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Finite groups with some weakly s-supplementally embedded subgroups

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Abstract: A subgroup H of G is said to be weakly s-supplementally embedded in G if there exist a subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$. In this paper, we investigate the influence of some weakly s-supplementally embedded subgroups on the structure of a finite group G. Some earlier results are unified and generalized.

Key words: s-permutable subgroup, weakly s-supplementally embedded subgroup, p-nilpotent group, formation

1. Introduction

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [11]. $\mathcal{F}$ stands for a formation, $\mathcal{N}_p$ and $\mathcal{N}$ denote the classes of all p-nilpotent groups and nilpotent groups, respectively. $G^{\mathcal{F}}$ denotes the $\mathcal{F}$-residual, $Z_{\mathcal{F}}(G)$ denotes the $\mathcal{F}$-hypercenter of G. For formation $\mathcal{N}$, we use the notation $Z_{\mathcal{N}}(G) = Z_{\infty}(G)$, the hypercenter of G. Fix a finite group G. How primary subgroups can be embedded in G is a question of particular interest in studying the structure of G. In fact, many results have been obtained. For example, Buckley [5] proved that if G is a group of odd order and all minimal subgroups of G are normal in G, then G is supersoluble. Itô proved that if G is a group of odd order and all minimal subgroups of G lie in the center of G, then G is nilpotent (see [11], III, 5.3). If all elements of G of order 2 and 4 lie in the center of G, then G is 2-nilpotent (see [11], IV, 5.5). Since then, a series of papers have dealt with generalizations of the results of Itô and Buckley by using the theory of formations and some generalized normal subgroups (see, for example, [1], [3], [6], [10], [13]).

Recall that a subgroup H of a group G is said to be s-permutable [12] (or s-quasinormal) in G, if $HP = PH$ for every Sylow subgroup P of G. Following Ballester-Bolinches and Pedraza-Aguilera [2], we say that a subgroup H of G is s-permutably embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. Recently, many other concepts were introduced successively, such as c-normal subgroup [20], c-supplemented subgroup [4], Q-supplemented subgroup [16], c^*-normal subgroup [21] etc. By assuming that some subgroups of G satisfying a certain kind of property, the authors have got many results about the structure of G. Furthermore, Skiba in [19] introduced weakly s-supplemented subgroup in which a subgroup H of a group G is said to be weakly s-supplemented in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is

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the maximal s-permutable subgroup of G contained in H. Following Li, Qiao and Wang in [15], we say that a subgroup H of a group G is weakly s-permutably embedded in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se}$, where H_{se} is an s-permutably embedded subgroup of G contained in H. In [23], the authors introduced the concept of weakly s-supplementally embedded subgroup, which extends all the generalized normal subgroups mentioned above properly. Motivated by [10], in this article, by assuming that $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geq 1$ and some subgroups of G with order p^n are weakly s-supplementally embedded in G, we give some criteria for (p-)nilpotency of G.

2. Preliminaries

In this section we list some basic definitions and known results which will be used below.

Definition 2.1 *A subgroup H of a group G is said to be weakly s-supplementally embedded in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se}$, where H_{se} is an s-permutably embedded subgroup of G contained in H.*

Remark Obviously, weakly s-supplemented subgroups and weakly s-permutably embedded subgroups are all weakly s-supplementally embedded subgroups. But the converse does not hold in general.

Example 1. Suppose that $G = A_5$ is the alternating group of degree 5. Then each Sylow 2-subgroup P of G is weakly s-supplementally embedded in G, since it is s-permutably embedded in G. But P is not weakly s-supplemented in G, as the only non-trivial s-permutable subgroup of G is itself.

