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GÖKHAN BİLHAN

AYŞE TUGBA GÜROĞLU

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A variation of supplemented modules

Gökhan BİLHAN,[1] Ayşe Tugba GÜROĞLU[2,*]
[1]Dokuz Eylül University, Faculty of Arts and Sciences, Department of Mathematics, Tınaztepe Y. Buca, İzmir, Turkey
[2]Celal Bayar University, Faculty of Arts and Sciences, Department of Mathematics, Muradiye Y. Manisa, Turkey

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Abstract: Over a general ring, an R-module is w-supplemented if and only if amply w-supplemented. It is proved that over a local Dedekind domain, all modules are w-supplemented and over a non-local Dedekind domain, an R-module M is w-supplemented if and only if $\mathrm{Soc}(M) \ll M$ or $M = S_0 \oplus (\bigoplus_{i\in I} K)$, where S_0 is a torsion, semisimple submodule of M and K is the field of quotients of R.

Key words: Modules, w-supplemented, Dedekind domain

1. Introduction

Since Kasch and Mares have defined the notions of perfect and semiperfect for modules, the notion of a *supplemented module* has been used extensively by many authors. A module M is called supplemented if, for every submodule A of M, there is a submodule B of M such that $M = A + B$ and $A \cap B$ is a small submodule of B. In early years, supplemented modules and two other generalizations, *Amply supplemented modules* and *weakly supplemented modules*, appeared in Helmut Zöschinger's works and he characterized their structure over local and non-local Dedekind domains [17],[18],[19],[20],[21]. After Zöschinger, we see more work on variations of supplemented modules. A. Harmancı, P. F. Smith, W. Xue and D. Keskin's works were on $\oplus$-supplemented modules [9],[12]. $\oplus$-supplemented modules are also studied by R. Tribak and A. Idelhadj in [10]. Cofinitely supplemented modules are studied by R. Alizade, P.F. Smith and G. Bilhan in [1]. Cofinitely weak supplemented modules are studied by R. Alizade and E. Büyükaşık in [2]. $\oplus$-cofinitely supplemented modules are studied by H. Çalışıcı and A. Pancar in [7]. Totally and totally cofinitely supplemented modules are studied by P.F. Smith and G. Bilhan in [3] and [14]. In recent years, rad-supplemented modules are studied by W. Yongduo and D. Nanging in [16] and by E. Büyükaşık, E. Mermut and S. Özdemir in [6] and cofinitely rad-supplemented modules are studied by E. Büyükaşık and C. Lomp in [5].

This paper is based on another variation of supplemented modules.

We shall say that a module M is *w-supplemented* if every semisimple submodule of M has a supplement in M.

Lemma 1.1 *Let $M = N + L$ where L is a submodule of M and N is a semisimple submodule of M. Then $M = N' \oplus L$ for some submodule N' of N.*

*Correspondence: tugba.guroglu@deu.edu.tr
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Proof Let N be a semisimple submodule of M. Then $N \cap L$ is direct summand in N. That is, $N = (N \cap L) \oplus N'$ for some submodule N' of N. Since $M = N + L$, then we have $M = ((N \cap L) \oplus N') + L = N' + L$. So $M = N' \oplus L$ because $(N \cap L) \cap N' = N' \cap L = 0$. □

Lemma 1.2 *([8], 2.8.(9)) Let M be an R-module and let $Rad(M)$ be the radical of M and let $Soc(M)$ be the socle of M. Then $Soc(Rad(M)) \ll M$.*

Lemma 1.3 *Let U be a semisimple submodule of M contained in $Rad(M)$. Then U is small in M.*

Proof Let $U \subseteq \mathrm{Rad}(M)$, where U is semisimple in M. Then $\mathrm{Soc}(U) \subseteq \mathrm{Soc}(\mathrm{Rad}(M))$. Since U is semisimple, $\mathrm{Soc}(U) = U$. Then $U \subseteq \mathrm{Soc}(\mathrm{Rad}(M))$. By 1.2 and by ([15], 19.3), $U \ll M$. □

Example 1.4 *Clearly, any module M with $Soc(M) = 0$ is w-supplemented. So, $\mathbb{Z}$-module $\mathbb{Z}$ is w-supplemented but not supplemented.*

We see weakly supplemented modules in Zöschinger's works. But defining weakly w-supplemented modules or some other variations of w-supplemented modules does not make sense, because of the following result.

