule algebra structure on A . We will write

$$\delta(a) = \sum_{\alpha \in \mathbb{A}} D_\alpha(a) \otimes x^\alpha, \quad \text{for all } a \in A.$$

Thus for all $\alpha \in \mathbb{A}$ and $a, b \in A$,

$$D_\alpha(ab) = \sum_{\substack{\beta + \gamma = \alpha \\ \beta, \gamma \in \mathbb{A}}} D_\beta(a)D_\gamma(b), \quad \text{and } D_{(0, \dots, 0)} = \text{id}.$$

For all i , let $\delta_i = (\delta_{ij})_{1 \leq j \leq n} \in \mathbb{A}$, where $\delta_{ij} = 1$, if $j = i$, and $\delta_{ij} = 0$, otherwise. We put $D_i = D_{\delta_i}, 1 \leq i \leq n$. Thus the linear maps $D_i : A \rightarrow A$ are derivations of the algebra A , and for all $a \in A$ we have

$$\delta(a) = a \otimes 1 + \sum_{1 \leq i \leq n} D_i(a) \otimes x_i + \sum_{\substack{\alpha \in \mathbb{A} \\ |\alpha| \geq 2}} D_\alpha(a) \otimes x^\alpha. \tag{1}$$

From now we will consider the Hopf algebra H_a of the additive group, that is

$$H_a = k[X_1, \dots, X_n] / (X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}) \quad n \geq 1, \quad s_1 \geq \dots \geq s_n \geq 1, \tag{2}$$

with comultiplication

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i, \quad 1 \leq i \leq n \tag{3}$$

and the Hopf algebra of the multiplicative group, that is

$$H_m = k[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}) \quad n \geq 1, \quad s_1 \geq \dots \geq s_n \geq 1, \tag{4}$$

with comultiplication

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i, \quad 1 \leq i \leq n \tag{5}$$

We call these algebras H_c , $c \in \{0, 1\}$, respectively. In the Lie algebra case of the additive group, that is

$$H_0 = k[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p), \tag{6}$$

coactions have a special form. Precisely they are derivations $D_1, \dots, D_n \in Der(A)$ with $D_i D_j = D_j D_i$, $D_i^p = 0$ and

$$D_\alpha = \frac{D_1^{\alpha_1}}{\alpha_1!} \dots \frac{D_n^{\alpha_n}}{\alpha_n!}, \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad 0 \leq \alpha_i < p, \quad 1 \leq i \leq n.$$

In the Lie algebra case of the multiplicative group, that is

$$H_1 = k[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p), \tag{7}$$

coactions are derivations $D_1, \dots, D_n \in Der(A)$ with $D_i D_j = D_j D_i$, $D_i^p = D_i$ and

$$\begin{aligned} D_\alpha &= \frac{\prod_{j_1=0}^{\alpha_1-1} (D_1 - j_1)}{\alpha_1!} \frac{\prod_{j_2=0}^{\alpha_2-1} (D_2 - j_2)}{\alpha_2!} \dots \frac{\prod_{j_n=0}^{\alpha_n-1} (D_n - j_n)}{\alpha_n!} = \\ &= \frac{\prod_{t=1}^n \prod_{j_t=0}^{\alpha_t-1} (D_t - j_t)}{\alpha!} \end{aligned}$$

with $\alpha = (\alpha_1, \dots, \alpha_n)$, $0 \leq \alpha_i < p$, $1 \leq i \leq n$ and $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$ (see [1], Theorem 3.3).

Theorem 2.2 *Let H_c , $c \in \{0, 1\}$, be the Hopf algebra in the Lie cases, defined as before and A a right H_c -comodule algebra with structure map $\delta : A \rightarrow A \otimes H_c$. Let $R = A^{coH_c}$. Assume, for $c = 1$, A is a commutative local algebra with maximal ideal m and $R + m = A$.*

(a) *The following are equivalent:*

(i) *$R \subset A$ is a faithfully flat H_c -Galois extension.*

(ii) *There are $y_1, \dots, y_n \in A$ with $\delta(y_i) = y_i \otimes 1 + (1 + y_i) \otimes x_i$, for all $1 \leq i \leq n$*

(b) *Suppose (ii) holds. Then*

$$R \otimes H_1 \rightarrow A, \quad r \otimes x^\alpha \mapsto ry^\alpha, \quad r \in R, \alpha \in \mathbb{A}, y^\alpha = y_1^{\alpha_1} \dots y_n^{\alpha_n}, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{A}$$

is a left R -linear and right H_c -colinear isomorphism.

