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A class of 3-dimensional almost cosymplectic manifolds

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Abstract: The main interest of the present paper is to classify the almost cosymplectic 3-manifolds that satisfy \( \| \text{grad} \lambda \| = \text{const.}(\neq 0) \) and \( \nabla \xi h = 2ah\phi \).

Key words: Almost cosymplectic manifold, cosymplectic manifold

1. Preliminaries

Let \( M \) be an almost contact metric manifold and let \( (\phi, \xi, \eta, g) \) be its almost contact metric structure. Thus \( M \) is a \((2n + 1)\)-dimensional differentiable manifold and \( \phi \) is a \((1, 1)\) tensor field, \( \xi \) is a vector field, and \( \eta \) is a 1-form on \( M \), such that

\[
\phi X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi) \tag{1}
\]

\[
\phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{2}
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}
\]

for any vector fields \( X, Y \) on \( M \).

The fundamental 2-form \( \Phi \) of an almost contact metric manifold \( (M, \phi, \xi, \eta, g) \) is defined by

\[
\Phi(X, Y) = g(X, \phi Y), \tag{4}
\]

for any vector fields \( X, Y \) on \( M \), and this form satisfies \( \eta \wedge \Phi^n \neq 0 \). \( M \) is said to be almost cosymplectic if the forms \( \eta \) and \( \Phi \) are closed, that is, \( d\eta = 0 \) and \( d\Phi = 0 \).

The theory of an almost cosymplectic manifold was introduced by Goldberg and Yano in [9]. The products of almost Kaehler manifolds and the real \( \mathbb{R} \) line or the circle \( S^1 \) are the simplest examples of almost cosymplectic manifolds. Topological and geometrical properties of almost cosymplectic manifolds have been studied by many mathematicians (see [4], [11], [5], [9], [15], and [18]).

For \( M \), define \((1, 1)\)-tensor fields \( \tilde{A} \) and \( h \) by ([7],[8],[15],[16])

\[
\tilde{A} X = -\nabla_X \xi, \tag{5}
\]

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\[ h = \frac{1}{2} \mathcal{L}_\xi \phi, \]  

where \( \mathcal{L} \) indicates the Lie differentiation operator and \( \nabla \) is the Levi-Civita connection determined by \( g \). The tensors \( \tilde{A} \) and \( h \) are related by 
\[ h = \tilde{A} \phi, \quad \tilde{A} = \phi h. \]  

The main algebraic properties of \( \tilde{A} \) and \( h \) are the following:
\[ g(\tilde{A} X, Y) = g(\tilde{A} Y, X), \quad \tilde{A} \phi + \phi \tilde{A} = 0, \quad \tilde{A} \xi = 0, \quad \eta \circ \tilde{A} = 0, \]
\[ g(hX, Y) = g(hY, X), \quad h\phi + \phi h = 0, \quad h\tilde{A} + \tilde{A} h = 0, \quad h\xi = 0, \quad \eta \circ h = 0. \]

The curvature tensor \( R \) of \( M \) is given by 
\[ R(X, Y)Z = [rX, rY]Z - r[X, Y]Z \]  
and the Ricci tensor \( Ric \) of \( M \) are defined by 
\[ Ric(X, Y) = Tr X \rightleftharpoons R(X, Y)Z \]  
of any vector field \( X, Y \) and \( Z \).

In [6], Dacko and Olszak proved the existence of a new class of almost cosymplectic manifolds, which is called \((\kappa, \mu, v)\)-spaces. This means that the curvature tensor \( R \) satisfies the condition 
\[ R(X, Y) \xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \]
\[ + \nu(\eta(Y)\phi h X - \eta(X)\phi h Y), \]  
where \( \kappa, \mu, \nu \) are smooth functions. Contact metric manifolds fulfilling Eq. (8) were investigated in [2], [1], [3], and [12].

This work was inspired by [14] and [13]. We carry on those studies to the 3-dimensional almost cosymplectic manifolds in this paper. The purpose of the present paper is to give a new local classification of 3-dimensional almost cosymplectic manifolds under some conditions. The paper is organized in the following way. Section 2 is devoted to some lemmas related to 3-dimensional almost cosymplectic manifolds for later use. In Section 3, we give our main theorem.