Proposition 1.5 *Let R be a ring and M be an R-module. Then the following statements are equivalent.*

1. *M is w-supplemented.*
2. *Every semisimple submodule of M has a supplement that is a direct summand.*
3. *Every semisimple submodule of M has a weak supplement.*
4. *Every semisimple submodule of M has a rad-supplement.*

Proof $(1 \Rightarrow 2)$ Let N be a semisimple submodule of M. By assumption, N has a supplement K in M for some submodule K of M. That is, $M = N + K$ and $N \cap K \ll K$. By 1.1, $M = N' \oplus K$ for some submodule N' of N.
$(2 \Rightarrow 3)$ Let N be a semisimple submodule of M. By (2), N has a supplement, so N has a weak supplement, since supplements are also weak supplements.
$(3 \Rightarrow 4)$ Let N be a semisimple submodule of M. Since N has a weak supplement, then there exists a submodule K of M such that $N + K = M$ and $N \cap K \ll M$. By 1.1, $M = N' \oplus K$ for some submodule N' of N. By ([15], 19.3(5)), $N \cap K \ll K$. This implies $N \cap K \leq \mathrm{Rad}(K)$. Thus K is rad-supplement of N in M.
$(4 \Rightarrow 1)$ Let N be a semisimple submodule of M. By assumption, N has a rad-supplement K in M. Then $M = N + K$ and $K \cap N \leq \mathrm{Rad}(K)$, also by ([15], 21.6(1)(i)) and considering inclusion map $i : K \to M$, we say $K \cap N \leq \mathrm{Rad}(M)$. Then by 1.3, $K \cap N \ll M$. Since N is semisimple, by 1.1, $M = N' \oplus K$ for some submodule N' of N. So we get $K \cap N \ll K$ by ([15], 19.3(5)). □

Proposition 1.6 *Any direct summand of a w-supplemented module is w-supplemented.*

Proof Let M be w-supplemented module and N be a direct summand of M so that $M = N \oplus K$ for some submodule K of M. Let S be a semisimple submodule of N. If $S = 0$, then N is trivially w-supplemented. Let $S \neq 0$, since $S \subseteq M$, then $M = S + T$ and $S \cap T \ll T$ for some submodule T of M. Then by the modular law, $N = S + (N \cap T)$ and consequently by 1.1, $N = S' \oplus (N \cap T)$ for some $S' \subseteq S$. That is, $N \cap T$ is a direct summand of N. If we are able to show that $S \cap (N \cap T) << N \cap T$, then we are done. Since $S \cap T \ll T$ by ([15], 19.3(4)) together with the inclusion map, $S \cap T << M$, and since $S \cap T \subseteq N$, then by ([15], 19.3(5)) $S \cap T << N$ and consequently $S \cap (N \cap T) << N \cap T$, because $(S \cap T) \cap N = S \cap T \subseteq N \cap T$. Therefore $N \cap T$ is a supplement of S in N. □

Proposition 1.7 *Any finite direct sum of w-supplemented modules is w-supplemented.*

Proof It is sufficient to prove for the case $M = M_1 \oplus M_2$ where M_1 and M_2 are w-supplemented modules, then result follows inductively. For $i = 1, 2$, let $p_i : M \to M_i$ be the projection map. Let L be a semisimple submodule of M. Then so are the modules $p_1(L) = (L + M_2) \cap M_1$ and $p_2(L) = (L + M_1) \cap M_2$. Then $p_1(L)$ and $p_2(L)$ have supplements H_1 and H_2 in M_1 and M_2 respectively. $M_1 + M_2 + L$ has a supplement 0 in M. By ([9], Lemma 1.3), H_2 is a supplement of $M_1 + L$ in M. Also we may say that $(L + H_2) \cap M_1 \subseteq (L + M_2) \cap M_1 = p_1(L)$ means $(L + H_2) \cap M_1$ is also semisimple, then has a supplement K in M_1. Again applying ([9], Lemma 1.3), $H_2 + K$ is a supplement of L in M. Hence M is w-supplemented. □

Lemma 1.8 *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence for R-modules. If L and N are w-supplemented and $L \ll M$, then M is w-supplemented.