In particular, the elements y^α , $\alpha \in \mathbb{A}$, form an R -basis of A as a left R -module.

Proof For $c = 0$, see [6], Theorem 3.1.

For $c = 1$, (a) is proved in [1], Proposition 4.2. To prove (b) we observe that the coradical C of H_1 is the k -subalgebra of H_1 :

$$C = k \oplus kx_1 \oplus \cdots \oplus kx_n, \quad x_i = X_i + (X_1^p, \dots, X_n^p).$$

For this, it is sufficient to prove for $i = 1$ that $C = k \oplus kx$, $H_1 = k[X]/(X^p) = k[x]$.

$$\Delta(1 + x) = \Delta(1) + \Delta(x) = 1 \otimes 1 + 1 \otimes x + x \otimes 1 = (1 + x) \otimes (1 + x) \in C \otimes C.$$

Moreover, the vector subspaces of H_1 , k and kx , are the only simple coalgebras of H_1 . Hence the assertion.

Suppose (ii) of (a) holds. Then we define a k -linear map $\gamma : H_1 \rightarrow A$ by $\gamma : x^\alpha \mapsto ry^\alpha$ for all $\alpha \in \mathbb{A}$. Since Δ and δ are algebra maps and, for all i ,

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i, \quad \delta(y_i) = y_i \otimes 1 + (1 + y_i) \otimes x_i,$$

γ is right H_1 -colinear. If we prove that the map γ is convolution invertible, the H_1 -extension $R \subset A$ is H_1 -cleft, hence H_1 -Galois and

$$R \otimes H_1 \rightarrow A, \quad r \otimes x^\alpha \mapsto ry^\alpha, \quad r \in R, \alpha \in \mathbb{A}$$

is bijective ([9], 8.2.4, 7.2.3). To prove that $\gamma \in \text{Hom}(A, A)$ is invertible with respect to the convolution $*$, it is sufficient to prove that $\gamma|_C$ is invertible as an element of $\text{Hom}(C, A)$. For $f \in \text{Hom}(C, A)$, $i = 1, \dots, n$, it results in

$$\begin{aligned} f * \gamma(1 + x_i) &= m(f \otimes \gamma)(\Delta(1 + x_i)) = m(f \otimes \delta(1 \otimes x_i + x_i \otimes 1 + x_i \otimes x_i)) \\ &= m(f(1) \otimes \gamma(x_i) + f(x_i) \otimes \delta(1) + f(x_i) \otimes \delta(x_i)) = 1_A y_i + f(x_i) + f(x_i) y_i \\ &= y_i + f(x_i)(1 + y_i) \end{aligned}$$

and

$$u\varepsilon(1 + x_i) = u(\varepsilon(1) + \varepsilon(x_i)) = u(1) = 1_A,$$

with $m : H_1 \otimes H_1 \rightarrow H_1$ and $u : k \rightarrow H_1$ being the multiplication and the unit maps of H_1 , respectively. If we put $f(x_i) = \frac{1-y_i}{1+y_i}$, we have $y_i + f(x_i)(1 + y_i) = 1$, γ is left invertible and its inverse map is f . Hence the conclusion follows. \square

Remark 2.3 The result contained in Theorem 2.2, (b) can be deduced from (ii), under the hypotheses that the elements $1 + y_i, 1 < i < n$, are invertible, A not necessarily local.

In the following, for $c = 1$, we will suppose that A is commutative, local and $A = R + m$, where R is the coinvariant subring of A with respect to the coaction δ and m is the maximal ideal of A .

Corollary 2.4 Let H_c be the Hopf Lie algebra of the group H_c , A an algebra and $\delta : A \rightarrow A \otimes H_c$ a coaction. Put D_1, \dots, D_n the derivations defined by (1) and $R := A^{coH_c}$. The following are equivalent:

- (1) $R \subset A$ is a faithfully flat H_c -Galois extension.
- (2) There are $y_1, \dots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_i)$, for all $1 \leq i, j \leq n$.