All manifolds considered in this paper are assumed to be connected and of class \( C^\infty \).

2. Three-dimensional almost cosymplectic manifolds

Now we shall give some essential Lemmas and notations.

Lemma 2.1 [10] Let \( M \) be a smooth manifold \( f : M \rightarrow \mathbb{R} \) be a smooth real function. Let \( V_1 \) and \( V_2 \) be open sets of \( M \) defined by 
\[ V_1 = \{ m \in M \mid f(m) \neq 0 \text{ in a neighborhood of } m \}, \]
\[ V_2 = \{ m \in M \mid f(m) = 0 \text{ in a neighborhood of } m \}. \]

Then \( V_1 \cup V_2 \) is open and dense in \( M \).

Let \((M, \phi, \xi, \eta, g)\) be an almost cosymplectic 3-manifold. Let 
\[ U = \{ p \in M \mid h(p) \neq 0 \text{ in a neighborhood of } p \} \subset M, \]
\[ U_0 = \{ p \in M \mid h(p) = 0 \text{ in a neighborhood of } p \} \subset M. \]
be open sets of $M$. Using Lemma 2.1, we can say that $U \cup U_0$ is an open and dense subset of $M$, and so any property satisfied in $U_0 \cup U$ is also satisfied in $M$. For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of $h$ in a neighborhood of $p$ (this we call a $\phi$-basis).

On $U$, we put $he = \lambda e$, $h\phi e = -\lambda \phi e$, where $\lambda$ is a nonvanishing smooth function assumed to be positive.

**Lemma 2.2** [17] On the open set $U$ we have

\begin{align*}
\nabla_\xi e & = -a \phi e, & \nabla_\phi e = be, & \nabla_{\phi e} e = -c \phi e + \lambda \xi, & (9) \\
\nabla_\xi \phi e & = ae, & \nabla_\phi \phi e = -be + \lambda \xi, & \nabla_{\phi e} \phi e = ce, & (10) \\
\nabla_\xi \xi & = 0, & \nabla_\phi \xi = -\lambda \phi e, & \nabla_{\phi e} \xi = -\lambda e, & (11) \\
\nabla_\xi h & = 2ah \phi + \xi(\lambda) s, & (12)
\end{align*}

where $a$ is a smooth function,

\begin{align*}
b & = \frac{1}{2\lambda}((\phi e)(\lambda) + A) \text{ with } A = \sigma(e) = \Ric(e, \xi), & (13) \\
c & = \frac{1}{2\lambda} (e(\lambda) + B) \text{ with } B = \sigma(\phi e) = \Ric(\phi e, \xi), & (14)
\end{align*}

and $s$ is the type $(1,1)$ tensor field defined by $s\xi = 0$, $se = e$, and $s\phi e = -\phi e$, and $\Ric$ is Ricci tensor field.

By Lemma 2.2, we can prove that

\begin{align*}
[e, \phi e] & = \nabla_\phi \phi e - \nabla_{\phi e} e = -be + c \phi e, & (15) \\
[e, \xi] & = \nabla_\phi \xi - \nabla_\xi e = (a - \lambda) \phi e, & (16) \\
[\phi e, \xi] & = \nabla_{\phi e} \xi - \nabla_\xi \phi e = -(a + \lambda) e. & (17)
\end{align*}

If we adapt Theorem 7 of [17] to a 3-dimensional almost cosymplectic manifolds, we get the following:

**Lemma 2.3** [17] Let $(M, \phi, \xi, \eta, g)$ be a 3-dimensional almost cosymplectic manifold. If $\sigma \equiv 0$, then the $(\kappa, \mu, \nu)$-structure always exists on every open and dense subset of $M$. This means that the Riemannian curvature tensor $R$ of $M$ satisfies

\begin{align*}
R(X, Y) \xi & = -\lambda^2(\eta(Y)X - \eta(X)Y) \\
& + 2a(\eta(Y)hX - \eta(X)hY) \\
& + \frac{\xi(\lambda)}{\lambda}(\eta(Y)\phi hX - \eta(X)\phi hY),
\end{align*}

for all vector fields $X$ and $Y$ on $M$.