Proof Let us consider N as $\frac{M}{L}$. Let U be a semisimple submodule of M. Then $\frac{U+L}{L}$ is a semisimple submodule in $\frac{M}{L}$. If $\frac{M}{L} = \frac{U+L}{L}$, then $M = U + L$. By 1.1, $M = U' \oplus L$ where $U' \subseteq U$, then M is w-supplemented as a finite direct sum of w-supplemented modules. Let $\frac{U+L}{L}$ be a proper submodule of $\frac{M}{L}$. By assumption, $\frac{U+L}{L}$ has a supplement $\frac{V}{L}$ in $\frac{M}{L}$. That is, $\frac{M}{L} = \frac{(U+L)}{L} + \frac{V}{L}$ and $\frac{(U+L)}{L} \cap \frac{V}{L} \ll \frac{V}{L}$. Therefore $M = U + V$ and with modular law $\frac{(U \cap V) + L}{L} \ll \frac{V}{L}$. By 1.1, $M = U' \oplus V$ for some submodule U' of U. Let us show $U \cap V \ll V$: Let $V = (U \cap V) + X$ for some submodule X of V. Then $\frac{V}{L} = \frac{(U \cap V) + L}{L} + \frac{X + L}{L}$. Since $\frac{(U \cap V) + L}{L} \ll \frac{V}{L}$, then $\frac{V}{L} = \frac{X + L}{L}$. It follows that $V = X + L$. Since V is a direct summand of M, by ([15], 19.3(5)) $V = X$. □

A module M is *amply supplemented*, if whenever $M = A + B$, then B contains a supplement of A. A module M is called *amply w-supplemented*, if $M = A + B$ where A is a semisimple submodule of M, then B contains a supplement of A.

In all variations of supplemented modules, amply supplemented versions are different than supplemented ones. For instance, (cofinitely) supplemented modules need the projective property to become amply (cofinitely) supplemented, (see [15], 41.15) and ([1], Proposition 2.14.) But for our modules, they are the same.

Proposition 1.9 *M is w-supplemented if and only if M is amply w-supplemented.*

Proof ($\Leftarrow$) Obvious.

($\Rightarrow$) Let $M = A+B$ where A is semisimple. Since A is semisimple, then so is $A\cap B$ and hence by Lemma 1.1, $M = Y_1 \oplus T$ for some submodule Y_1 of $A\cap B$ and some supplement T of $A\cap B$ in M. By the modular law, $A\cap B = Y_1 \oplus (A\cap B\cap T)$. Let's call $A\cap B\cap T = S$ and by applying the modular law once more to $M = Y_1\oplus T$, we get $B = Y_1 \oplus (B\cap T)$. Let's call $B\cap T = Y_2$. We consider the projection mapping $\pi : Y_1 \oplus Y_2 \to Y_2$, then

$$A\cap B = Y_1 \oplus S = Y_1 \oplus (A\cap(B\cap T)) = Y_1 \oplus (A\cap Y_2) = Y_1 \oplus (A\cap Y_2 \cap B)$$

since $Y_2 \subseteq B$. Then $A\cap Y_2 = A\cap B\cap Y_2 = \pi(A\cap B) = \pi(Y_1+S) = \pi(S)$. Then by ([15], 19.3(4)), $A\cap Y_2 = \pi(S)$ is small in $\pi(T)$ and consequently in Y_2. Also $M = A+B = A+Y_1+Y_2 = A+Y_2$. Therefore A has supplement Y_2 contained in B. □

A module M is totally (cofinitely) supplemented, if every submodule is (cofinitely) supplemented ([3],[14]). We shall say that a module M is *totally w-supplemented module* if every submodule of M is w-supplemented. Maybe, after Prop.1.5 and Prop.1.9, it is expected that w-supplemented modules are also totally w-supplemented. But unfortunately, it is not the case.

Here is an example:

Let R be a commutative ring with identity 1 and M be an R-module. Then it is not difficult to check that

$$S = \left\{ \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} : a\in R, m\in M \right\}$$

is a ring with ordinary addition and multiplication. Furthermore, S is a commutative ring with identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

An R-module M is *faithful*, if for all $r\in R$, $Mr = 0$ implies that $r = 0$.

