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## Polynomial root separation in terms of the Remak height

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**Abstract:** We investigate some monic integer irreducible polynomials which have two close roots. If  $P(x)$  is a separable polynomial in  $\mathbb{Z}[x]$  of degree  $d \geq 2$  with the Remak height  $\mathcal{R}(P)$  and the minimal distance between the quotient of two distinct roots and unity  $\text{Sep}(P)$ , then the inequality  $1/\text{Sep}(P) \ll \mathcal{R}(P)^{d-1}$  is true with the implied constant depending on  $d$  only. Using a recent construction of Bugeaud and Dujella we show that for each  $d \geq 3$  there exists an irreducible monic polynomial  $P \in \mathbb{Z}[x]$  of degree  $d$  for which  $\mathcal{R}(P)^{(2d-3)(d-1)/(3d-5)} \ll 1/\text{Sep}(P)$ . For  $d = 3$  the exponent  $3/2$  is improved to  $5/3$  and it is shown that the exponent 2 is optimal in the class of cubic (not necessarily monic) irreducible polynomials in  $\mathbb{Z}[x]$ .

**Key words:** Polynomial root separation, Mahler's measure, Remak height, discriminant

### 1. Introduction

Let

$$P(x) := a_d x^d + \cdots + a_1 x + a_0 = a_d (x - \alpha_1) \cdots (x - \alpha_d) \in \mathbb{C}[x], \quad a_d, a_0 \neq 0,$$

be a separable polynomial of degree  $d \geq 2$ . Throughout, let

$$\Delta(P) := a_d^{2d-2} \prod_{1 \leq i < j \leq d} (\alpha_i - \alpha_j)^2$$

be its *discriminant*,

$$H(P) := \max_{1 \leq j \leq d} |a_j|$$

its *height*,

$$M(P) := |a_d| \prod_{j=1}^d \max(1, |\alpha_j|)$$

its *Mahler measure* and

$$\mathcal{R}(P) := |a_d| \prod_{j=1}^d |\alpha_j|^{(d-j)/(d-1)},$$

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where  $\alpha_1, \dots, \alpha_d$  are labeled so that  $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_d|$ , its *Remak height*. The last quantity in the context of polynomials first appeared in the paper of Remak [21] who proved the inequality

$$\sqrt{|\Delta(P)|} \leq d^{d/2} \mathcal{R}(P)^{d-1}. \tag{1}$$

This quantity also appears in [15], [20], [24] and is studied in detail in [9], [10], where it is named after Remak. In [8], it is shown that if  $a_{ij} \in \mathbb{C}$  for  $1 \leq i, j \leq d$  and the complex numbers  $z_j$  satisfy  $|z_1| \geq |z_2| \geq \dots \geq |z_d|$ , then

$$|\det(a_{ij} z_j^{i-1})_{1 \leq i, j \leq d}| \leq |z_1|^{d-1} |z_2|^{d-2} \dots |z_{d-1}| \prod_{j=1}^d \left( \sum_{i=1}^d |a_{ij}|^2 \right)^{1/2}. \tag{2}$$

This implies both (1) and Hadamard’s inequality.

Note that in view of

$$\sqrt{M(P) \min(|a_d|, |a_0|)} \leq \mathcal{R}(P) \leq M(P) \tag{3}$$

(see [10]) the inequality (1) is at least as good as Mahler’s inequality

$$\sqrt{|\Delta(P)|} \leq d^{d/2} M(P)^{d-1}.$$

In [16] Mahler also proved that

$$\text{sep}(P) > \frac{\sqrt{3|\Delta(P)|}}{d^{d/2+1} M(P)^{d-1}}, \tag{4}$$

where

$$\text{sep}(P) := \min_{i \neq j} |\alpha_i - \alpha_j|$$

is the minimal distance between two distinct roots of  $P$ . After the paper of Mahler various aspects of polynomial root separation have been investigated in [1]–[5], [7], [11]–[13], [18]–[20], [22].

In fact, in (4) one cannot replace  $M(P)$  by  $\mathcal{R}(P)$  (see the first example in Section 2 below), but instead finds the following.

**Theorem 1** *For each  $d \geq 2$  and each polynomial  $P \in \mathbb{C}[x]$  of degree  $d$ ,  $P(0) \neq 0$ , we have*

$$\text{Sep}(P) > \frac{c_d \sqrt{|\Delta(P)|}}{\mathcal{R}(P)^{d-1}}, \tag{5}$$

where  $\text{Sep}(P) := \min_{i \neq j} |1 - \alpha_j/\alpha_i|$  and

$$c_d := \frac{\sqrt{3}}{d^{d/2+1} \sqrt{(1 - 1/d)(1 - 1/2d)}}. \tag{6}$$

The inequality (5) is due to Mignotte [19] (see also [7]). We shall give its short proof based on (2) in Section 4.