### 3. Main theorem and proof

In this section, we will give our main theorem and prove it.

**Theorem 3.1** (Main theorem) Let $M(\phi, \xi, \eta, g)$ be a 3-dimensional almost cosymplectic manifold with $\|\text{grad } \lambda\| = 1$ and $\nabla_\xi h = 2ah \phi$. Then at any point $p \in M$ there exists a chart $(U, (x, y, z))$ such that $\lambda = f(z) \neq 0$ and
A = 0, B = F(y, z) or A = F(y, z), B = 0. In the first case \((A = \text{Ric}(e, \xi) = 0, B = \text{Ric}(\phi e, \xi) = F(y, z))\), the following are valid:

\[
\xi = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y} \quad \text{and} \quad e = k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y} + k_3 \frac{\partial}{\partial z}, \quad k_3 \neq 0.
\]

In the second case \((A = \text{Ric}(e, \xi) = F(y, z), B = \text{Ric}(\phi e, \xi) = 0)\), the following are valid:

\[
\xi = \frac{\partial}{\partial x}, \quad e = \frac{\partial}{\partial y} \quad \text{and} \quad \phi e = k_1' \frac{\partial}{\partial x} + k_2' \frac{\partial}{\partial y} + k_3' \frac{\partial}{\partial z}, \quad k_3' \neq 0,
\]

where

\[
\begin{align*}
k_1(x, y, z) &= r(z) = k_1'(x, y, z), \\
k_2(x, y, z) &= k_2'(x, y, z) = 2xf(z) - \frac{H(y, z) + y}{2f(z)} + \beta(z), \\
k_3(x, y, z) &= k_3'(x, y, z) = t(z) + \delta, \quad \frac{\partial H(y, z)}{\partial y} = F(y, z),
\end{align*}
\]

and \(r, \beta\) are smooth functions of \(z\) and \(\delta\) is constant. Furthermore, \(f(z) = \int \frac{1}{k_3(z)} \, dz\).

**Proof.** By virtue of Lemma 2.2, it can be easily proven that the assumption \(\nabla \xi h = 2ah\phi\) is equivalent to \(\xi(\lambda) = 0\). From the definition of a gradient of a differentiable function, we get

\[
\begin{align*}
gr\lambda &= e(\lambda)e + (\phi e)(\lambda)\phi e + \xi(\lambda)\xi \\
&= e(\lambda)e + (\phi e)(\lambda)\phi e. \quad \text{(18)}
\end{align*}
\]

Using Eq. (18) and \(\|\text{grad } \lambda\| = 1\) we have

\[
(e(\lambda))^2 + ((\phi e)(\lambda))^2 = 1. \quad \text{(19)}
\]

Differentiating (19) with respect to \(\xi\) and using Eqs. (16) and (17) and \(\xi(\lambda) = 0\), we obtain

\[
\begin{align*}
\xi(e(\lambda))e(\lambda) + \xi((\phi e)(\lambda))(\phi e)(\lambda) &= 0, \\
([\xi, e](\lambda))e(\lambda) + ([\xi, \phi e](\lambda)) (\phi e)\lambda &= 0, \\
\lambda e(\lambda)(\phi e)(\lambda) &= 0,
\end{align*}
\]

and since \(\lambda \neq 0\),

\[
e(\lambda)(\phi e)\lambda = 0. \quad \text{(20)}
\]

To study this system, we consider the open subsets of \(U\):

\[
\begin{align*}
U' &= \{p \in U \mid e(\lambda)(p) \neq 0, \ \text{in a neighborhood of } p\}, \\
U'' &= \{p \in U \mid (\phi e)(\lambda)p \neq 0, \ \text{in a neighborhood of } p\}.
\end{align*}
\]

From Lemma 2.1 we have that \(U' \cup U''\) is open and dense in the closure of \(U\). We distinguish 2 cases.
Case 1: We suppose that $p \in U'$. By virtue of Eqs. (19) and (20), we have $(\phi e)(\lambda) = 0$, and $e(\lambda) = \mp 1$. Changing to the basis $(\xi, -e, -\phi e)$ if necessary, we can assume that $e(\lambda) = 1$. The Eqs. (15), (16), (17), and (13), Eq. (14) reduces to

$$\begin{align*}
[e, \phi e] &= -be + c\phi e \\
[e, \xi] &= -2\lambda \phi e \\
[\phi e, \xi] &= 0, \quad \lambda = -a
\end{align*}$$

(21)

$$b = \frac{A}{2\lambda}, \quad e = \frac{B + 1}{2\lambda}, \quad a = -\lambda.$$  

(24)

respectively.