Lemma 1.10 *If M is faithful right R-module, then*

$$Soc(_SS) = \begin{pmatrix} 0 & Soc(_RM) \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in S : m\in Soc(_RM) \right\}$$

and $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \ll_S S$.

Proof We note that if N is any R-submodule of M, then $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ is an ideal of the ring S. Also, an R-submodule N of M is simple if and only if $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ is a simple ideal of S and hence $\begin{pmatrix} 0 & \mathrm{Soc}_R(M) \\ 0 & 0 \end{pmatrix} \subseteq \mathrm{Soc}(S)$. Conversely, let I be any nonzero simple ideal of S and let $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \in I$ be nonzero. Then $I = S.\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}$. Since $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ is an ideal of S, $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}.\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & aM \\ 0 & 0 \end{pmatrix}$ is an ideal of S contained in I. Then, by the minimality of I, $\begin{pmatrix} 0 & aM \\ 0 & 0 \end{pmatrix} = I$ or zero. In the former case we get that $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \in \begin{pmatrix} 0 & aM \\ 0 & 0 \end{pmatrix}$ which implies that $a = 0$, $m \in aM$ and hence $m = 0$ which implies that $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} = 0$ which is a contradiction. So $\begin{pmatrix} 0 & aM \\ 0 & 0 \end{pmatrix} = 0$ which implies $aM = 0$. Since, by hypothesis, M is faithful R-module, it follows that $a = 0$. Thus

$$I = S.\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \{\begin{pmatrix} a' & m' \\ 0 & a' \end{pmatrix}.\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} : a' \in R, m' \in M\}$$

$$= \{\begin{pmatrix} 0 & am \\ 0 & 0 \end{pmatrix} : a \in R\} = \begin{pmatrix} 0 & Rm \\ 0 & 0 \end{pmatrix}$$

Since I is a simple ideal of S, it follows by above observation that Rm is a simple R-submodule of M and hence $I \subseteq \begin{pmatrix} 0 & \mathrm{Soc}_R(M) \\ 0 & 0 \end{pmatrix}$. This proves that $\mathrm{Soc}(S) \subseteq \begin{pmatrix} 0 & \mathrm{Soc}_R(M) \\ 0 & 0 \end{pmatrix}$ and hence $\mathrm{Soc}(S) = \begin{pmatrix} 0 & \mathrm{Soc}_R(M) \\ 0 & 0 \end{pmatrix}$. For the other part, $J(S) \ll S$ is always true, since S has identity. □

Example 1.11 *Let R be a ring and M be a faithful right R-module with the properly that $Soc({}_RM)$ has no supplement in M. For example, we consider a $\mathbb{Z}$-module $M = \prod_{p-prime} \mathbb{Z}_p$ is faithful but the torsion submodule $Soc(M) = \bigoplus_{p-prime} \mathbb{Z}_p$ has no supplement in M, as it is explained in Example 2.7 where $\mathbb{Z}_p$'s are $\mathbb{Z}/p\mathbb{Z}$'s for various prime p's. Then ${}_SS = \left\{ \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} : a \in R, m \in M \right\}$ is w-supplemented but the submodule ${}_SN = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ is not w-supplemented. Because, by 1.10, $Soc({}_SS) = \begin{pmatrix} 0 & Soc({}_RM) \\ 0 & 0 \end{pmatrix}$ and $Soc({}_SS) \ll_S S$. So, $Soc({}_SS)$ has supplement ${}_SS$ in ${}_SS$. $\begin{pmatrix} 0 & Soc({}_RM) \\ 0 & 0 \end{pmatrix}$ is a semisimple submodule of $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ but it has no supplement in ${}_SN$, because if $\begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$ was a supplement of $Soc({}_SS)$, then obviously we will have $Soc({}_RM) + L = M$ and $Soc({}_RM) \cap L \ll L$. That is, L is a supplement of $Soc({}_RM)$ in M, a contradiction.*

Lemma 1.12 *If every semisimple submodule of $\frac{M}{Rad(M)}$ is a direct summand, then M is w-supplemented.*