Example 2. (See [9]) Put $H = \langle a, b | a^5 = b^5 = 1, a \neq b \textit{ and } ab = ba\rangle$ and let α be an automorphism of H of order 3 satisfying that $a^\alpha = b$, $b^\alpha = a^{-1}b^{-1}$. Let $H_1 = H$, $H_2 = \langle c, d\rangle$ be a copy of H_1 and $G = [H_1 \times H_2]\langle\alpha\rangle$. Then H_1, H_2 are minimal normal subgroups of G of order 25. Let $A = \langle ad, bc\rangle$ be a subgroup of G of order 25. Then it is not difficult to see that $H_1 \cap A = 1$. This shows that $T = [H_1]\langle\alpha\rangle$ is a complement of A in G and thereby A is weakly s-supplementally embedded in G. Now we prove that A is not weakly s-permutably embedded in G. First, we show that $\langle\alpha\rangle^G = G$, hence there exists no non-trivial normal subgroup of G containing $\langle\alpha\rangle$. In fact, since $|\alpha| = 3$, $\alpha^2 = \alpha^{-1}$. We have $a^{\alpha^{-1}} = a^{\alpha^2} = b^\alpha = a^{-1}b^{-1}$ and $b^{\alpha^{-1}} = a$, so $\alpha^{a^i} = a^{-i}\alpha a^i = a^{-i}(a^{\alpha^{-1}})^i\alpha = a^{-2i}b^{-i}\alpha$ and $\alpha^{b^i} = b^{-i}\alpha b^i = b^{-i}(b^{\alpha^{-1}})^i\alpha = a^ib^{-i}\alpha$. Then $(\alpha^{a^i})^{-1} = \alpha^{-1}a^{2i}b^i$ and thereby $\alpha^{b^i}(\alpha^{a^j})^{-1} = a^ib^{-i}\alpha\alpha^{-1}a^{2j}b^j = a^{2j+i}b^{j-i}$. Let $i = j = 1$, we get that $a^3 \in \langle\alpha\rangle^G$ and so $a \in \langle\alpha\rangle^G$. Let $j = 2$ and $i = 1$, we obtain that $b \in \langle\alpha\rangle^G$. Similarly, we can obtain that $c, d \in \langle\alpha\rangle^G$. Hence $G = \langle\alpha\rangle^G$. Suppose that there exists some subnormal subgroup T of G such that $G = AT$ and $A \cap T \leq A_{se}$. Then we can easily deduce that $\langle\alpha\rangle \leq T$, which implies that $T = G$. Therefore, $A = A_{se}$ is s-permutably embedded in G, then A is a Sylow 5-subgroup of some s-permutable subgroup K of G. Since K cannot contain $\langle\alpha\rangle$, $K = A$ is s-permutable in G. Hence $A = A\langle\alpha\rangle \cap (H_1 \times H_2) \trianglelefteq A\langle\alpha\rangle$, which is a contradiction.

Lemma 2.2 *([18, Lemma A]) If H is an s-permutable p-subgroup of G for some prime p, then $N_G(H) \geq O^p(G)$.*

Lemma 2.3 *([14, Lemma 2.4]) Suppose that P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

Lemma 2.4 *([10, Lemma 2.5]) Let G be a group and p a prime such that $p^{n+1} \nmid |G|$ for some integer $n \geq 1$. If $(|G|,(p-1)(p^2-1)\cdots(p^n-1))=1$, then G is p-nilpotent.*

Lemma 2.5 *([23, Lemma 2.6]) Let U be a weakly s-supplementally embedded subgroup and N a normal subgroup of G. Then we have the following:*

(1) If $U \leq H \leq G$, then U is weakly s-supplementally embedded in H.

(2) If $N \leq U$, then U/N is weakly s-supplementally embedded in G/N.

(3) If $(|U|,|N|)=1$, then UN/N is weakly s-supplementally embedded in G/N.

Lemma 2.6 *([22, Lemma 2.3]) Let the p'-group H act on the p-group P. If H acts trivially on $\Omega_1(P)$ and P is quaternion-free if $p=2$, then H acts trivially on P.*

Lemma 2.7 *([22, Lemma 2.2]) Let G be a group and let p be a prime number dividing $|G|$, with $(|G|,p-1)=1$. Then*

(1) If N is normal in G of order p, then N lies in $Z(G)$.