Since $[\phi e, \xi] = 0$, the distribution that is spanned by $\phi e$ and $\xi$ is integrable, and so for any $p \in U'$ there exists a chart $\{V, (x, y, z)\}$ at $p$, such that

$$\xi = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y}, \quad e = k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y} + k_3 \frac{\partial}{\partial z}$$

(25)

where $k_1, k_2, k_3$ are smooth functions on $V$. Since $e, \phi e$ are linearly independent we have $k_3 \neq 0$ at any point of $V$.

Using Eqs. (21), (22) and (25), we get the following partial differential equations:

$$\begin{align*}
\frac{\partial k_1}{\partial y} &= \frac{A}{2\lambda} k_1, & \frac{\partial k_2}{\partial y} &= \frac{1}{2\lambda} [Ak_2 - B - 1], & \frac{\partial k_3}{\partial y} &= \frac{A}{2\lambda} k_3, \\
\frac{\partial k_1}{\partial x} &= 0, & \frac{\partial k_2}{\partial x} &= 2\lambda, & \frac{\partial k_3}{\partial x} &= 0.
\end{align*}$$

(26)

(27)

Moreover, we know that

$$\frac{\partial \lambda}{\partial x} = 0, \quad \frac{\partial \lambda}{\partial y} = 0.$$  

(28)

Differentiating the equation $\frac{\partial k_3}{\partial x} = 0$ with respect to $\frac{\partial}{\partial y}$, and using $\frac{\partial k_3}{\partial y} = \frac{A}{2\lambda} k_3$, we find

$$0 = \frac{\partial^2 k_3}{\partial y \partial x} = \frac{\partial^2 k_3}{\partial x \partial y} = \frac{1}{2\lambda} \frac{\partial}{\partial x} k_3 + \frac{1}{2\lambda} A \frac{\partial k_3}{\partial x} = \frac{1}{2\lambda} A \frac{\partial k_3}{\partial x}.$$  

So,

$$\frac{\partial A}{\partial x} = 0.$$  

(29)

Differentiating $\frac{\partial k_2}{\partial y} = 2\lambda$ with respect to $\frac{\partial}{\partial y}$, and using $\frac{\partial k_2}{\partial y} = \frac{1}{2\lambda} [Ak_2 - B - 1]$ and Eq. (29), we prove that

$$\frac{\partial^2 k_2}{\partial y \partial x} = 0 = \frac{\partial^2 k_2}{\partial x \partial y} = \frac{1}{2\lambda} \left[ \frac{\partial}{\partial x} k_2 + A \frac{\partial k_2}{\partial x} - \frac{\partial B}{\partial x} \right].$$

So,

$$\frac{\partial B}{\partial x} = 2\lambda A.$$  

(30)
From Eq. (28) we have the following solution:

\[ \lambda(z) = f(z) + d, \]  

(31)

where \( d \) is constant. For the sake of shortness, we will use \( \tilde{f}(z) \) instead of \( f(z) + d \). Using \( e(\lambda) = k_1 \frac{\partial \lambda}{\partial z} + k_2 \frac{\partial \lambda}{\partial y} + k_3 \frac{\partial \lambda}{\partial z} = 1 \) and Eq. (28), we get

\[ \frac{\partial \lambda}{\partial z} = \frac{1}{k_3}, \quad k_3 \neq 0. \]  

(32)

If we differentiate Eq. (32) with respect to \( \frac{\partial}{\partial y} \) because of the equation \( \frac{\partial \lambda}{\partial y} = 0 \), we obtain

\[ 0 = \frac{\partial^2 \lambda}{\partial z \partial y} = \frac{\partial^2 \lambda}{\partial y \partial z} = -\frac{1}{k_3^2} \frac{\partial k_3}{\partial y}. \]  

(33)

Since \( k_3 \neq 0 \), Eq. (33) reduces and then we obtain

\[ \frac{\partial k_3}{\partial y} = 0. \]  

(34)

Combining Eqs. (26) and (34), we deduced that

\[ A = 0. \]  

(35)

Using Eqs. (30) and (35), we have

\[ \frac{\partial B}{\partial x} = 0. \]  

(36)

It follows from Eq. (36) that

\[ B = F(y, z). \]  

(37)

By virtue of Eqs. (35), (26), and (27), we easily see that

\[ k_1 = r(z), \]  

(38)

where \( r(z) \) is an integration function.

