Turkish Journal of Mathematics

Volume 37 | Number 4 Article 15

1-1-2013

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 704 – 718
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doi:10.3906/mat-1109-6

L^p solutions of infinite time interval BSDEs and the corresponding g-expectations and g-martingales

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Received: 04.09.2011 • **Accepted:** 07.07.2012 • **Published Online:** 12.06.2013 • **Printed:** 08.07.2013

Abstract: In this paper we study the existence and uniqueness theorem for L^p $(1 < p < 2)$ solutions for a class of infinite time interval backward stochastic differential equations (BSDEs). Furthermore, we introduce generalized g-expectations and generalized g-martingales via the L^p solutions and prove the stability theorem of generalized g-expectations.

Key words: Backward stochastic differential equation (BSDE), comparison theorem, generalized g-expectation, generalized g-martingale

1. Introduction

The theory of backward stochastic differential equations (BSDEs) was developed by Pardoux and Peng [24], from which we know that there exists a unique adapted and square integrable solution to a BSDE of the type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s, \qquad t \in [0, T], \tag{1}$$

provided the function g (also called the generator) is Lipschitz in both variables y and z, and ξ and $(g(t, 0, 0))_{0\leq t\leq T}$ are square integrable. Later, many researchers developed the theory of BSDEs and their applications in a series of papers (for example, see Briand et al. [3], Hu and Peng [16], Lepeltier and San Martin [19], Pardoux [22, 23], El Karoui et al. [13] and the references therein) under some other assumptions on the coefficients but for a fixed terminal time $T > 0$. Let us mention the contribution of Lepeltier and San Martin [19], which dealt with the quadratic of growth generator g in z and got the existence and uniqueness result in L^2. Let us mention also that when the generator g is Lipschitz continuous, a result of El Karoui et al. [13] provides for a solution when the data ξ and $\left\{(g(t, 0, 0)_{t\in[0,T]}\right\}$ are in L^p even for $p \in (1, 2)$. In 2003, Briand et al. [3] was devoted to the generalization of this result to the case of a monotone generator for BSDEs on a fixed time interval.

In 1997, Peng [27] introduced the notions of g-expectation and g-martingale via the L^2 solution of BSDE (1). Peng's g-expectation is a kind of nonlinear expectation, which can be considered as a nonlinear extension of the well-known Girsanov transformations. The original motivation for studying Peng's g-expectation comes from the theory of expected utility. Since the notion of Peng's g-expectation was introduced, many properties of Peng's g-expectation have been studied by Briand et al. [2], Chen [4], Chen and Wang [5], Chen and Epstein

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2000 *AMS Mathematics Subject Classification:* 60H10.

[6], Chen, Kulperger and Jiang [7, 8], Chen et al. [9], Coquet et al. [10], Hu [15], Jiang [20, 21], Rosazza Gianin [28] and in the references therein. In 2010, Hu and Chen [14] gave the extensions of Peng's g-expectations which are called generalized Peng's g-expectations, and investigated their related properties.

In this paper, we investigate generalized g-expectations and generalized g-martingales via L^p $(1 < p < 2)$ solutions of infinite time interval BSDEs. One difficulty of this problem is how to study the existence and uniqueness of BSDE (1) when $T \equiv \infty$ in L^p. In fact, such a problem in L^p $(1 < p \leq 2)$ has been investigated by Briand et al. [3], Peng [26], Pardoux [22], Darling and Pardoux [11], Pardoux and Zhang [25] and other researchers under the assumption that terminal value $\xi = 0$ or $E[\mathrm{e}^{p\rho T}|\xi|^p] < \infty$ for some constant ρ and random terminal time T (i.e. T is a stopping time).

Let us mention the contribution of Briand et al. [3] which dealt with a monotone generator g in y and got the existence and uniqueness result in L^p $(1 < p < 2)$ on a random time interval. Furthermore, Briand et al. [3] strongly pointed out that their existence and uniqueness result covered the case of $T \equiv \infty$ (see the first paragraph of Section 5 and Remark 5.3 in [3]).

Let us mention also the contribution of Hu and Tessitore [17]. In 2007, Hu and Tessitore [17] studied the existence and uniqueness of mild solutions to a possibly degenerate elliptic partial differential equation $\mathcal{L}u(x) + \psi(x, u(x), \nabla u(x)G(x)) - \lambda u(x) = 0$ in Hilbert spaces. The main tool was existence, uniqueness and regular dependence on parameters of a bounded solution to a suitable BSDE with a random terminal time T.

In 2000, Chen and Wang [5] obtained the existence and uniqueness theorem for L^2 solutions of infinite time interval BSDEs when $T \equiv \infty$, by the martingale representation theorem and fixed point theorem. But in L^p $(1 < p < 2)$, there is no martingale representation theorem. In order to get rid of this difficulty, we give a new a priori estimate (Lemma 3.1). The main idea of this a priori estimate comes from Proposition 3.2 in Briand et al. [3]. Using this a priori estimate, we study the existence and uniqueness of L^p solutions to infinite time interval BSDEs. In fact, the difference between [3] and this paper is not the time horizon over which the problem is formulated but the assumptions on the function that appear in BSDE (1) (this paper's g and [3]'s f), in which λ and μ appearing in (H2) of [3] are constant, while our α and β are integrable Lipschitz functions on time t. These integrability conditions are introduced in [5]. In this paper, we also introduce generalized g-expectations and generalized g-martingales via L^p solutions of infinite time interval BSDEs. Furthermore, we give the stability theorem of generalized g-expectations.

This paper is organized as follows. In Section 2, we introduce some notations, assumptions and lemmas. In Section 3, we prove the existence and uniqueness theorem for L^p solutions of infinite time interval BSDEs. In Section 4, we introduce generalized g-expectations and generalized g-martingales via L^p solutions of infinite time interval BSDEs and prove the stability theorem of generalized g-expectations.

2. Preliminaries

In this section, we shall present some notations, assumptions and lemmas that are used in this paper.

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space, $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on this space and $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by Brownian motion $(W_t)_{t\geq 0}$, that is,

$$\mathcal{F}_t := \sigma\{W_s; s \leq t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. Furthermore, we assume $\mathcal{F} := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$.

For simplicity, we just consider the case that $d=1$, but our method can be easily extended to the other cases.