(2) If G has cyclic Sylow p-subgroups, then G is p-nilpotent.

(3) If M is a subgroup of G with index p, then M is normal in G.

Lemma 2.8 *([11, X. 13]) Let G be a group and M a subgroup of G.*

(1) If M is normal in G, then $F^*(M) \leq F^*(G)$.

(2) $F^*(G) \neq 1$, if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G)=F(G)$.

(4) If $K \leq Z(G)$, then $F^*(G/K)=F^*(G)/K$.

3. Main results

Theorem 3.1 *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|,p-1)=1$. Suppose that every minimal subgroup of $P \cap G^{\mathcal{N}_p}$ not having a p-nilpotent supplement in G is weakly s-supplementally embedded in G. If $p=2$, assume, in addition, that P is quaternion-free or every cyclic subgroup of $P \cap G^{\mathcal{N}_p}$ with order 4 not having a p-nilpotent supplement in G is weakly s-supplementally embedded in G. Then G is p-nilpotent.*

Proof Suppose that the result is false and let G be a counterexample of minimal order. Pick a proper subgroup M of G. Since $M/(M \cap G^{\mathcal{N}_p}) \cong MG^{\mathcal{N}_p}/G^{\mathcal{N}_p} \leq G/G^{\mathcal{N}_p}$ is p-nilpotent, $M^{\mathcal{N}_p} \leq M \cap G^{\mathcal{N}_p}$. Now let M_p be a Sylow p-subgroup of M. Without loss of generality, we may assume that $M_p \leq P$. Then $M_p \cap M^{\mathcal{N}_p} \leq P \cap G^{\mathcal{N}_p}$. Hence, every minimal subgroup of $M_p \cap M^{\mathcal{N}_p}$ not having a p-nilpotent supplement in M is weakly s-supplementally embedded in M by hypothesis and Lemma 2.5. Moreover, when $p=2$ we have that every cyclic subgroup of order 4 of $M_p \cap M^{\mathcal{N}_p}$ not having a p-nilpotent supplement in M is weakly

s-supplementally embedded in M or M_p is quaternion-free. Thus M satisfies the hypothesis of the theorem. The minimal choice of G implies that M is p-nilpotent and G is a minimal non-p-nilpotent group. By [11, IV, Theorem 5.4], G has a normal Sylow p-subgroup P and a non-normal cyclic Sylow q-subgroup Q such that $G = PQ$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Moreover, P is of exponent p if $p > 2$ and of exponent at most 4 if $p = 2$. On the other hand, the minimal choice of G implies that $G^{\mathcal{N}_p} = P$.

Let H be a minimal subgroup of P and T a supplement of H in G. If $T < G$, then T is a subgroup of G with index p. Lemma 2.7(3) shows that T is normal in G. From the nilpotency of T, it follows that Q is normal in G, a contradiction. Therefore, we may suppose that G is the unique supplement of H in G. Since G is not p-nilpotent, by hypothesis we know that $H = H_{se}$ is s-permutably embedded in G for every minimal subgroup H of P. Lemma 2.3 shows that every minimal subgroup of P is s-permutable in G. Then for any minimal subgroup $\langle x \rangle$ of P, $\langle x \rangle Q$ is a proper subgroup of G. Thus $\langle x \rangle Q$ is p-nilpotent and $\langle x \rangle \leq C_G(Q)$. If P has exponent p, then $P = \Omega_1(P)$ and $G = P \times Q$, a contradiction. Hence we may assume that $p = 2$ and $\exp P = 4$.