(3) If A is local there are $y_1, \dots, y_n \in A$ such that for all $1 \leq m \leq n$, the $m \times m$ matrix $(D_i(y_j))_{1 \leq i, j \leq m}$ over A is invertible.

Proof For $c = 0$ the result is in [6], Corollary 3.3 and Theorem 4.1.

For $c = 1$, (1) \iff (2) by Theorem 1.8(a), (1) \iff (3) by Theorem 4.1 in [6]. □

Recall that an H -Galois extension $R \subset A$ is faithfully flat if A is faithfully flat over R as a left (or equivalently right) module over R . Recently Schauenburg and Schneider ([9], Theorem 4.5.1) have proved a theorem which allows one to reduce questions about faithfully flat Hopf Galois extensions for H to the case of Hopf subalgebras and quotient algebras of H . We use it to prove the following:

Theorem 2.5 *Let A be a k -algebra, k a field of characteristic $p > 0$ and let $\{D_1, \dots, D_n\} \subset \text{Der}_k(A)$ such that $D_i D_j = D_j D_i$, $D_i^p = c D_i$, $c \in \{0, 1\}$, for all $i, j, 1 \leq i, j \leq n$. Suppose that*

- 1) *There exist $z_1, \dots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}(1 + cz_i)$, $1 \leq i, j \leq n - 1$.*
- 2) *There exists $y \in A$ such that $D_n(y) = 1 + cy$.*

Then $R := A^{\text{co}H_c} \subset A$ is a faithfully flat H_c -Galois extension and, consequently, there are $y_1, \dots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_i)$ for all $1 \leq i, j \leq n$.

Proof The set of derivations comes from a comodule structure of A on H_c , $H = k[x_1, \dots, x_n]$, $x_i^p = 0$, given by $\delta : A \rightarrow A \otimes H_c$,

$$\delta(a) = a \otimes 1 + \sum_{1 \leq i \leq n} D_i(a) \otimes x_i + \sum_{\substack{\alpha \in \mathbb{A} \\ |\alpha| \geq 2}} D_\alpha(a) \otimes x^\alpha. \tag{8}$$

$\alpha \in \mathbb{N}^n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Let $R = A^{\text{co}H_c}$ be the coinvariant subring of A with respect to δ and let $\overline{H}_c = k[x_n]$, $x_n^p = 0$, $B = A^{\text{co}\overline{H}_c}$ the coinvariant subring of A with respect to $\overline{\delta} : A \rightarrow A \otimes \overline{H}_c$, $\overline{H}_c = H_c/K^+H_c$, $K = k[x_1, \dots, x_{n-1}]$, $x_i^p = 0$, $i = 1, \dots, n - 1$, $K^+ = (x_1, \dots, x_{n-1})$. Consider the extension $R \subset B \subset A$.

$B \subset A$ is \overline{H} -Galois extension (Corollary 2.4). By hypothesis 2) and by Corollary 2.4, $R \subset B$ is a K -Galois extension. By Theorem 4.5.1 [9], $R \subset A$ is a faithfully flat H_c -Galois extension and, by Corollary 2.4, there exist $y_1, \dots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_i)$ for all $1 \leq i, j \leq n$. By 2) the assertion follows. □

3. A constructive theorem

We will describe, in the additive case, the special elements y_1, \dots, y_n that appear in Theorem 2.5 and satisfy a strong condition on the derivability. Following the same direction of research contained in the papers by Matsumura, Restuccia and Utano [5], [8], where the elements are computed, we obtain the result contained in [8] without the hypotheses that A is local, regular and k a separably closed field, but requiring that the last derivation evaluates to one on an element $t \in U(A)$.

Theorem 3.1 *Let A be a commutative k -algebra, k a field of characteristic $p > 0$ and let $\{D_1, \dots, D_n\} \subset \text{Der}_k(A)$ such that $D_i D_j = D_j D_i$, $D_i^p = 0$ for all $i, j, 1 \leq i, j \leq n$. Suppose that*

- 1) *There exist $z_1, \dots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}$, $1 \leq i, j \leq n - 1$.*
- 2) *There exists $y \in A$ such that $D_n(y) = 1$.*

Then there exists $t \in A$ such that $D_n(t) = 1$ and $D_i(t) = 0$, for all $i = 1, \dots, n - 1$.