We consider the following spaces:

$L^p(\Omega,\mathcal{F},P):=\{\xi:\xi$ is $\mathcal{F}$-measurable random variable such that $E[|\xi|^p]<\infty, p\geq 1\}$;

$\mathcal{L}(\Omega,\mathcal{F},P):=\bigcup_{p>1} L^p(\Omega,\mathcal{F},P)$;

$\mathcal{S}^p(\mathbb{R}):=\{V:V_t$ is $\mathcal{F}_t$-adapted process such that $E[\sup_{t\geq 0}|V_t|^p]<\infty, p\geq 1\}$;

$\mathcal{S}(\mathbb{R}):=\bigcup_{p>1}\mathcal{S}^p(\mathbb{R})$;

$\mathcal{L}^p(\mathbb{R}):=\{V:V_t$ is $\mathcal{F}_t$-adapted process such that $E[(\int_0^\infty |V_s|^2\mathrm{d}s)^{\frac{p}{2}}]<\infty, p\geq 1\}$;

$\mathcal{L}(\mathbb{R}):=\bigcup_{p>1}\mathcal{L}^p(\mathbb{R})$.

In the sequel, we assume that $1<p<2$.

Consider the following infinite time interval BSDE:

$$Y_t=\xi+\int_t^\infty g(s,Y_s,Z_s)\mathrm{d}s+V_\infty-V_t-\int_t^\infty Z_s\mathrm{d}W_s. \tag{2}$$

Let

$$g:\Omega\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$$

such that for any $(y,z)\in\mathbb{R}\times\mathbb{R}$, $g(\cdot,y,z)$ is $\mathcal{F}_t$-progressively measurable. We make the following assumptions:

(A.1) $E\left[\left(\int_0^\infty |g(t,0,0)|\mathrm{d}t\right)^2\right]<\infty$;

(A.2) There exists two positive non-random functions $\alpha(t)$ and $\beta(t)$, such that for all $y_1,y_2\in\mathbb{R}$, $z_1,z_2\in\mathbb{R}$,

$$|g(t,y_1,z_1)-g(t,y_2,z_2)|\leq\alpha(t)|y_1-y_2|+\beta(t)|z_1-z_2|,$$

where $\alpha(t)$ and $\beta(t)$ satisfy that $\int_0^\infty\alpha(t)\mathrm{d}t<\infty$, $\int_0^\infty\beta(t)\mathrm{d}t<\infty$, $\int_0^\infty\beta^2(t)\mathrm{d}t<\infty$;

(A.3) There exists some constant $T\in[0,\infty)$ such that

$$E\left[\left(\int_0^T |g(t,0,0)|\mathrm{d}t\right)^p\right]<\infty,$$

$$E\left[\left(\int_T^\infty |g(t,0,0)|\mathrm{d}t\right)^2\right]<\infty.$$

(A.4) $(V_t)_{t\geq 0}$ is an RCLL process (i.e. $(V_t)_{t\geq 0}$ has sample paths which are right continuous with left limits) with $(V_t)_{t\geq 0}\in\mathcal{S}^2(\mathbb{R})$.

The following lemmas are very useful in this paper.

Lemma 2.1 *Let $\{K_t\}_{t\geq 0}$ and $\{H_t\}_{t\geq 0}$ be two progressively measurable processes with values in $\mathbb{R}$ such that P-a.s.,*

$$\int_0^\infty(|K_t|+|H_t|^2)\mathrm{d}t<+\infty.$$

We consider the $\mathbb{R}$*-valued semi-martingale* $\{X_t\}_{t\geq 0}$ *defined by*

$$X_t = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s, \qquad 0 \leq t \leq \infty.$$

Then, for any $p \geq 1$*, we have*

$$\begin{array}{rcl} |X_\infty|^p & \geq & |X_t|^p + p\int_t^\infty |X_s|^{p-1}\frac{X_s}{|X_s|}1_{(X_s\neq 0)}K_s\mathrm{d}s + p\int_t^\infty |X_s|^{p-1}\frac{X_s}{|X_s|}1_{(X_s\neq 0)}H_s\mathrm{d}W_s \\ & + & c(p)\int_t^\infty |X_s|^{p-2}1_{(X_s\neq 0)}|H_s|^2\mathrm{d}s, \end{array} \tag{3}$$

where $c(p) = \frac{p[1\wedge(p-1)]}{2}$, $1\wedge(p-1) := \min\{1,(p-1)\}$.

The proof of Lemma 2.1 is very similar to that of Lemma 2.2 in [3]. It is almost verbatim adapted from [3]. Now we briefly give the idea of the proof of Lemma 2.1. Since the function $x \mapsto |x|^p$ *is not smooth enough (for* $p \in [1,2)$*) to apply Itô's formula, we use an approximation. Let* $\varepsilon > 0$ *and let us consider the function* $u_\varepsilon(x) := (|x|^2+\varepsilon^2)^{\frac{1}{2}}$*. Obviously, it is a smooth function. Itô's formula leads to the following equality:*

$$\begin{array}{rcl} u_\varepsilon^p(X_\infty) & = & u_\varepsilon^p(X_t) + p\int_t^\infty u_\varepsilon^{p-2}(X_s)X_sK_s\mathrm{d}s + p\int_t^\infty u_\varepsilon^{p-2}(X_s)X_sH_s\mathrm{d}W_s \\ & + & \frac{1}{2}p\int_t^\infty \left[u_\varepsilon^{p-2}(X_s) + (p-2)u_\varepsilon^{p-4}(X_s)X_s^2\right]H_s^2\mathrm{d}s. \end{array} \tag{4}$$

Letting $\varepsilon \to 0$ *in (4) and applying convergence, we can obtain (3).*

Lemma 2.2 *If* (Y,Z) *is a solution of the following BSDE:*

$$Y_t = \xi + \int_t^\infty g(s,Y_s,Z_s)\mathrm{d}s - \int_t^\infty Z_s\mathrm{d}W_s, \qquad 0 \leq t \leq \infty, \tag{5}$$

then we have

$$\begin{array}{rl} & |Y_t|^p + \frac{p(p-1)}{2}\int_t^\infty |Y_s|^{p-2}1_{(Y_s\neq 0)}|Z_s|^2\mathrm{d}s \\ \leq & |\xi|^p + p\int_t^\infty |Y_s|^{p-1}\frac{Y_s}{|Y_s|}1_{(Y_s\neq 0)}g(s,Y_s,Z_s)\mathrm{d}s \\ - & p\int_t^\infty |Y_s|^{p-1}\frac{Y_s}{|Y_s|}1_{(Y_s\neq 0)}Z_s\mathrm{d}W_s. \end{array} \tag{6}$$

Proof Noting that

$$Y_t = Y_0 - \int_0^t g(s,Y_s,Z_s)\mathrm{d}s + \int_0^t Z_s\mathrm{d}W_s, \quad 0 \leq t \leq \infty,$$

then, together with (3), we obtain (6). □

3. Existence and uniqueness

In this section, we prove the existence and uniqueness theorem for L^p solutions of infinite time interval BSDEs which generalizes the result of [5] and give the corresponding comparison theorem.