Proof Let N be a semisimple submodule of M, then $\frac{N + \mathrm{Rad}(M)}{\mathrm{Rad}(M)}$ is semisimple too. Then $\frac{M}{\mathrm{Rad}(M)} = \frac{N + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} \oplus \frac{K}{\mathrm{Rad}(M)}$ for some submodule K of M containing $\mathrm{Rad}(M)$. So, $M = N + K$ and $N \cap K \subseteq \mathrm{Rad}(M)$. By 1.3, $N \cap K \ll M$. Furthermore, by 1.1, K is a direct summand of M containing $N \cap K$. Then by ([15], 19.3(5)), $N \cap K \ll K$. Therefore, M is w-supplemented. □

Lemma 1.13 *Let M be an R-module with $Soc(M) \ll M$, then M is w-supplemented.*

Proof Obviously, if $\mathrm{Soc}(M) = 0$, then M is w-supplemented. Let N be a nonzero semisimple submodule of M, then $N \subseteq \mathrm{Soc}(M)$, so N is small in M too, then $M = M + N$ and $M \cap N = N \ll M$. □

Lemma 1.14 *M is w-supplemented if and only if Soc(M) has a supplement in M.*

Proof $(\Rightarrow)$ Straightforward.
$(\Leftarrow)$ Let V be a supplement of $\mathrm{Soc}(M)$ in M, then $M = \mathrm{Soc}(M) + V$ and $\mathrm{Soc}(V) = \mathrm{Soc}(M) \cap V \ll V$, then by 1.13, V is w-supplemented. Since $M = S \oplus V$ where S is a semisimple submodule of M by 1.1, then by 1.7, M is w-supplemented. □

A submodule N of a module M is said to be *radical*, if $\mathrm{Rad}(N) = N$.

Proposition 1.15 *Every radical module M is w-supplemented.*

Proof Let M be a radical module, that is, $M = \mathrm{Rad}(M)$, then $\mathrm{Soc}(M) = \mathrm{Soc}(\mathrm{Rad}(M)) \ll M$ by 1.3 and then by 1.13, M is w-supplemented. □

2. W-supplemented modules over commutative domains

Proposition 2.1 *Over a proper (not a field) Dedekind domain R, if an R-module M is torsion free, then Soc$(M) = 0$.*

Proof Let S be a nonzero simple submodule of M where M is torsion-free, then for any prime element p of R, $ps \neq 0$ for any nonzero s of S. Then, obviously $< ps >= S$. Let r_0 be another nonzero nonunit element of R that is not an associate of p and then $< r_0 s >= S$ indeed. Then $ps = r'(r_0 s)$ for some $r' \in R$, though $(p - r'r_0)s = 0$ implies $p = r'r_0$; but primes are irreducible, so r' is a unit, a contradiction. □

Corollary 2.2 *Over a Dedekind domain, all torsion-free modules are w-supplemented.*

Proof All modules with zero socle are w-supplemented. If R is a field, then all submodules are semisimple, and so w-supplemented. □

Proposition 2.3 *Over a Dedekind domain R all torsion modules are w-supplemented.*

Proof Let M be a torsion R-module, then by ([4], Corollary 2.7), $\dfrac{M}{\mathrm{Rad}(M)}$ is semisimple, then M is w-supplemented by 1.12. □

We may give now an example of a w-supplemented module that is not supplemented and with nontrivial proper socle.

Example 2.4 *Let $M = \bigoplus_{k=1}^{\infty} \mathbb{Z}_{p^k}$ where p is fixed prime. Then M is torsion when considered as a $\mathbb{Z}$-module. Note that Soc$(M) \cong \bigoplus^{\infty} \mathbb{Z}_p \neq 0$. In Example 2.14 of [4], it is showed that M is not weakly supplemented. Since all supplemented modules are weakly supplemented, then M is not supplemented. By 2.3, M is w-supplemented.*

Proposition 2.5 *If R is a local Dedekind domain, then all R-modules are w-supplemented.*

Proof Let M be an R-module with unique maximal ideal P, then $\frac{M}{\mathrm{Rad}(M)} = \frac{M}{PM}$ is semisimple, then by 1.12 M is w-supplemented. □

Proposition 2.6 *Let R be a Dedekind domain, then every divisible R-module is w-supplemented.*

Proof Let D be a divisible R-module, then $\mathrm{Rad}(D) = D$, that is, D is a radical module. So by 1.15, D is w-supplemented. □

The following example shows that not all modules are w-supplemented over a Dedekind domain.