Proof The set of derivations comes from a comodule structure of A on H , $H = k[x_1, \dots, x_n]$, $x_i^p = 0$, x_i primitive, given by $\delta : A \rightarrow A \otimes H$,

$$\delta(a) = a \otimes 1 + \sum_{1 \leq i \leq n} D_i(a) \otimes x_i + \sum_{\substack{\alpha \in \Lambda \\ |\alpha| \geq 2}} D_\alpha(a) \otimes x^\alpha. \tag{9}$$

$\alpha \in \mathbb{N}^n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Let $R = A^{\text{co}H}$ be the coinvariant subring of A with respect to δ and let $\overline{H} = k[x_n]$, $x_n^p = 0$, $B = A^{\text{co}\overline{H}}$ the coinvariant subring of A with respect to $\overline{\delta} : A \rightarrow A \otimes \overline{H}$, $\overline{H} = H/K^+H$, $K = k[x_1, \dots, x_{n-1}]$, $x_i^p = 0$, $i = 1, \dots, n - 1$. Consider the extensions $R \subset B \subset A$. By 2), $B \subset A$ is \overline{H} -Galois and $1, y, y^2, \dots, y^{p-1}$ is a basis of A on $B = A^{D_n}$. By 1), $R \subset B$ is K -Galois and the monomials $z_1^{j_1} \dots z_{n-1}^{j_{n-1}}$, $1 \leq j_i \leq p - 1$, $i = 1, \dots, n - 1$, are a basis of B on R . We want to find $t \in A$ such that $D_n(t) = 1$ and $D_i(t) = 0$ for all $i = 1, \dots, n - 1$. Put $t = \sum_{i=0}^{p-1} b_i y^i$. Then $D_n(t) = 1 = \sum_{i=0}^{p-1} b_i i y^{i-1}$ implies $b_1 = 1$ and $b_i = 0$, for all $i > 1$. We can rewrite $t = b_0 + y$ as $t = y - b$, $b \in B$. Then we need an element $b \in B$ such that $D_i(y) = D_i(b)$, $i = 1, \dots, n - 1$. Moreover for $i = 1, \dots, n - 1$, $D_i(y) \in B$, since $D_n(D_i(y)) = D_i(D_n(y)) = D_i(1) = 0$, for all $i = 1, \dots, n - 1$. Then we can write:

$$D_j(y) = \sum_{0 \leq i_j \leq p-1} s_{j, i_1, \dots, i_{n-1}} z_1^{i_1} \dots z_{n-1}^{i_{n-1}}, \quad j = 1, \dots, n - 1, s_{j, i_1, \dots, i_{n-1}} \in R.$$

Since $D_j^p = 0$, for all $j = 1, \dots, n - 1$, we have:

$$\left\{ \begin{array}{l} D_1^{p-1}(D_1(y)) = 0 = \sum_{0 \leq i_j \leq p-1} s_{1, i_1, \dots, i_{n-1}} D_1^{p-1}(z_1^{i_1}) \dots z_{n-1}^{i_{n-1}}, \\ \dots \\ D_{n-1}^{p-1}(D_{n-1}(y)) = 0 = \sum_{0 \leq i_j \leq p-1} s_{n-1, i_1, \dots, i_{n-1}} z_1^{i_1} \dots D_{n-1}^{p-1}(z_{n-1}^{i_{n-1}}). \end{array} \right.$$

Hence we get the relations

$$\left\{ \begin{array}{l} 0 = \sum_{\substack{0 \leq i_j \leq p-1 \\ j \neq 1}} s_{1, p-1, i_2, \dots, i_{n-1}} (p-1)! z_2^{i_2} \dots z_{n-1}^{i_{n-1}}, \\ \dots \\ 0 = \sum_{\substack{0 \leq i_j \leq p-1 \\ j \neq n-1}} s_{n-1, i_1, \dots, p-1} (p-1)! z_1^{i_1} \dots z_{n-2}^{i_{n-2}}, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} s_{1, p-1, i_2, \dots, i_{n-1}} = 0 \quad 0 \leq i_2, \dots, i_{n-1} \leq p - 1 \\ \dots \\ s_{n-1, i_1, \dots, i_{n-2}, p-1} = 0 \quad 0 \leq i_1, \dots, i_{n-2} \leq p - 1. \end{array} \right.$$