Note that for  $d = 2$  we have

$$\text{Sep}(P) = \frac{\sqrt{|\Delta(P)|}}{\mathcal{R}(P)},$$

which is better than (5). For  $d = 3$  the constant  $c_3 = 1/3\sqrt{5} = 0.14907\dots$  given in (6) can be improved to  $1/4$ . Furthermore, as in [22], the latter constant is best possible even if we restrict to the class of monic irreducible polynomials in  $\mathbb{Z}[x]$ .

**Theorem 2** *If  $P(x) \in \mathbb{C}[x]$  is a separable cubic polynomial,  $P(0) \neq 0$ , then*

$$\text{Sep}(P) > \frac{\sqrt{|\Delta(P)|}}{4\mathcal{R}(P)^2}. \tag{7}$$

Furthermore, for each  $\varepsilon > 0$  there is a monic cubic irreducible polynomial  $P(x) \in \mathbb{Z}[x]$  for which

$$\text{Sep}(P) < (1 + \varepsilon) \frac{\sqrt{|\Delta(P)|}}{4\mathcal{R}(P)^2}. \tag{8}$$

Note that, inequality (5) (unlike (4)) is symmetric with respect to the map  $x \mapsto 1/x$  in the sense that we can replace  $P(x)$  by its reciprocal polynomial  $P^*(x) = \pm x^d P(1/x)$ . Then  $|\Delta(P)| = |\Delta(P^*)|$  and  $\mathcal{R}(P) = \mathcal{R}(P^*)$ , by Prop. 3.3 in [10]. Furthermore,  $\text{Sep}(P)$  is the minimal number among the following  $d(d-1)/2$  real numbers

$$|1 - \alpha_2/\alpha_1|, |1 - \alpha_3/\alpha_1|, \dots, |1 - \alpha_d/\alpha_{d-1}|,$$

because  $|\alpha_1| \geq \dots \geq |\alpha_d|$  implies  $|1 - \alpha_i/\alpha_j| \geq |1 - \alpha_j/\alpha_i|$  for  $i < j$ . So is also  $\text{Sep}(P^*)$ , since the roots of  $P^*$  are  $1/\alpha_d, \dots, 1/\alpha_1$ . Hence  $\text{Sep}(P) = \text{Sep}(P^*)$ . Of course,  $\text{sep}(P)$  and  $\text{sep}(P^*)$  can be different.

Below, when the degree of  $P$ , i.e.,  $d$  will be fixed, we shall write  $u \ll v$  for positive quantities  $u, v$  if the inequality  $u \leq cv$  holds with some constant  $c = c(d)$  depending on  $d$  only. With this notation, one has

$$H(P) \leq 2^d M(P) \ll M(P) \leq \sqrt{\sum_{j=0}^d |a_j|^2} \leq \sqrt{(d+1)} H(P) \ll H(P), \tag{9}$$

so  $H(P)$  and  $M(P)$  are of the same size. Hence, for a separable polynomial  $P(x) \in \mathbb{Z}[x]$  of degree  $d$ , from (4), (9) and (5) using  $|\Delta(P)| \geq 1$  we find that

$$1/\text{sep}(P) \ll H(P)^{d-1} \quad \text{and} \quad 1/\text{Sep}(P) \ll \mathcal{R}(P)^{d-1}. \tag{10}$$

To investigate how sharp is the exponent  $d-1$  in the first inequality of (10) the quantity

$$e_{\text{irr}}(d) := \limsup_{H(P) \rightarrow \infty} \frac{\log(1/\text{sep}(P))}{\log H(P)},$$

where the limsup is taken over all integer irreducible polynomials  $P$  of degree  $d$ , is introduced. Of course, by the first inequality of (10), it satisfies  $e_{\text{irr}}(d) \leq d-1$ . A similar quantity, where the polynomial  $P$  is, in addition, monic, is denoted by  $e_{\text{irr}}^*(d)$ . Clearly,