If P is quaternion-free, by Lemma 2.6 we can get that $P \leq C_G(Q)$ and so $Q \trianglelefteq G$, a contradiction. Now assume that every cyclic subgroup of $P \cap G^{\mathcal{N}_p} = P$ with order 4 not having a p-nilpotent supplement in G is weakly s-supplementally embedded in G. Let $P_1 = \langle x \rangle$ be a cyclic subgroup of P with order 4 and let K be a supplement of P_1 in G. Then $P = P \cap P_1K = P_1(P \cap K)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ and $(P \cap K)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, we have $P \cap K = P$ or $P \cap K \leq \Phi(P)$. If the latter case happens, then $P = P_1(P \cap K) = P_1$ is a cyclic Sylow 2-subgroup of G, which implies that G is 2-nilpotent. If $P \cap K = P$, then $K = G$ and $P_1 = (P_1)_{se}$ is s-permutably embedded in G. Lemma 2.3 implies that P_1 is s-permutable in G. If $P_1Q = G$, then G is p-nilpotent by Lemma 2.7(2). If $P_1Q < G$, then P_1Q is p-nilpotent by the former discussion. Therefore, $P_1 \leq C_G(Q)$ for any cyclic subgroup P_1 of P with order 4. Since P has exponent 4, $P \leq C_G(Q)$ and so $Q \trianglelefteq G$, a contradiction. This contradiction completes the proof of the theorem. $\square$

Next, we prove that:

Theorem 3.2 *Let p be a prime and G a group with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n > 1$. Suppose that all the subgroups H of G with order p^n not having a p-nilpotent supplement in G are weakly s-supplementally embedded in G. Then G is p-nilpotent.*

Proof Suppose that the result is false and let G be a counterexample of minimal order. We break the proof into the following steps:

(1) $p^{n+1} \mid |G|$ and every proper subgroup of G is p-nilpotent.

The fact that $p^{n+1} \mid |G|$ follows from Lemma 2.4. Let L be a proper subgroup of G, then $(|L|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$. If $p^{n+1} \nmid |L|$, then by Lemma 2.4 we know L is p-nilpotent. Now assume that $p^{n+1} \mid |L|$. Let H be a subgroup of L with order p^n. Then by hypothesis, H either has a p-nilpotent supplement T in G or is weakly s-supplementally embedded in G. In the former case, $L = L \cap HT = H(L \cap T)$ and $L \cap T$ is a p-nilpotent supplement of H in L. In the latter case, by Lemma 2.5, H is weakly s-supplementally embedded in L. This shows that L satisfies the hypothesis of the theorem. The minimal choice of G implies that L is p-nilpotent. Thus, by [11, IV, Theorem 5.4] we have: $G = PQ$, where P is a normal Sylow p-subgroup and Q a non-normal cyclic Sylow q-subgroup of G for some prime $q \neq p$; $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$; $\exp P = p$ when $p > 2$, while $\exp P$ is at most 4 when $p = 2$.

(2) Every subgroup H of P with order p^n is s-permutable in G.

Let T be any supplement of H in G, then $HT = G$ and so $P = P \cap HT = H(P \cap T)$. Since $P/\Phi(P)$ is a chief factor of G, $P/\Phi(P)$ is an elementary abelian p-group and hence $(P \cap T)\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. It follows that $P \cap T \leq \Phi(P)$ or $P \cap T = P$. If $P \cap T \leq \Phi(P)$, then $H = P$ is of order p^n, which contradicts (1). If $P \cap T = P$, then $T = G$ is not p-nilpotent. Thus, H is weakly s-supplementally embedded in G by the hypothesis. Therefore, $H = H \cap T = H_{se}$ is s-permutably embedded in G. Since $H \leq P \leq O_p(G)$, by Lemma 2.3 we know that H is s-permutable in G.

(3) Final contradiction.

By our hypothesis and (2), we know that all subgroups H of P with order p^n are s-permutable in G. Then HQ is a proper subgroup of G for any such subgroup H. Hence HQ is p-nilpotent by (1), which implies that $H \leq N_G(Q)$. By the facts that $\exp P = p$ or $\exp P = 4$, and every subgroup of P with order p or 4 is contained in some subgroup H of P with order p^n, we know Q is normalized by P and so $Q \trianglelefteq G$. This final contradiction completes the proof of the theorem. □

By Theorem 3.1 and Theorem 3.2, we have the following theorem.