Writing

$$b = \sum_{0 \leq j_i \leq p-1} t_{j_1, \dots, j_{n-1}} z_1^{j_1} \dots z_{n-1}^{j_{n-1}},$$

b is uniquely determined by coefficients $t_{j_1, \dots, j_{n-1}}$, $0 \leq j_i \leq p-1$. By derivation, we obtain

$$\left\{ \begin{array}{l} D_1(b) = \sum_{\substack{0 \leq j_i \leq p-1}} t_{j_1, \dots, j_{n-1}} j_1 z_1^{j_1-1} \dots z_{n-1}^{j_{n-1}}, \\ = \sum_{\substack{0 \leq j_1 \leq p-2 \\ 0 \leq j_i \leq p-1, i \neq 1}} t_{j_1+1, j_2, \dots, j_{n-1}} (j_1 + 1) z_1^{j_1} \dots z_{n-1}^{j_{n-1}}. \\ \dots \dots \dots \\ D_{n-1}(b) = \sum_{\substack{0 \leq j_i \leq p-1}} t_{j_1, j_2, \dots, j_{n-1}} j_{n-1} z_1^{j_1} \dots z_{n-1}^{j_{n-1}-1} \\ = \sum_{\substack{0 \leq j_{n-1} \leq p-2 \\ 0 \leq j_i \leq p-1, i \neq n-1}} t_{j_1, \dots, j_{n-1}+1} (j_{n-1} + 1) z_1^{j_1} \dots z_{n-1}^{j_{n-1}}. \end{array} \right.$$

From $D_i(y) = D_i(b)$, for $i = 1, \dots, n-1$, it follows

$$\left\{ \begin{array}{l} \sum_{\substack{0 \leq j_1 \leq p-2 \\ 0 \leq j_i \leq p-1, i \neq 1}} t_{j_1+1, j_2, \dots, j_{n-1}} (j_1 + 1) z_1^{j_1} \dots z_{n-1}^{j_{n-1}} = \sum_{\substack{0 \leq j_1 \leq p-2 \\ 0 \leq j_i \leq p-1, i \neq 1}} s_{1, j_1, \dots, j_{n-1}} z_1^{j_1} \dots z_{n-1}^{j_{n-1}}, \\ \dots \dots \dots \\ \sum_{\substack{0 \leq j_{n-1} \leq p-2 \\ 0 \leq j_i \leq p-1, i \neq n-1}} t_{j_1, j_2, \dots, j_{n-1}+1} (j_{n-1} + 1) z_1^{j_1} \dots z_{n-1}^{j_{n-1}} = \sum_{\substack{0 \leq j_{n-1} \leq p-2 \\ 0 \leq j_i \leq p-1, i \neq n-1}} s_{n-1, j_1, \dots, j_{n-1}} z_1^{j_1} \dots z_{n-1}^{j_{n-1}}. \end{array} \right.$$

Hence we get the relations

$$\left\{ \begin{array}{l} t_{j_1+1, j_2, \dots, j_{n-1}} (j_1 + 1) = s_{1, j_1, \dots, j_{n-1}} \quad 0 \leq j_1 \leq p-2, 0 \leq j_i \leq p-1, i \neq 1, \\ \dots \dots \dots \\ t_{j_1, j_2, \dots, j_{n-1}+1} (j_{n-1} + 1) = s_{n-1, j_1, \dots, j_{n-1}} \quad 0 \leq j_{n-1} \leq p-2, 0 \leq j_i \leq p-1, i \neq n-1 \end{array} \right. \tag{10}$$