Theorem 3.1 *Under assumptions (A.2)–(A.4), if* $\xi \in L^p(\Omega,\mathcal{F},P)$*, then BSDE (2) has a unique solution* $(Y,Z) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$.

In order to prove Theorem 3.1, we give an a priori estimate.

Lemma 3.1 *Suppose that (A.2) holds for g. Furthermore, each ϕ_i satisfies that*

$$E\left[\left(\int_0^\infty |\phi_i(s)|\mathrm{d}s\right)^p\right] < \infty.$$

Let $\xi_i \in L^p(\Omega, \mathcal{F}, P)$, $(Y^i, Z^i) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$ satisfy the following BSDEs:

$$Y_t^i = \xi_i + \int_t^\infty \left[g\left(s, Y_s^i, Z_s^i\right) + \phi_i(s)\right]\mathrm{d}s - \int_t^\infty Z_s^i \mathrm{d}W_s, \quad i = 1, 2.$$

Then

$$\begin{array}{rl} & E\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ \leq & C_p E\left[|\xi_1 - \xi_2|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right], \end{array}$$

where C_p is a positive constant depending only on p.

Proof It is easy to check that

$$\int_0^\infty \left(\left|g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right| + \left|Z_s^1 - Z_s^2\right|^2\right)\mathrm{d}s < \infty,$$

so applying Itô's formula to $\left(Y_s^1 - Y_s^2\right)^2$, we have

$$\begin{array}{rl} & \left|Y_0^1 - Y_0^2\right|^2 + \int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s \\ = & |\xi_1 - \xi_2|^2 + 2\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right)\mathrm{d}s \\ - & 2\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s. \end{array}$$

From the Lipschitz assumption (A.2) on g, we have

$$\begin{array}{rl} & 2\left(Y_s^1 - Y_s^2\right)\left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right) \\ \leq & 2\alpha(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\beta(s)\left|Y_s^1 - Y_s^2\right|\left|Z_s^1 - Z_s^2\right| \\ \leq & 2\alpha(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\beta^2(s)\left|Y_s^1 - Y_s^2\right|^2 + \frac{1}{2}\left|Z_s^1 - Z_s^2\right|^2 \\ \leq & 2\left(\alpha(s) + \beta^2(s)\right)\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^2 + \frac{1}{2}\left|Z_s^1 - Z_s^2\right|^2. \end{array}$$

It follows that

$$\begin{array}{rl} & \frac{1}{2}\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s \\ \leq & \left[1 + 2\left(\int_0^\infty \alpha(s)\mathrm{d}s + \int_0^\infty \beta^2(s)\mathrm{d}s\right)\right]\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^2 \\ + & 2\int_0^\infty \left|Y_s^1 - Y_s^2\right||\phi_1(s) - \phi_2(s)|\mathrm{d}s + 2\left|\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|. \end{array}$$

Since $2\int_0^\infty \left|Y_s^1 - Y_s^2\right||\phi_1(s) - \phi_2(s)|\mathrm{d}s \leq \sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^2 + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^2$, we have

$$\begin{array}{rl} & \int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s \\ \leq & 4\left(\left[1 + \left(\int_0^\infty \alpha(s)\mathrm{d}s + \int_0^\infty \beta^2(s)\mathrm{d}s\right)\right]\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^2\right) \\ + & 4\left(\left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^2 + \left|\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|\right). \end{array}$$

Using the fact that if $b,\ a_i \geq 0$ and $b \leq \sum_{i=1}^{n} a_i$, then $b^p \leq \sum_{i=1}^{n} a_i^p$ for any $p \in (0,1)$ (see, e.g., Kuang [18, page 132]), we have

$$\begin{aligned}\left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}} &\leq c_p\left(\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right)\\ &+ c_p\left(\left|\int_0^\infty (Y_s^1 - Y_s^2)(Z_s^1 - Z_s^2)\mathrm{d}W_s\right|^{\frac{p}{2}}\right),\end{aligned} \tag{7}$$

where c_p is a positive constant depending only on p. By the Burkholder-Davis-Gundy inequality (see, e.g., Barlow et al. [1, Table 4.1 page 162]), we get

$$\begin{aligned}c_pE\left[\left|\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|^{\frac{p}{2}}\right] &\leq d_pE\left[\left(\int_0^\infty \left|Y_s^1 - Y_s^2\right|^2\left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right]\\ &\leq d_pE\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^{\frac{p}{2}}\left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right]\end{aligned}$$

and thus

$$\begin{aligned}c_pE\left[\left|\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|^{\frac{p}{2}}\right] &\leq \tfrac{1}{2}E\left[\left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]\\ &+ \tfrac{d_p^2}{2}E\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p\right],\end{aligned} \tag{8}$$

where d_p is a positive constant depending only on p. From (7) and (8), we have

$$E\left[\left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \leq CE\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right], \tag{9}$$

where C is a positive constant depending only on p.

Now, we prove that

$$E\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p\right] \leq C^{'}E\left[|\xi_1 - \xi_2|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right], \tag{10}$$

where $C^{'}$ is a positive constant depending only on p. The proof of (10) is similar to that of Proposition 3.2 of Briand et al. [3]. Let us fix $\theta(t) := \alpha(t) + \frac{\beta^2(t)}{p-1}$ and define $\overline{\xi} := \mathrm{e}^{\int_0^\infty \theta(s)\mathrm{d}s}\xi$, $\overline{Y}_t^i := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s}Y_t^i$, $\overline{Z}_t^i := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s}Z_t^i$, $i = 1,2$, which solve the following BSDEs, respectively:

$$\overline{Y}_t^i = \overline{\xi}_i + \int_t^\infty \left[\overline{g}\left(s, \overline{Y}_s^i, \overline{Z}_s^i\right) + \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}\phi_i(s)\right]\mathrm{d}s - \int_t^\infty \overline{Z}_s^i\mathrm{d}W_s, \quad i = 1,2,$$

where $\overline{g}(t,y,z) := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s}g\left(t, \mathrm{e}^{-\int_0^t \theta(s)\mathrm{d}s}y, \mathrm{e}^{-\int_0^t \theta(s)\mathrm{d}s}z\right) - \theta(t)y$.