Combining Eqs. (27) and (34), we get

\[ k_3 = t(z) + \delta, \]  

(39)

where \( \delta \) is constant.

If we use Eqs. (27), (31), (35), and (37) in Eq. (26),

\[ \frac{\partial k_2}{\partial x} = 2\tilde{f}(z), \quad \frac{\partial k_2}{\partial y} = \frac{-(B + 1)}{2\lambda} = \frac{-(F(y, z) + 1)}{2\tilde{f}(z)}. \]  

(40)

It follows from this last partial differential equation that

\[ k_2 = 2x\tilde{f}(z) - \frac{(H(y, z) + y)}{2\tilde{f}(z)} + \beta(z), \]  

(41)
where
\[
\frac{\partial H(y, z)}{\partial y} = F(y, z).
\] (42)

Because of Eq. (32), there is a relation between \(\lambda(z) = \tilde{f}(z)\) and \(k_3(z)\) such that \(\tilde{f}(z) = \int \frac{1}{k_3(z)} dz\). We will calculate the tensor fields \(\eta, \phi, g\) with respect to the basis \(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\). For the components \(g_{ij}\) of the Riemannian metric \(g\), using Eq. (25) we have
\[
g_{11} = 1, \quad g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = \frac{-k_1}{k_3}, \quad g_{23} = g_{32} = \frac{-k_2}{k_3}, \quad g_{33} = \frac{1 + k_1^2 + k_2^2}{k_3^2}.
\]

The components of the tensor field \(\phi\) are immediate consequences of
\[
\phi(\xi) = \phi\left(\frac{\partial}{\partial x}\right) = 0, \quad \phi\left(\frac{\partial}{\partial y}\right) = -k_1 \frac{\partial}{\partial x} - k_2 \frac{\partial}{\partial y} - k_3 \frac{\partial}{\partial z},
\]
\[
\phi\left(\frac{\partial}{\partial z}\right) = \frac{k_1 k_2}{k_3} \frac{\partial}{\partial x} + \frac{k_1^2}{k_3} \frac{\partial}{\partial y} + \frac{k_2}{k_3} \frac{\partial}{\partial z}.
\]

The expression of the 1-form \(\eta\) immediately follows from \(\eta(\xi) = 1, \eta(\phi) = \eta(\phi e) = 0\).
\[
\eta = dx - \frac{k_1}{k_3} dz.
\]

Now we calculate the components of tensor field \(h\) with respect to the basis \(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\).
\[
h(\xi) = h\left(\frac{\partial}{\partial x}\right) = 0, \quad h\left(\frac{\partial}{\partial y}\right) = -\lambda \frac{\partial}{\partial y},
\]
\[
h\left(\frac{\partial}{\partial z}\right) = \lambda \frac{k_1}{k_3} \frac{\partial}{\partial x} + 2\lambda \frac{k_2}{k_3} \frac{\partial}{\partial y} + \lambda \frac{\partial}{\partial z}.
\]

**Case 2:** Now we suppose that \(p \in U''\). As in Case 1, we can assume that \((\phi e)(\lambda) = 1\). The Eqs. (15), (16) , (17), and (13), Eq. (14) reduces to
\[
[e, \phi e] = -be + c\phi e,
\]
\[
[e, \xi] = 0,
\]
\[
[\phi e, \xi] = -2\lambda e,
\]
\[
b = \frac{A + 1}{2\lambda}, \quad c = \frac{B}{2\lambda}, \quad a = \lambda,
\] (46)

respectively. Because of Eq. (44), we find that there exists a chart \(\{V', (x, y, z)\}\) at \(p \in U''\) such that
\[
\xi = \frac{\partial}{\partial x}, \quad \phi e = k_1' \frac{\partial}{\partial x} + k_2' \frac{\partial}{\partial y} + k_3' \frac{\partial}{\partial z}, \quad e = \frac{\partial}{\partial y},
\] (47)
where $k_1', k_2',$ and $k_3'$ ($k_3' \neq 0$), are smooth functions on $V'$.