Example 2.7 *(Comes from [4], Example 2.11.) Let R be a non-local Dedekind domain which has infinitely many maximal ideals and $\{P_i\}_{i\in I}$ be an infinite collection of distinct maximal ideals of R. Let $M = \prod_{i\in I}(R/P_i)$. Let $T = \bigoplus_{i\in I}(R/P_i)$ be the torsion submodule of M. By Lemma 2.9 of [4], M/T is divisible and isomorphic to $K^{(J)}$ for some index set J where K is the field of quotients. Thus, M/T has a submodule N/T such that $N/T \cong K$. We claim that N is not w-supplemented: T is semisimple but doesn't have a supplement in N. Since $Rad(N) = 0$, if T had a supplement T' in N, then it would be a direct summand in N. But it is not, because whenever $N = T \oplus T'$, then $N/T \cong T'$ is divisible; since $Rad(N) = 0$, then $Rad(T') = 0$, a contradiction with $Rad(T') = T'$.*

Proposition 2.8 *Let M be a w-supplemented module. Either $Soc(M)$ is a small submodule of M, or $M = S_0 \oplus V_0$ for some nonzero greatest semisimple submodule S_0 of M containing no nonzero small submodule of M and for some submodule V_0 of M with $Soc(V_0) = 0$.*

Proof Let us construct $\Gamma = \{X \subseteq M \mid X$ is semisimple and $X \cap \mathrm{Rad}(M) = 0\}$. Since $\{0_M\} \in \Gamma$, then $\Gamma \neq \varnothing$. Clearly, any chain $\{X_i\}_{i\in I}$ for some index set I of Γ has an upper bound $\bigcup_{i\in I} X_i = X_0$, because X_0 is semisimple and $X_0 \cap \mathrm{Rad}(M) = 0$. Then by Zorn's Lemma Γ has a maximal element S_0. If $S_0 = 0$, then all simple submodules of M are also small, that is $\mathrm{Soc}(M) \subseteq \mathrm{Rad}(M)$. So by 1.3, $\mathrm{Soc}(M) \ll M$. Let $S_0 \neq 0$. Since M is w-supplemented, $M = S_0 + V$ and $S_0 \cap V \ll V$. Then, $S_0 \cap V \subseteq \mathrm{Rad}(M)$ and also is a submodule of S_0, consequently it is semisimple. But then by construction of S_0, $S_0 \cap V = 0$. Hence $M = S_0 \oplus V$. By 1.6, V is also w-supplemented, then $V = \mathrm{Soc}(V) + V_0$ and $\mathrm{Soc}(V) \cap V_0 \ll V_0$ for some submodule V_0 of V. Then $M = S_0 \oplus (\mathrm{Soc}(V) + V_0)$. By maximality of S_0, $\mathrm{Soc}(V) \subseteq \mathrm{Rad}(M)$, then by 1.3, $\mathrm{Soc}(V) \ll M$. Thus $M = S_0 \oplus V_0$. □

Lemma 2.9 *Any semisimple module over a non-local Dedekind domain is torsion.*

Proof Let R be a non-local Dedekind domain and let M be a semisimple R-module. Let S be a simple submodule of M, then $S \cong R/I$ for some ideal I of R. Since simple modules are local, then $I \neq 0$ because R is non-local. Then S is torsion, consequently $M = \bigoplus S$ is torsion, too. □

A module M is called *reduced*, if for any nonzero submodule N of M, $RadN \neq N$.