Theorem 3.3 *Let p be a prime and G a group with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geq 1$. Suppose that every subgroup H of $P \in Syl_p(G)$ with order p^n or cyclic of order 4 (if P is a non-abelian 2-group and $n = 1$) not having a p-nilpotent supplement in G is weakly s-supplementally embedded in G, then G is p-nilpotent.*

Now we can prove that:

Theorem 3.4 *Let p be a prime and $\mathcal{F}$ a saturated formation containing the class $\mathcal{N}_p$ of all p-nilpotent groups. Suppose that G is a group with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$, for some integer $n \geq 1$. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and for a Sylow p-subgroup P of E, there exists a subgroup D of P such that $1 < |D| < p^{n+1}$ and all subgroups H of P with order $|D|$ or cyclic of order 4 (if P is a non-abelian 2-group and $|D| = 2$) not having a p-nilpotent supplement in G are weakly s-supplementally embedded in G.*

Proof Only the sufficiency needs to be verified. Suppose that the result is false and let G be a counterexample of minimal order. Then obviously, $(|E|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$. By Lemma 2.5, we know that for every subgroup H of $P \in \mathrm{Syl}_p(E)$ with order $|D|$ or cyclic of order $2|D| = 4$, either H has a p-nilpotent supplement in E or H is weakly s-supplementally embedded in E. Now, Theorem 3.3 implies that E is p-nilpotent. Let P be a Sylow p-subgroup and T a normal p-complement of E, then T is normal in G. Next, we break the proof into the following steps:

(1) $T = 1$.

If $T \neq 1$, we claim that G/T (with respect to E/T) satisfies the hypothesis of the theorem. In fact, $(G/T)/(E/T) \cong G/E \in \mathcal{F}$. Let H/T be an arbitrary subgroup of E/T with $|H/T| = |DT/T|$ or cyclic with $|H/T| = 2|DT/T| = 4$. Then $H = LT$, where L is a Sylow p-subgroup of H. Thus, $|L| = |D|$ or $|L| = 2|D| = 4$. By the hypothesis, either L has a p-nilpotent supplement K in G or L is weakly s-supplementally embedded in G. This means that either $H/T = LT/T$ has a p-nilpotent supplement KT/T in G/T or H/T is weakly s-supplementally embedded in G/T by Lemma 2.5. Hence, G/T satisfies the hypothesis of the theorem. Then the minimal choice of G implies that $G/T \in \mathcal{F}$. Let f and F be the canonical definitions of $\mathcal{N}_p$ and $\mathcal{F}$, respectively. Since T is a normal p'-subgroup of G, $G/C_G(T_{i+1}/T_i) \in f(q)$ for every chief factor T_{i+1}/T_i of G with $T_i \leq T$ and every prime q dividing $|T_{i+1}/T_i|$. Since $\mathcal{N}_p \subseteq \mathcal{F}$, $f(q) \subseteq F(q)$ by [7, IV, Proposition 3.11]. It follows that $G/C_G(T_{i+1}/T_i) \in F(q)$. Therefore, $G \in \mathcal{F}$ because $G/T \in \mathcal{F}$. This contradiction shows that $T = 1$.

(2) $C_G(P) \geq O^p(G)$.

Since $T = 1$, $P = E \trianglelefteq G$. Let Q be a Sylow q-subgroup of G, where $q \neq p$. Then PQ is a subgroup of G. Obviously, D is a subgroup of PQ and every subgroup H of PQ with order $|D|$ or $2|D|$ (when P is a non-abelian 2-group and $|D| = 2$) not having a p-nilpotent supplement in PQ is weakly s-supplementally embedded in PQ by Lemma 2.5. Hence by Theorem 3.3, PQ is p-nilpotent. It follows that $PQ = P \times Q$ and so $Q \leq C_G(P)$.