$$e_{\text{irr}}^*(d) \leq e_{\text{irr}}(d) \leq d-1.$$

It is straightforward that  $e_{\text{irr}}(2) = 1$  and  $e_{\text{irr}}^*(2) = 0$ . It is also known that  $e_{\text{irr}}(3) = 2$  (see [12], [22]). The lower bounds for  $e_{\text{irr}}(d)$ ,  $d \geq 4$ , and for  $e_{\text{irr}}^*(d)$ ,  $d \geq 3$ , have been obtained in [1]–[4]. Currently, the best bound on  $e_{\text{irr}}(d)$  for each  $d \geq 4$  is due to Bugeaud and Dujella [2]

$$e_{\text{irr}}(d) \geq \frac{d}{2} + \frac{d-2}{4(d-1)}.$$

As for  $e_{\text{irr}}^*(d)$ , their example gives the lower bound

$$e_{\text{irr}}^*(d) \geq \frac{d}{2} + \frac{d-2}{4(d-1)} - 1$$

for  $d \geq 7$ , but for  $d = 3, 5$  and  $d \geq 4$  even, the best bounds are due to Bugeaud and Mignotte [4]

$$e_{\text{irr}}^*(3) \geq 3/2, \quad e_{\text{irr}}^*(5) \geq 7/4 \quad \text{and} \quad e_{\text{irr}}^*(d) \geq (d-1)/2,$$

respectively.

By (9), the height  $H(P)$  and the Mahler measure  $M(P)$  are essentially of the same size, so we will not get anything new by considering a corresponding quantity with  $M(P)$  in place of  $H(P)$ . However, by (3), the Remak height  $\mathcal{R}(P)$  can be significantly smaller, i.e.,  $\sqrt{H(P)} \ll \mathcal{R}(P) \ll H(P)$ . So one can study

$$g_{\text{irr}}(d) := \limsup_{\mathcal{R}(P) \rightarrow \infty} \frac{\log(1/\text{Sep}(P))}{\log \mathcal{R}(P)}$$

(resp.  $g_{\text{irr}}^*(d)$ ), where the limsup is taken over all (resp. all monic) integer irreducible polynomials  $P$  of degree  $d$ . Now, by the second inequality of (10), we obtain

$$g_{\text{irr}}^*(d) \leq g_{\text{irr}}(d) \leq d - 1$$

for each  $d \geq 2$ .

A simple example,

$$x^2 - (2t + 1)x + t^2 + t - 1 = \left(x - t - \frac{1 + \sqrt{5}}{2}\right) \left(x - t - \frac{1 - \sqrt{5}}{2}\right)$$

with  $t \in \mathbb{N}$  tending to infinity, shows that  $g_{\text{irr}}^*(2) \geq 1$ , hence

$$g_{\text{irr}}(2) = g_{\text{irr}}^*(2) = 1.$$

For  $d \geq 3$ , by a construction based on the example of Bugeaud and Dujella [2], we can come closer to the upper bound  $d - 1$  with the quantity  $g_{\text{irr}}^*(d)$  compared to the quantities  $e_{\text{irr}}(d)$  and  $e_{\text{irr}}^*(d)$ .

**Theorem 3** *We have*

$$g_{\text{irr}}^*(d) \geq \frac{(2d-3)(d-1)}{3d-5}$$

for each  $d \geq 3$ .

The next theorem sharpens the inequality of this theorem for  $d = 3$  and evaluates the corresponding quantity for not necessarily monic polynomials.

**Theorem 4** *We have  $g_{\text{irr}}(3) = 2$  and  $g_{\text{irr}}^*(3) \geq 5/3$ .*

Clearly, for monic polynomials  $P$  of degree  $d$  we have

$$\mathcal{R}(P) \leq |\overline{P}|^{d/2},$$

where  $|\overline{P}| := \max_{\alpha: P(\alpha)=0} |\alpha|$  is the *house* of  $P$ . Thus (10) implies

$$1/\text{Sep}(P) \ll |\overline{P}|^{d(d-1)/2}$$

for monic integer separable polynomials  $P$  of degree  $d$ . In the opposite direction we prove the following.

**Theorem 5** *For each  $d \geq 4$  there are infinitely many monic integer irreducible polynomials  $P \in \mathbb{Z}[x]$  of degree  $d$  for which  $|\overline{P}|^{d(d-2)/4} \ll 1/\text{Sep}(P)$ . Furthermore, there are infinitely many monic cubic integer irreducible polynomials  $P \in \mathbb{Z}[x]$  for which  $|\overline{P}|^{5/2} \ll 1/\text{Sep}(P)$ .*

For monic cubic polynomials we have  $\mathcal{R}(P)^{5/3} \leq |\overline{P}|^{5/2}$ , and so Theorem 5 implies the inequality  $g_{\text{irr}}^*(3) \geq 5/3$  of Theorem 4. In fact, by Proposition 7 below, the equality  $g_{\text{irr}}^*(3) = 5/3$  holds (and also the constant  $5/2$  in Theorem 5 is optimal) if and only if Hall's conjecture [14] (asserting that there is an absolute constant  $c > 0$  such that the Diophantine inequality  $0 < |x^3 - y^2| < c\sqrt{x}$  has no solutions in positive integers) is true. A corresponding result for the equality  $e_{\text{irr}}^*(3) = 3/2$  is given in [4].