Using Eqs.(43), (45), and (47), we get the following partial differential equations:

$$\frac{\partial k_1'}{\partial y} = \frac{B}{2\lambda} k_1', \quad \frac{\partial k_2'}{\partial y} = \frac{1}{2\lambda} [Bk_2' - A - 1], \quad \frac{\partial k_3'}{\partial y} = \frac{B}{2\lambda} k_3'.$$

Moreover, we know that

$$\frac{\partial \lambda}{\partial x} = 0, \quad \frac{\partial \lambda}{\partial y} = 0.$$

As in Case 1, if we solve the partial differential equations Eq. (48) and Eq. (49), then we find

$$B = 0, \quad A = F'(y, z)$$

$$\lambda(z) = f'(z) + d' = \tilde{f}'(z), \quad k_1' = r'(z), \quad k_3' = t'(z) + \delta'$$

$$k_2' = 2x\tilde{f}'(z) - \frac{H'(y, z) + y}{2f(z)} + \beta'(z)$$

$$\frac{\partial H'(y, z)}{\partial y} = F'(y, z)$$

where $d'$ and $\delta'$ are constants.

By the help of Eq. (51), the equation $(\phi e)(\lambda) = 1$ implies

$$\lambda(z) = \tilde{f}'(z) = \int \frac{1}{k_3'(z)} \, dz.$$ 

As in Case 1, we can directly calculate the tensor fields $g, \phi, \eta,$ and $h$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

$$g = \begin{pmatrix} 1 & 0 & -\frac{k_1'}{k_3'} \\ 0 & 1 & -\frac{k_2'}{k_3'} \\ -\frac{k_1'}{k_3'} & \frac{k_2'}{k_3'} & 1 + \frac{k_1'^2}{k_3'^2} \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & \frac{k_1'}{k_3'} & \frac{k_1'k_2'}{k_3'^2} \\ 0 & k_2' & \frac{1 + k_2'^2}{k_3'} \\ 0 & \frac{k_2'}{k_3'} & -k_2' \end{pmatrix},$$

$$\eta = dx - \frac{k_1'}{k_3'} dz \quad \text{and} \quad h = \begin{pmatrix} 0 & 0 & -\lambda \frac{k_1'}{k_3'} \\ 0 & \lambda & -2\lambda \frac{k_1'}{k_3'} \\ 0 & 0 & -\lambda \end{pmatrix}.$$ 

□

**Example 3.2**

$$M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$$

and the vector fields

$$\xi = \frac{\partial}{\partial x}, \quad e = \frac{\partial}{\partial y}, \quad \phi e = z \frac{\partial}{\partial x} + (2xz - 1) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$
The 1-form $\eta = dx - zdz$ is closed and the characteristic vector field is $\xi = \frac{\partial}{\partial x}$. Let $g$, $\phi$ be the Riemannian metric and the $(1,1)$-tensor field given by

$$
g = \begin{pmatrix} 1 & 0 & -a_1 \\ 0 & 1 & a_2 \\ -a_1 & a_2 & 1 + a_1^2 + (a_2)^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & a_1 & a_1a_2 \\ 0 & -a_2 & -1 + (1 + a_2^2) \\ 0 & 1 & a_2 \end{pmatrix},
$$

$$
h = \begin{pmatrix} 0 & 0 & -\lambda a_1 \\ 0 & \lambda & 2\lambda a_2 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad \lambda = z,
$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, where $a_1 = z$ and $a_2 = 1 - 2xz$.