(3) Final contradiction.

Let M be an arbitrary non-trivial normal subgroup of G contained in $P \leq G_p \in \mathrm{Syl}_p(G)$. By (2), we know $O^p(G) \leq C_G(M)$ and so $[M, G] = [M, G_pO^p(G)] = [M, G_p] \trianglelefteq G$. Since $[M, G_p] < M$, there exists a normal subgroup N of G contained in M such that M/N is a chief factor of G and $[M, G] \leq N$. This implies that $M/N \leq Z(G/N)$. Thus $G/C_G(M/N) = 1 \in F(p)$. The arbitrary choice of M implies that there exists a normal chain of G contained in P such that every G-chief factor M/N is F-central. Since $G/P \in \mathcal{F}$, it follows that $G \in \mathcal{F}$. This final contradiction completes the proof of the theorem. □

Remarks: (1) Theorem 3.4 cannot be improved by taking a smaller number of subgroups of order p^n, say with the subgroups of the generalized Fitting subgroup $F^*(E)$. For example, we can consider the following special case ($p = 2$ and $n = 1$):

Suppose that $G = [(C_3 \times C_3 \times C_3)A_4] \times (C_2 \times C_2)$, where A_4 acts on $C_3 \times C_3 \times C_3$ as an irreducible and faithful module over the field of 3 elements. Then $F^*(G) = (C_3 \times C_3 \times C_3) \times (C_2 \times C_2)$ and $Z(G) = C_2 \times C_2 \in Syl_2(F^*(G))$. Therefore, there exists a subgroup D of $P = Z(G) \in Syl_2(F^*(G))$ of order 2 such that $1 < |D| < p^2 = |P|$ and all subgroups H of P with order 2 are normal in G, but G is not 2-nilpotent.

(2) From Theorem 3.4, we know that [17, Theorem 3.1], [9, Theorem **C**] and [10, Theorem 3.3] are true.

In [23], the authors prove that:

Lemma 3.5 *([23, Theorem 3.4]) Let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{N}_p$ of all p-nilpotent groups. If every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is weakly s-supplementally embedded in G, then $G \in \mathcal{F}$ if and only if every cyclic subgroup of $G^{\mathcal{F}}$ of prime order lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G.*

With this result, now we can prove this next theorem.

Theorem 3.6 *A group G is nilpotent if and only if every minimal subgroup of $F^*(G^{\mathcal{N}})$ lies in $Z_\infty(G)$ and every cyclic subgroup of $F^*(G^{\mathcal{N}})$ with order 4 is weakly s-supplementally embedded in G.*

Proof Only the sufficiency needs to be verified. Suppose that the result is false and let G be a counterexample of minimal order. Then we have:

(1) Every proper normal subgroup of G is nilpotent.

Let M be a proper normal subgroup of G. Since $M/(M \cap G^{\mathcal{N}}) \cong MG^{\mathcal{N}}/G^{\mathcal{N}} \leq G/G^{\mathcal{N}}$ is nilpotent and $M^{\mathcal{N}} \trianglelefteq M \cap G^{\mathcal{N}} \trianglelefteq G^{\mathcal{N}}$, by Lemma 2.8, we have $F^*(M^{\mathcal{N}}) \leq F^*(M \cap G^{\mathcal{N}}) \leq F^*(G^{\mathcal{N}})$. Moreover, $M \cap Z_\infty(G) \leq Z_\infty(M)$. Now we can see easily that M satisfies the hypothesis of the theorem. The minimal choice of G implies that M is nilpotent.

(2) $F(G)$ is the unique maximal normal subgroup of G.

Pick a maximal normal subgroup M of G. Then M is nilpotent by (1). Since the class of all nilpotent groups is a Fitting class, the nilpotency of M implies that $M = F(G)$ is the unique maximal normal subgroup of G.

(3) $G^{\mathcal{N}} = G = G^{'}$ and $F^*(G) = F(G) < G$.