By Lemma 2.2, we can get the inequality

$$\begin{aligned}&\left|\overline{Y}_t^1 - \overline{Y}_t^2\right|^p + \tfrac{p(p-1)}{2}\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)}\left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\\ \leq\ &\left|\overline{\xi}_1 - \overline{\xi}_2\right|^p + p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1}\tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|}1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)}\left(\overline{g}\left(s, \overline{Y}_s^1, \overline{Z}_s^1\right) - \overline{g}\left(s, \overline{Y}_s^2, \overline{Z}_s^2\right)\right)\mathrm{d}s\\ +\ &p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1}\mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}|\phi_1(s) - \phi_2(s)|\mathrm{d}s\\ -\ &p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1}\tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|}1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)}\left(\overline{Z}_s^1 - \overline{Z}_s^2\right)\mathrm{d}W_s.\end{aligned} \tag{11}$$

From the Lipschitz assumption (A.2) on g and with the help of

$$\theta(t) := \alpha(t) + \frac{\beta^2(t)}{p-1},$$

$$\overline{Y}_t^i := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} Y_t^i, \quad \overline{Z}_t^i := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} Z_t^i, \quad i = 1, 2$$

and

$$\overline{g}(t, y, z) := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} g\left(t, \mathrm{e}^{-\int_0^t \theta(s)\mathrm{d}s} y, \mathrm{e}^{-\int_0^t \theta(s)\mathrm{d}s} z\right) - \theta(t) y,$$

we have

$$\begin{aligned}
& p \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{|\overline{Y}_s^1 - \overline{Y}_s^2|} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \left(\overline{g}\left(s, \overline{Y}_s^1, \overline{Z}_s^1\right) - \overline{g}\left(s, \overline{Y}_s^2, \overline{Z}_s^2\right)\right) \\
= & \; p \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{|\overline{Y}_s^1 - \overline{Y}_s^2|} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} \left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right) \\
- & \; p \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{|\overline{Y}_s^1 - \overline{Y}_s^2|} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \theta(s) \left(\overline{Y}_s^1 - \overline{Y}_s^2\right) \\
\leq & \; p \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} \left|g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right| \\
- & \; p \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{|\overline{Y}_s^1 - \overline{Y}_s^2|} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \theta(s) \left(\overline{Y}_s^1 - \overline{Y}_s^2\right) \\
= & \; p\alpha(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} \left|Y_s^1 - Y_s^2\right| \\
+ & \; p\beta(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} \left|Z_s^1 - Z_s^2\right| - p\theta(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p \\
= & \; p\alpha(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p + p\beta(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right| - p\theta(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p \\
= & \; p\beta(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right| - \tfrac{p\beta^2(s)}{p-1} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p .
\end{aligned} \tag{12}$$

Noting that

$$\begin{aligned}
& p\beta(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right| \\
= & \; p\beta(s) \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{\frac{p}{2}} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{\frac{p}{2}-1} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right| \\
\leq & \; \tfrac{p\beta^2(s)}{p-1} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p + \tfrac{p(p-1)}{4} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2,
\end{aligned}$$

(where the inequality comes from the fact that if $a,\ b \geq 0$, then $ab \leq a^2 + \frac{b^2}{4}$), we have

$$\begin{aligned}
& p \int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{|\overline{Y}_s^1 - \overline{Y}_s^2|} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \left(\overline{g}\left(s, \overline{Y}_s^1, \overline{Z}_s^1\right) - \overline{g}\left(s, \overline{Y}_s^2, \overline{Z}_s^2\right)\right) \mathrm{d}s \\
\leq & \; \tfrac{p(p-1)}{4} \int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s.
\end{aligned} \tag{13}$$

Thus from (11) and (13), we obtain the following inequality:

$$\begin{aligned}
& \left|\overline{Y}_t^1 - \overline{Y}_t^2\right|^p + \tfrac{p(p-1)}{4} \int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s \\
\leq & \; \left|\overline{\xi}_1 - \overline{\xi}_2\right|^p + p \int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)| \mathrm{d}s \\
- & \; p \int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{|\overline{Y}_s^1 - \overline{Y}_s^2|} 1_{(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0)} \left(\overline{Z}_s^1 - \overline{Z}_s^2\right) \mathrm{d}W_s.
\end{aligned} \tag{14}$$

Denote

$$M_t := \int_0^t \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \frac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left(\overline{Z}_s^1 - \overline{Z}_s^2\right) \mathrm{d}W_s.$$

By the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]) and Young's inequality (i.e. $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, $a \geq 0$, $b \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, see, e.g., Kuang [18, page 136]), we have

$$\begin{aligned}
E[|M_t|] &\leq E\left[\left(\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{2p-2} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq E\left[\sup_{s \geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \left(\int_0^\infty \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq \tfrac{p-1}{p} E\left[\sup_{s \geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p\right] + \tfrac{1}{p} E\left[\left(\int_0^\infty \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\
&< \infty.
\end{aligned}$$

It then follows that $\{M_t\}_{t \geq 0}$ is a martingale. For notational simplification, let

$$X := \int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s.$$

Coming back to inequality (14), we get both

$$\frac{p(p-1)}{4} E[X] \leq E\left[\left|\overline{\xi}_1 - \overline{\xi}_2\right|^p\right] + pE\left[\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)| \mathrm{d}s\right] \tag{15}$$

and

$$\begin{aligned}
& E\left[\sup_{s \geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p\right] \\
\leq\ & E\left[\left|\overline{\xi}_1 - \overline{\xi}_2\right|^p + p \int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)| \mathrm{d}s\right] \\
+\ & D_p E[|M_\infty|],
\end{aligned} \tag{16}$$

where D_p is a positive constant depending only on p. Applying the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]) again, we have

$$\begin{aligned}
D_p E[|M_\infty|] &\leq D_p E\left[\left(\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{2p-2} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq D_p E\left[\sup_{s \geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{\frac{p}{2}} \left(\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq \tfrac{1}{2} E\left[\sup_{s \geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p\right] + \tfrac{D_p^2}{2} E[X].
\end{aligned}$$