$$
\eta = dx - zdz, \quad dh = 0,
$$

$$
\Phi = -dy \wedge dz, \quad d\Phi = 0.
$$

By a straightforward calculation, we obtain

$$
\nabla_\xi h = 2zh\phi, F(y, z) = -1, \|\text{grad } \lambda\| = 1.
$$

**Remark 3.3** Let $M(\phi, \xi, \eta, g)$ be an almost cosymplectic manifold. A $D_\alpha$-homothetic transformation [19] is the transformation

$$
\bar{\eta} = \alpha \eta, \quad \bar{\xi} = \frac{1}{\alpha} \xi, \quad \bar{\phi} = \phi, \quad \bar{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta
$$

(54)

of the structure tensors, where $\alpha$ is a positive constant. It is well known [19] that $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost cosymplectic manifold. When 2 contact structures $(\phi, \xi, \eta, g)$ and $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ are related by Eq. (54), we will say that they are $D_\alpha$-homothetic. We can easily show that $\bar{h} = \frac{1}{\alpha} h$ so $\bar{\lambda} = \frac{1}{\alpha} \lambda$.

(a) As a result, an almost cosymplectic manifold with $\|\text{grad } \lambda\|_g = d \neq 0$ (const.) is $D_\alpha$-deformed in another almost cosymplectic manifold with $\|\text{grad } \lambda\|_g = d\alpha^{-\frac{1}{2}}$ and choosing $\alpha = d^2$, it is enough to study those almost cosymplectic manifolds with $\|\text{grad } \lambda\| = 1$.

(b) If $d = 0$, then $\lambda$ is constant. As a result, if $\lambda = 0$, then $M$ is a cosymplectic manifold.

**Remark 3.4** There are no compact 3-dimensional almost cosymplectic manifolds with $\|\text{grad } \lambda\| = \text{const} \neq 0$. In fact, if such a manifold is compact, then the smooth function $\lambda$ will attain a maximum value at some point $p$ of $M$. Then $\text{grad } \lambda$ vanishes at $p$, contrary to the requirement that $\text{grad } \lambda$ is a nonzero constant.

**Remark 3.5** Using Theorem 3.1, we can produce infinitely many possible examples about 3-dimensional almost cosymplectic manifolds. If we add the condition $F(y, z) = 0$ to Theorem 3.1, we have $A = 0$ and $B = 0$. Thus, by Lemma 2.3, we can state that a 3-dimensional almost cosymplectic manifold under the same conditions of Theorem 3.1 is a 3-dimensional almost cosymplectic $(\kappa, \mu)$ manifold.

Now we will give an example satisfying Remark 3.5.
Example 3.6 We consider the 3-dimensional manifold

\[ M = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} \]

and the vector fields

\[ \xi = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y}, \quad e = z^2 \frac{\partial}{\partial x} + \left(2xz - z + \frac{y^2}{2z}\right) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \]

The 1-form \( \eta = dx - z^2dz \) is closed and the characteristic vector field is \( \xi = \frac{\partial}{\partial x} \). Let \( g, \phi \) be the Riemannian metric and the \((1,1)\)-tensor field given by

\[
\begin{pmatrix}
1 & 0 & -\frac{a_1}{a_3} \\
0 & 1 & -\frac{a_2}{a_3} \\
-\frac{a_1}{a_3} & -\frac{a_2}{a_3} & \frac{1}{a_3^2} + \frac{a_2^2}{a_3^2}
\end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & -a_1 & \frac{a_1a_2}{a_3} \\
0 & -a_2 & \frac{a_3}{a_2} \\
0 & -a_3 & a_2 \end{pmatrix},
\]

\[ \eta = dx - \frac{a_1}{a_3} dz, \quad \text{and} \quad h = \begin{pmatrix} 0 & 0 & \lambda \frac{a_2}{a_3} \\
0 & -\lambda & 2\lambda \frac{a_2}{a_3} \\
0 & 0 & \lambda \end{pmatrix}, \]

with respect to the basis \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \), where \( a_1 = z^2, \ a_2 = 2xz - \frac{z + y}{z}, \ a_3 = 1, \ \lambda = z \).

By direct computations, we get

\[ \| \text{grad} \lambda \| = 1, \nabla_\xi h = -2z \text{h} \phi, \ F(y, z) = 0 \]

and

\[ R(X, Y)\xi = (-z^2)(\eta(Y)X - \eta(X)Y) - 2z(\eta(Y)hX - \eta(X)hY) \]

for any vector field \( X, Y \) on \( M \).

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References