Theorem 2.10 *Let R be a non-local Dedekind domain, then an R-module M is w-supplemented if and only if either $Soc(M) \ll M$ or $M = S_0 \oplus (\bigoplus_{i\in I} K)$ where S_0 is torsion, semisimple and reduced submodule of M and K is the field of quotients of R.*

Proof ($\Rightarrow$) We may write M as $M = D \oplus A$ where D is divisible and A is reduced part of M. Also by 2.8 and 2.9, $M = S_0 \oplus V_0$ where S_0 is semisimple torsion submodule of M and V_0 is a submodule of M with zero socle. For any prime p, the divisible submodule $R(p^\infty)$ cannot lie in S_0, because its simple submodule is also small and since $R(p^\infty)$ is indecomposable then torsion divisible part of M must completely lie in V_0. But actually it cannot be in V_0 either, because V_0 has no simple submodules. Therefore no $R(p^\infty)$ exists in M. Thus, $D = \bigoplus_{i\in I} K$. But then, since D becomes torsion-free, $D \subseteq V_0$ by 2.9. Therefore $M = S_0 \oplus (\bigoplus_{i\in I} K)$ where S_0 is torsion, semisimple and reduced. If $S_0 = 0$, then obviously $\mathrm{Soc}(M) \ll M$.
($\Leftarrow$) By 2.6, $(\bigoplus_{i\in I} K)$ is w-supplemented. Then by 1.7 M is w-supplemented. Or 1.13 implies M is w-supplemented. □

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References

[1] Alizade, R., Bilhan, G., Smith, P. F.: Modules whose maximal submodules have supplements. Communications in Algebra. 29(6), 2389–2405 (2001).

[2] Alizade, R., Büyükaşık, E.: Cofinitely Weak Supplemented Modules. Communication in Algebra. 31, 5377–5390 (2003).

[3] Bilhan, G.: Totally Cofinitely Supplemented Modules. Int. Electron. J. Algebra. 2, 106–113 (2007).

[4] Büyükaşık, E., & Alizade, R.: Extensions of Weakly Supplemented Modules. Math.Scand. 103, 161–168 (2008).

[5] Büyükaşık, E., Lomp, C.: On a Recent Generalization of Semiperfect Rings. Bull. Aust. Math. Soc. 78, 317–325 (2008).

[6] Büyükaşık, E., Mermut, E. and Özdemir, S.: Rad–supplemented Modules. Rend. Sem. Mat. Univ. PADOVA. 124 (2010).

[7] Çalışıcı, H., Pancar, A.: ⊕-cofinitely Supplemented Modules. Czech. Math. J. 54, 1083–1088 (2004).

[8] Clark, J., Lomp, C., Vanaja, N., Wisbauer, R.: Lifting Modules. Birkhauser 2006.

[9] Harmancı, A., Keskin, D., Smith, P.F.: On ⊕-supplemented Modules. Acta Math. Hungarica. 83, 161–169 (1999).

[10] Idelhadj, A., Tribak, R.: On Sum Properties of ⊕-supplemented Modules. Int. J. Math. Sci. 69, 4373–4387 (2003).

[11] Kasch, F., Mares, E. A.: Eine Kennzeichnung semi-perfekter Moduln. Nagoya Math. J. 27, 525–529 (1966).

[12] Keskin, D., Smith, P. F., Xue, W.: Rings whose modules are ⊕-supplemented. J. Algebra. 218, 470–487 (1999).

[13] Lam, T. Y.: Lectures on modules and rings. vol.189, Graduate Texts in Mathematics, New-York: Springer-Verlag 1999.

[14] Smith, P. F.: Finitely generated supplemented modules are amply supplemented. Arab. J. Sci. Eng. Sect. C Theme Issues. 25(2), 69–79 (2000).

[15] Wisbauer, R.: Foundations of Modules and Rings. Gordon and Breach 1991.

[16] Yongduo, W., Nanging, D.: Generalized Supplemented Modules. Taiwanese J. Math. 10, 1589–1601 (2006).

[17] Zöschinger, H.: Komplementierte Moduln über Dedekindringen. Journal of Algebra. 29, 42–56 (1974).

[18] Zöschinger, H.: Komplemente als direkte Summanden. Arch. Math. (Basel). 25, 241–253 (1974).

[19] Zöschinger, H.: Komplemente fur zyklische Moduln uber Dedekindringen. Arch. Math. (Basel). 32, 143–148 (1979).

[20] Zöschinger, H.: Komplemente als direkte Summanden II. Arch. Math. (Basel). 38(4), 324–334 (1982a).

[21] Zöschinger, H.: Komplemente als direkte Summanden III. Arch. Math. (Basel). 46(2), 125–135 (1986).