It then follows from (15) and (16) that

$$E\left[\sup_{s \geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p\right] \leq K_p E\left[\left|\overline{\xi}_1 - \overline{\xi}_2\right|^p + p \int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)| \mathrm{d}s\right], \tag{17}$$

where K_p is a positive constant depending only on p. Applying once again Young's inequality, we get

$$\begin{aligned}
& pK_pE\left[\int_0^\infty \left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}|\phi_1(s)-\phi_2(s)|\mathrm{d}s\right] \\
\leq\ & pK_pE\left[\sup_{s\geq 0}\left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p-1} \int_0^\infty \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}|\phi_1(s)-\phi_2(s)|\mathrm{d}s\right] \\
\leq\ & \tfrac{1}{2}E\left[\sup_{s\geq 0}\left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p}\right]+M_pE\left[\left(\int_0^\infty \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}|\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right] \\
\leq\ & \tfrac{1}{2}E\left[\sup_{s\geq 0}\left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p}\right]+M_p\left(\mathrm{e}^{\int_0^\infty \theta(s)\mathrm{d}s}\right)^p E\left[\left(\int_0^\infty |\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right],
\end{aligned}$$

where M_p is a positive constant depending only on p. From this, we deduce that

$$E\left[\sup_{s\geq 0}\left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p}\right]\leq C^{'}E\left[\left|\overline{\xi}_1-\overline{\xi}_2\right|^p+\left(\int_0^\infty |\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right], \tag{18}$$

where $C^{'}$ is a positive constant depending only on p.

Combining (9) with (18), we get

$$\begin{aligned}
& E\left[\sup_{s\geq 0}\left|Y_s^1-Y_s^2\right|^{p}+\left(\int_0^\infty \left|Z_s^1-Z_s^2\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
\leq\ & C_pE\left[|\xi_1-\xi_2|^p+\left(\int_0^\infty |\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right],
\end{aligned}$$

where C_p is a positive constant depending only on p. The proof of Lemma 3.1 is complete. □

Lemma 3.2 *([5]) Let $\xi\in L^2(\Omega,\mathcal{F},P)$ be given. Suppose that (A.1) and (A.2) hold for g, then BSDE*

$$Y_t=\xi+\int_t^\infty g\left(s,Y_s,Z_s\right)\mathrm{d}s-\int_t^\infty Z_s\mathrm{d}W_s \tag{19}$$

has a unique solution $(Y,Z)\in\mathcal{S}^2(\mathbb{R})\times\mathcal{L}^2(\mathbb{R})$.

Proof of Theorem 3.1. We prove this theorem in two steps.

Step 1. We prove the existence and uniqueness to BSDE (19). Let $\xi^n:=(\xi\wedge n)\vee(-n)$ and $g_n(t,y,z):=g(t,y,z)-g(t,0,0)+f_n(g(t,0,0))$, where $f_n(g(t,0,0)):=\frac{g(t,0,0)n}{|g(t,0,0)|\vee n}$, if $t\leq T$; $f_n(g(t,0,0))=g(t,0,0)$, if $t>T$. It is easy to check that for each n, the function g_n satisfies (A.1) and (A.2). Then by Lemma 3.2, BSDE

$$Y_t^n=\xi^n+\int_t^\infty g_n\left(s,Y_s^n,Z_s^n\right)\mathrm{d}s-\int_t^\infty Z_s^n\mathrm{d}W_s$$

has a unique solution $(Y^n,Z^n)\in\mathcal{S}^2(\mathbb{R})\times\mathcal{L}^2(\mathbb{R})$. Using Lemma 3.1, we have

$$\begin{aligned}
& E\left[\sup_{t\geq 0}\left|Y_t^{n+m}-Y_t^n\right|^{p}+\left(\int_0^\infty \left|Z_s^{n+m}-Z_s^n\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
\leq\ & C_pE\left[|\xi^{n+m}-\xi^n|^p+\left(\int_0^\infty |f_{n+m}(g(s,0,0))-f_n(g(s,0,0))|\,\mathrm{d}s\right)^p\right].
\end{aligned}$$

The right-hand side of the above inequality clearly tends to 0, as $n \to \infty$, uniformly in m, so we have a Cauchy sequence and the limit is a solution to BSDE (19). Let us consider (Y, Z) and $(Y^{'}, Z^{'})$ to be two solutions to BSDE (19). Using Lemma 3.1 again, we get immediately $(Y, Z) = (Y^{'}, Z^{'})$.

Step 2. Let $\hat{\xi} := \xi + V_\infty$ and $\hat{Y}_t := Y_t + V_t$, then BSDE (2) can be rewritten as

$$\hat{Y}_t = \hat{\xi} + \int_t^\infty \hat{g}\left(s, \hat{Y}_s, Z_s\right) \mathrm{d}s - \int_t^\infty Z_s \mathrm{d}W_s, \tag{20}$$

where $\hat{g}(t, y, z) := g\,(t, y - V_t, z)$. It is easy to check that $\hat{g}(t, y, z)$ satisfies (A.2), (A.3) and $\hat{\xi} \in L^p(\Omega, \mathcal{F}, P)$. By Step 1, there exists a unique pair $\left(\hat{Y}, Z\right)$ of adapted processes in $\mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$ solving BSDE (20). Using the fact $|Y_t|^p \le 2^p(|\hat{Y}_t|^p + |V_t|^p)$, we have $(Y, Z) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$. The proof of Theorem 3.1 is complete.

Remark 3.1 If $g(t, 0, 0) \equiv 0$, then by Theorem 3.1, we have: Under assumptions (A.2) and (A.4), for each given $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, BSDE (2) has a unique solution $(Y, Z) \in \mathcal{S}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$.