Suppose that $G^{\mathcal{N}} < G$. Then $G^{\mathcal{N}}$ is nilpotent by (1). Thus, we have $F^*(G^{\mathcal{N}}) = G^{\mathcal{N}}$. Now Lemma 3.5 implies immediately that G is nilpotent, a contradiction. Hence, we must have $G^{\mathcal{N}} = G$. Since $G^{\mathcal{N}} \leq G'$, it follows that $G' = G$. Hence $G/F(G)$ cannot be cyclic of prime order. Thus $G/F(G)$ is a non-abelian simple group. If $F(G) < F^*(G)$, then $F^*(G^{\mathcal{N}}) = F^*(G) = G$ by (2). Again by Lemma 3.5, we have G is nilpotent, which is a contradiction.

(4) Final contradiction.

Since $F(G) = F^*(G) \neq 1$, we may choose the smallest prime divisor p of $|F(G)|$ such that $O_p(G) \neq 1$. For any Sylow q-subgroup Q of G, where $q \neq p$, we consider $G_0 = O_p(G)Q$. It is clear that $G_0^{\mathcal{N}} \leq O_p(G)$ and $G_0 \cap Z_\infty(G) \leq Z_\infty(G_0)$. Hence, every minimal subgroup of $G_0^{\mathcal{N}}$ lies in $Z_\infty(G_0)$ and every cyclic subgroup of $G_0^{\mathcal{N}}$ with order 4 is weakly s-supplementally embedded in G_0. By Lemma 3.5, G_0 is nilpotent. Hence, $G_0 = O_p(G) \times Q$ and $Q \leq C_G(O_p(G))$. Consequently, $G/C_G(O_p(G))$ is a p-group. Thus we have $C_G(O_p(G)) = G$ by (3), namely $O_p(G) \leq Z(G)$. Now we consider the factor group $\overline{G} = G/O_p(G)$. First we have $F^*(\overline{G}) = F^*(G)/O_p(G)$ by Lemma 2.8(4). For any element $\overline{x}$ of odd prime order in $F^*(\overline{G})$, since $O_p(G)$ is the Sylow p-subgroup of $F^*(G)$, $\overline{x}$ can be viewed as the image of an element x of odd prime order in $F^*(G)$. It follows that x lies in $Z_\infty(G)$ and $\overline{x}$ lies in $Z_\infty(\overline{G})$, for $Z_\infty(G/O_p(G)) = Z_\infty(G)/O_p(G)$. This shows that $\overline{G}$ satisfies the hypothesis of the theorem. By the minimal choice of G, we conclude that $\overline{G}$ is nilpotent and so G is nilpotent, as required. □

Theorem 3.7 *Let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{N}$ of all nilpotent groups. Then $G \in \mathcal{F}$ if and only if every minimal subgroup of $F^*(G^{\mathcal{F}})$ lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G and every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is weakly s-supplementally embedded in G.*

Proof Only the sufficiency needs to be verified. By [7, IV, 6.10], $G^{\mathcal{F}} \cap Z_{\mathcal{F}}(G) \leq Z(G^{\mathcal{F}}) \leq Z_\infty(G^{\mathcal{F}})$. Consequently, every minimal subgroup of $F^*(G^{\mathcal{F}})$ is contained in $Z_\infty(G^{\mathcal{F}})$. By the hypothesis and Lemma 2.5, every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is weakly s-supplementally embedded in $G^{\mathcal{F}}$. By applying Theorem 3.6, we see that $G^{\mathcal{F}}$ is nilpotent and so $F^*(G^{\mathcal{F}}) = G^{\mathcal{F}}$. Now by Lemma 3.5, we deduce that $G \in \mathcal{F}$. This completes the proof of the theorem. □

Remark From our Theorem 3.7, we can deduce that [13, Theorem 4.2], [4, Theorem 4.3] and [3, Theorem 3.1] are true.

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