From the conditions $D_k D_\ell = D_\ell D_k$ for $1 \leq \ell < k \leq n-1$ we obtain the compatibility relations

$$j_k s_{\ell, j_1, j_2, \dots, j_\ell, \dots, j_k, \dots, j_{n-1}} = (j_\ell + 1) s_{k, j_1, j_2, \dots, j_\ell+1, \dots, j_k-1, \dots, j_{n-1}} \tag{11}$$

with $0 \leq j_\ell \leq p-2, 1 \leq j_k \leq p-1, 0 \leq j_i \leq p-1, i \neq \ell, k, 1 \leq \ell < k \leq n-1$. The first two relations of (10) give, for $\ell = 1, k = 2$

$$t_{j_1+1, j_2, \dots, j_{n-1}} (j_1 + 1) = s_{1, j_1, \dots, j_{n-1}} \quad 0 \leq j_i \leq p-1, i \neq 1, 0 \leq j_1 \leq p-2, \tag{12}$$

$$t_{j_1, j_2+1, \dots, j_{n-1}} (j_2 + 1) = s_{2, j_1, \dots, j_{n-1}} \quad 0 \leq j_i \leq p-1, i \neq 2, 0 \leq j_2 \leq p-2. \tag{13}$$

We rewrite the relations (12) and (13)

$$t_{j_1, j_2, \dots, j_{n-1}} j_1 = s_{1, j_1-1, j_2, \dots, j_{n-1}} \quad 0 \leq j_i \leq p-1, i \neq 1, 1 \leq j_1 \leq p-2,$$

$$t_{j_1, j_2, \dots, j_{n-1}} j_2 = s_{2, j_1, j_2-1, \dots, j_{n-1}}, \quad 0 \leq j_i \leq p-1, i \neq 2, 1 \leq j_2 \leq p-2,$$

obtaining

$$j_1 j_2 t_{j_1, j_2, \dots, j_{n-1}} = j_2 s_{1, j_1-1, j_2, \dots, j_{n-1}} = j_1 s_{2, j_1, j_2-1, \dots, j_{n-1}}.$$

Likewise, we can deduce

$$j_1 \dots j_{n-1} t_{j_1, j_2, \dots, j_{n-1}} = j_2 \dots j_{n-1} s_{1, j_1-1, j_2, \dots, j_{n-1}} = j_1 j_3 \dots j_{n-1} s_{2, j_1, j_2-1, \dots, j_{n-1}} =$$

$$\cdots = j_1 j_2 \cdots j_{n-2} s_{n-1, j_1, j_2, \dots, j_{n-1}+1}, \quad \text{for } 0 \leq j_i \leq p-1.$$

Hence, the elements $t_{j_1, j_2, \dots, j_{n-1}}$ are determined and, as a consequence, the element b is obtained. □

Corollary 3.2 *Let A be a k -algebra, k a field of characteristic $p > 0$ and let $\{D_1, \dots, D_n\} \subset \text{Der}_k(A)$ such that $D_i D_j = D_j D_i$, $D_i^p = 0$ for all $i, j, 1 \leq i, j \leq n$.*

Suppose that

- 1) *There exist $z_1, \dots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}$, $1 \leq i, j \leq n-1$.*
- 2) *There exists $y \in A$ such that $D_n(y) = 1$.*

Then there exist z_1, \dots, z_{n-1}, z_n such that $D_i(z_j) = \delta_{ij}$.

Proof Follows from Theorem 3.1, with $z_n = t$. □

Corollary 3.3 *Let A be a k -algebra, k a field of characteristic $p > 0$ and let $\{D_1, \dots, D_n\} \subset \text{Der}_k(A)$ such that $D_i D_j = D_j D_i$, $D_i^p = 0$ for all $i, j, 1 \leq i, j \leq n$.*

Suppose that:

- 1) *There exist $z_1, \dots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}$, $1 \leq i, j \leq n-1$.*
- 2) *There exists $y \in A$ such that $D_n(y) = 1$.*

Then the set $\{z_1, \dots, z_{n-1}\}$ of p -independent elements of A on the subring of constants $A^{\{D_1, \dots, D_n\}}$ can be completed to a p -basis B of n elements of A on $A^{\{D_1, \dots, D_n\}}$ and $A = A^{\{D_1, \dots, D_n\}}[B]$.

Proof It is easy to prove that z_1, \dots, z_{n-1}, z_n , with z_n as in Corollary 3.2, verify $D_i(z_j) = \delta_{ij}$, $1 \leq i \leq j \leq n$, $D_i^p = 0$, $[D_i, D_j] = 0$ and form a p -basis of A on $A^{\{D_1, \dots, D_n\}}$. The structure of A follows by definition of p -basis [3]. □

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