Example 3.1 *Suppose that* $1 < p < 2$. *Consider the BSDE:*

$$Y_t = \exp\left(\frac{W_1^2}{2p} - W_1\right) 1_{(W_1 \ge p)} + \int_t^\infty \frac{1}{(1+s)^2}(Y_s + Z_s)\mathrm{d}s - \int_t^\infty Z_s \mathrm{d}W_s. \tag{21}$$

For notational simplification, let $\xi := \exp\left(\frac{W_1^2}{2p} - W_1\right) 1_{(W_1 \ge p)}$, $g(t, y, z) := \frac{1}{(1+t)^2}(y + z)$, $\alpha(t) := \frac{1}{(1+t)^2}$, $\beta(t) := \frac{1}{(1+t)^2}$. *Obviously,* g *satisfies (A.2) and (A.3). On the other hand,*

$$E[|\xi|^p] = \int_p^\infty \exp\left(\frac{x^2}{2} - px\right) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \mathrm{d}x = \frac{1}{\sqrt{2\pi}p} \mathrm{e}^{-p^2} < \infty,$$

and

$$E[|\xi|^2] = \infty.$$

Hence, $\xi \in L^p(\Omega, \mathcal{F}, P)$, $\xi \notin L^2(\Omega, \mathcal{F}, P)$. *But by Theorem 3.1, we have: BSDE (21) has a unique solution* $(Y, Z) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$.

The following comparison theorem is very useful. Since the proof is very similar to that of Theorem 2.2 in [13], we omit it.

Theorem 3.2 *(Comparison Theorem) We make the same assumptions as in Theorem 3.1. Let* $\left(\overline{Y}, \overline{Z}\right)$ *be the solution of the BSDE*

$$\overline{Y}_t = \overline{\xi} + \int_t^\infty \overline{g}\left(s, \overline{Y}_s, \overline{Z}_s\right) \mathrm{d}s + \overline{V}_\infty - \overline{V}_t - \int_t^\infty \overline{Z}_s \mathrm{d}W_s,$$

where $\overline{g}(t, y, z)$ *satisfies (A.2) and (A.3),* $\overline{V}_t$ *satisfies (A.4) and* $\overline{\xi} \in L^p(\Omega, \mathcal{F}, P)$. *If we suppose that*

$$\hat{\xi} := \xi - \overline{\xi} \ge 0, \quad \hat{g}_t := g\left(t, \overline{Y}_t, \overline{Z}_t\right) - \overline{g}\left(t, \overline{Y}_t, \overline{Z}_t\right) \ge 0, \quad a.s.,$$
$$\hat{V}_t := V_t - \overline{V}_t \quad \text{is an RCLL increasing process,}$$

then

$$Y_t \geq \overline{Y}_t, \quad a.s., \quad \forall t \in [0,\infty).$$

Moreover, if $P\left(\hat{\xi} > 0\right) > 0$*, then* $P\left(Y_t > \overline{Y}_t\right) > 0$*, for all* $t \geq 0$*. In particular,* $Y_0 > \overline{Y}_0$*.*

4. Generalized g-expectation and generalized g-martingale

In this section, we make an additional assumption on the function g:

(A.5) $g(\cdot, y, 0) \equiv 0$, $\forall y \in \mathbb{R}$.

For any given g, the solution (Y,Z) of BSDE (19) depends on terminal value ξ. Referring to Definition 36.1 in [27] or Definition 3.1 in [14], now we introduce the definitions of generalized g-expectation and generalized conditional g-expectation via the solution of BSDE (19).

Definition 4.1 *(Generalized g-expectation) Suppose g satisfies (A.2) and (A.5). For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, let (Y,Z) be the solution of BSDE (19). Consider the mapping $\mathcal{E}_g[\cdot] : \mathcal{L}(\Omega, \mathcal{F}, P) \mapsto \mathbb{R}$ denoted by $\mathcal{E}_g[\xi] := Y_0$. We call $\mathcal{E}_g[\xi]$ generalized g-expectation of ξ.*

Definition 4.2 *(Generalized conditional g-expectation) Suppose g satisfies (A.2) and (A.5). Generalized conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by*

$$\mathcal{E}_g[\xi|\mathcal{F}_t] := Y_t.$$

Generalized g-expectation has the following property.

Proposition 4.1 *$\mathcal{E}_g[\xi|\mathcal{F}_t]$ is the unique random variable η in $\mathcal{L}(\Omega, \mathcal{F}_t, P)$ such that*

$$\mathcal{E}_g[1_A \xi] = \mathcal{E}_g[1_A \eta], \qquad \forall A \in \mathcal{F}_t.$$

By Theorem 3.2 and (A.5), we can prove Proposition 4.1 by using the same method as that of Proposition 36.4 in [27], so we omit the proof.

The following proposition will tell us that generalized conditional g-expectations that we introduced meet some basic properties of Peng's conditional g-expectations.

Proposition 4.2 *Suppose $\xi, \quad \xi_1, \quad \xi_2 \in \mathcal{L}(\Omega, \mathcal{F}, P)$, then*

(i) *If ξ is $\mathcal{F}_t$-measurable, then $\mathcal{E}_g[\xi|\mathcal{F}_t] = \xi$;*

(ii) *For all stopping times τ and σ, $\mathcal{E}_g[\mathcal{E}_g[\xi|\mathcal{F}_\tau]|\mathcal{F}_\sigma] = \mathcal{E}_g[\xi|\mathcal{F}_{\tau\wedge\sigma}]$;*

(iii) *If $\xi_1 \geq \xi_2$ a.s., then $\mathcal{E}_g[\xi_1|\mathcal{F}_t] \geq \mathcal{E}_g[\xi_2|\mathcal{F}_t]$; if, moreover, $P(\xi_1 > \xi_2) > 0$, then*

$$P\left(\mathcal{E}_g[\xi_1|\mathcal{F}_t] > \mathcal{E}_g[\xi_2|\mathcal{F}_t]\right) > 0;$$

(iv) *For each $B \in \mathcal{F}_t$, $\mathcal{E}_g[1_B \xi|\mathcal{F}_t] = 1_B \mathcal{E}_g[\xi|\mathcal{F}_t]$;*

(v) *If g does not depend on y, then for any $(\xi, \eta) \in \mathcal{L}(\Omega, \mathcal{F}, P) \times \mathcal{L}(\Omega, \mathcal{F}_t, P)$,*

$$\mathcal{E}_g[\xi + \eta|\mathcal{F}_t] = \mathcal{E}_g[\xi|\mathcal{F}_t] + \eta.$$

By Theorem 3.2 and using the similar arguments as that of Lemma 36.6 in [27] and Lemma 4.2 in [2], we can prove Proposition 4.2.

Now we shall prove the stability theorem of generalized g-expectations.

Theorem 4.1 *(Stability Theorem) Suppose g satisfies (A.2) and (A.5). For $\xi, \eta_n \in \mathcal{L}(\Omega,\mathcal{F},P)$, where $n=1,2,\cdots$, if $E[|\xi-\eta_n|^p|\mathcal{F}_t]\to 0$, a.s., $t\in[0,\infty)$, then*

$$\lim_{n\to\infty}\mathcal{E}_g[\eta_n|\mathcal{F}_t]=\mathcal{E}_g[\xi|\mathcal{F}_t], \quad a.s., \ t\in[0,\infty).$$

Proof From Theorem 3.1, we know that

$$\begin{array}{l}\mathcal{E}_g[\eta_n|\mathcal{F}_t]=\eta_n+\int_t^\infty g\left(s,\mathcal{E}_g[\eta_n|\mathcal{F}_s],Z_s^n\right)\mathrm{d}s-\int_t^\infty Z_s^n\mathrm{d}W_s, \quad n=1,2,\cdots,\\ \mathcal{E}_g[\xi|\mathcal{F}_t]=\xi+\int_t^\infty g\left(s,\mathcal{E}_g[\xi|\mathcal{F}_s],Z_s\right)\mathrm{d}s-\int_t^\infty Z_s\mathrm{d}W_s.\end{array}$$

Then

$$\begin{array}{rcl}\mathcal{E}_g[\xi|\mathcal{F}_t]-\mathcal{E}_g[\eta_n|\mathcal{F}_t] & = & \xi-\eta_n+\int_t^\infty\left[a_s\left(\mathcal{E}_g[\xi|\mathcal{F}_s]-\mathcal{E}_g[\eta_n|\mathcal{F}_s]\right)+b_s\left(Z_s-Z_s^n\right)\right]\mathrm{d}s\\ & - & \int_t^\infty(Z_s-Z_s^n)\mathrm{d}W_s,\end{array} \tag{22}$$

where

$$\begin{array}{l}a_s:=\frac{g(s,\mathcal{E}_g[\xi|\mathcal{F}_s],Z_s)-g(s,\mathcal{E}_g[\eta_n|\mathcal{F}_s],Z_s)}{\mathcal{E}_g[\xi|\mathcal{F}_s]-\mathcal{E}_g[\eta_n|\mathcal{F}_s]}1_{(\mathcal{E}_g[\xi|\mathcal{F}_s]-\mathcal{E}_g[\eta_n|\mathcal{F}_s]\neq 0)},\\ b_s:=\frac{g(s,\mathcal{E}_g[\eta_n|\mathcal{F}_s],Z_s)-g(s,\mathcal{E}_g[\eta_n|\mathcal{F}_s],Z_s^n)}{Z_s-Z_s^n}1_{(Z_s-Z_s^n\neq 0)},\end{array}$$

which imply $|a_t|\le\alpha(t)$, $|b_t|\le\beta(t)$.

Relation (22) can be rewritten as follows:

$$\mathcal{E}_g[\xi|\mathcal{F}_t]-\mathcal{E}_g[\eta_n|\mathcal{F}_t]=\xi-\eta_n+\int_t^\infty a_s\left(\mathcal{E}_g[\xi|\mathcal{F}_s]-\mathcal{E}_g[\eta_n|\mathcal{F}_s]\right)\mathrm{d}s-\int_t^\infty\left(Z_s-Z_s^n\right)\mathrm{d}\overline{W}_s, \tag{23}$$

where $\overline{W}_t=W_t-\int_0^t b_s\mathrm{d}s$. By the Girsanov theorem, we know that $\left(\overline{W}_t\right)_{t\ge 0}$ is Q^b-Brownian motion, where $\frac{\mathrm{d}Q^b}{\mathrm{d}P}=\mathrm{e}^{-\frac{1}{2}\int_0^\infty|b_s|^2\mathrm{d}s+\int_0^\infty b_s\mathrm{d}W_s}$.

Solving (23), we obtain

$$\mathcal{E}_g[\xi|\mathcal{F}_t]-\mathcal{E}_g[\eta_n|\mathcal{F}_t]=(\xi-\eta_n)\mathrm{e}^{\int_t^\infty a_s\mathrm{d}s}-\int_t^\infty\left(Z_s-Z_s^n\right)\mathrm{e}^{\int_t^s a_r\mathrm{d}r}\mathrm{d}\overline{W}_s. \tag{24}$$

By the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]), Hölder's inequality and noting the fact that

$$E\left[\mathrm{e}^{-\frac{1}{2}\int_0^\infty|b_s|^2\mathrm{d}s+\int_0^\infty b_s\mathrm{d}W_s}\right]=1$$

and

$$E\left[\mathrm{e}^{-\frac{1}{2}\int_0^\infty|qb_s|^2\mathrm{d}s+\int_0^\infty qb_s\mathrm{d}W_s}\right]=1,$$

we have

$$\begin{array}{rl} & E_{Q^b}\left[\left|\int_0^t\left(Z_s-Z_s^n\right)\mathrm{e}^{\int_0^s a_r\mathrm{d}r}\mathrm{d}\overline{W}_s\right|\right]\\ \le & \mathrm{e}^{\int_0^\infty\alpha(t)\mathrm{d}t}E_{Q^b}\left[\left(\int_0^\infty|Z_s-Z_s^n|^2\mathrm{d}s\right)^{\frac{1}{2}}\right]\\ \le & \mathrm{e}^{\int_0^\infty\alpha(t)\mathrm{d}t}\left(E\left[\left(\int_0^\infty|Z_s-Z_s^n|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\left(E\left[\left(\frac{\mathrm{d}Q^b}{\mathrm{d}P}\right)^q\right]\right)^{\frac{1}{q}}\\ \le & \mathrm{e}^{\left[\frac{1}{2}(q-1)\int_0^\infty\beta^2(t)\mathrm{d}t+\int_0^\infty\alpha(t)\mathrm{d}t\right]}\left(E\left[\left(\int_0^\infty|Z_s-Z_s^n|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\\ < & \infty,\end{array}$$

where $\frac{1}{p}+\frac{1}{q}=1$. It then follows that $\left(\int_0^t (Z_s - Z_s^n)\, e^{\int_0^s a_r dr} d\overline{W}_s\right)_{t\geq 0}$ is a martingale with respect to Q^b. Hence $E_{Q^b}\left[\int_0^t (Z_s - Z_s^n)\, e^{\int_0^s a_r dr} d\overline{W}_s\right] = 0$. Taking conditional expectation $E_{Q^b}[\cdot|\mathcal{F}_t]$ on both sides of (24), we have

$$\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta_n|\mathcal{F}_t] = E_{Q^b}\left[(\xi-\eta_n)e^{\int_t^\infty a_s ds}|\mathcal{F}_t\right].$$

Note that $|a_t|\leq \alpha(t)$ and hence

$$|\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta_n|\mathcal{F}_t]| \leq e^{\int_0^\infty \alpha(t)dt} E_{Q^b}[|\xi-\eta_n||\mathcal{F}_t].$$

By Hölder's inequality, we obtain

$$E_{Q^b}[|\xi-\eta_n||\mathcal{F}_t] = \frac{E\left[|\xi-\eta_n|\frac{dQ^b}{dP}|\mathcal{F}_t\right]}{E\left[\frac{dQ^b}{dP}|\mathcal{F}_t\right]} \leq \frac{\left(E\left[|\xi-\eta_n|^p|\mathcal{F}_t\right]\right)^{\frac{1}{p}}\left(E\left[(\frac{dQ^b}{dP})^q|\mathcal{F}_t\right]\right)^{\frac{1}{q}}}{E\left[\frac{dQ^b}{dP}|\mathcal{F}_t\right]}.$$

Since $\left(e^{-\frac{1}{2}\int_0^t |b_s|^2 ds + \int_0^t b_s dW_s}\right)_{t\geq 0}$ and $\left(e^{-\frac{1}{2}\int_0^t |qb_s|^2 ds + \int_0^t qb_s dW_s}\right)_{t\geq 0}$ are both martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$, hence

$$\frac{\left(E\left[(\frac{dQ^b}{dP})^q|\mathcal{F}_t\right]\right)^{\frac{1}{q}}}{E\left[\frac{dQ^b}{dP}|\mathcal{F}_t\right]} \leq e^{\frac{1}{2}(q-1)\int_0^\infty \beta^2(t)dt}\frac{\left(e^{-\frac{1}{2}\int_0^t |qb_s|^2 ds + \int_0^t qb_s dW_s}\right)^{\frac{1}{q}}}{e^{-\frac{1}{2}\int_0^t |b_s|^2 ds + \int_0^t b_s dW_s}} \leq e^{\frac{1}{2}(q-1)\int_0^\infty \beta^2(t)dt}.$$

Thus for all $t\in[0,\infty)$,

$$|\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta_n|\mathcal{F}_t]| \leq e^{\left[\frac{1}{2}(q-1)\int_0^\infty \beta^2(t)dt + \int_0^\infty \alpha(t)dt\right]}\left(E[|\xi-\eta_n|^p|\mathcal{F}_t]\right)^{\frac{1}{p}}. \tag{25}$$

Noting that $E[|\xi-\eta_n|^p|\mathcal{F}_t]\to 0$, as $n\to\infty$, $t\in[0,\infty)$, then

$$|\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta_n|\mathcal{F}_t]| \to 0, \quad \text{as} \quad n\to\infty.$$

The proof of Theorem 4.1 is complete. □

Remark 4.1 (i) *In Theorem 4.1, if we replace (A.5) by (A.3), the following result* $\lim_{n\to\infty} Y_t^n = Y_t$, *a.s.,* $t\in[0,\infty)$ *holds.*

(ii) *For any* $\xi\in\mathcal{L}(\Omega,\mathcal{F},P)$, *let* $\xi^n := (\xi\wedge n)\vee(-n)$, $n=1,2,\cdots$, *then by Theorem 4.1, we have:*

$$\lim_{n\to\infty}\mathcal{E}_g[\xi^n|\mathcal{F}_t] = \mathcal{E}_g[\xi|\mathcal{F}_t], \quad a.s., \quad \forall t\in[0,\infty).$$

(iii) *By the proof of Theorem 4.1, we have: if* $\xi\in L^p(\Omega,\mathcal{F},P)$, *then there exists a constant* $C>0$ *such that* $\mathcal{E}_g[|\xi||\mathcal{F}_t]\leq C(E[|\xi|^p|\mathcal{F}_t])^{\frac{1}{p}}$, $\forall t\in[0,\infty)$.

At the end of the paper, we introduce the definition of generalized g*-martingale (resp. generalized* g*-supermartingale, generalized* g*-submartingale).*

Definition 4.3 *Suppose g satisfies (A.2) and (A.5). A process $(X_t)_{t\geq 0}$ satisfying that for each t, $X_t \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$ is called a generalized g-martingale (resp. generalized g-supermartingale, generalized g-submartingale), if for any t and s satisfying $t \leq s$,*

$$\mathcal{E}_g[X_s|\mathcal{F}_t] = X_t \quad (resp. \leq X_t, \geq X_t), \quad a.s.$$

Example 4.1 *Suppose that $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $(A_t)_{t\geq 0}$ is an RCLL increasing process with $(A_t)_{t\geq 0} \in \mathcal{S}^2(\mathbb{R})$. Consider the BSDE:*

$$Y_t = \xi + \int_t^{\infty} \frac{1}{(1+s)^2}|Z_s|\mathrm{d}s + A_\infty - A_t - \int_t^{\infty} Z_s \mathrm{d}W_s. \tag{26}$$

Let $g(t, y, z) := \frac{1}{(1+t)^2}|z|$. Obviously, g satisfies (A.2) and (A.5). By Theorem 3.2, for any t and s satisfying $t \leq s$, $\mathcal{E}_g[Y_s|\mathcal{F}_t] \leq Y_t$, a.s.. Thus $(Y_t)_{t\geq 0}$ is a generalized g-supermartingale.

Acknowledgments

The author would like to thank the anonymous referee for his/her constructive suggestions and comments on the previous version of this paper. This work was supported partly by the National Natural Science Foundation of China (No. 11171179), the Research Foundation for the Doctoral Program of Higher Education (No. 20093705110002) and the Natural Science Foundation of Shandong Province (No. ZR2012AQ009).

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