In Section 2 we give some examples (introduced in [16], [18], [2] or their variations) and prove the first statement of Theorem 5 and Theorem 3. In Section 3 prove Theorem 4 and the second statement of Theorem 5. Finally, in Section 4 we will prove Theorems 1 and 2.

## 2. Three examples

The following lemma is well known (see [17] or [23]).

**Lemma 6** *Suppose  $\lambda$  is a root of the polynomial  $x^d + \sum_{i=0}^{d-1} c_i x^i$  of multiplicity  $m$  and  $\varepsilon > 0$ . Then for  $|c_i - c'_i|$ ,  $i = 0, \dots, d-1$ , sufficiently small the polynomial  $x^d + \sum_{i=0}^{d-1} c'_i x^i$  has exactly  $m$  roots within  $\varepsilon$  of  $\lambda$ .*

As an illustration of his results in [16] Mahler considered the polynomial  $x^d - 1$ . Let us consider the polynomial

$$S_t(x) := x^d - t,$$

where  $t$  is a positive integer such that  $S_t$  is irreducible. (For instance,  $t$  can be a prime number.) Since  $\alpha_j = e^{2\pi i(j-1)/d} t^{1/d}$  for each  $j = 1, \dots, d$ , we have

$$\mathcal{R}(S_t) = t^{1/2}, \quad M(S_t) = H(S_t) = t,$$

$$\sqrt{|\Delta(S_t)|} = d^{d/2} t^{(d-1)/2},$$

$$\text{sep}(S_t) = 2 \sin(\pi/d)t^{1/d}, \quad \text{Sep}(S_t) = 2 \sin(\pi/d).$$

Hence

$$\frac{\text{Sep}(S_t)\mathcal{R}(S_t)^{d-1}d^{d/2+1}}{\sqrt{|\Delta(S_t)|}} = 2 \sin(\pi/d)d < 2\pi.$$

In particular, the constant  $\sqrt{3}$  in (6) cannot be replaced by the constant  $2\pi$ . Moreover, from  $\mathcal{R}(S_t^*) = \mathcal{R}(S_t) = t^{1/2}$ ,  $\sqrt{|\Delta(S_t^*)|} = \sqrt{|\Delta(S_t)|} = d^{d/2}t^{(d-1)/2}$  and  $\text{sep}(S_t^*) = 2 \sin(\pi/d)t^{-1/d}$  we deduce that

$$\frac{\text{sep}(S_t^*)\mathcal{R}(S_t^*)^{d-1}}{\sqrt{|\Delta(S_t^*)|}} = \frac{2 \sin(\pi/d)}{d^{d/2}t^{1/d}} < \varepsilon$$

for  $t$  large enough, so one cannot replace  $M(P)$  by  $\mathcal{R}(P)$  in (4).

The next example is due to Mignotte [18]. Fix a prime number  $p$  and consider the monic polynomial

$$Q_t(x) := x^d - p(tx - 1)^2 \in \mathbb{Z}[x],$$

where  $t$  is a sufficiently large positive integer. This polynomial is irreducible, by Eisenstein’s criterion. We claim that this polynomial has  $d - 2$  ‘large’ roots  $\alpha_1, \dots, \alpha_{d-2}$  satisfying

$$\alpha_j \sim e^{2\pi i(\tau(j)-1)/(d-2)}p^{1/(d-2)}t^{2/(d-2)} \quad \text{as } t \rightarrow \infty, \tag{11}$$

where  $\tau$  is a permutation of the set  $\{1, 2, \dots, d - 2\}$ , and two ‘small’ positive roots  $\alpha_{d-1} > \alpha_d$  satisfying

$$\alpha_{d-1} - \frac{1}{t} \sim \frac{1}{\sqrt{pt^{d/2+1}}}, \quad \alpha_d - \frac{1}{t} \sim -\frac{1}{\sqrt{pt^{d/2+1}}} \quad \text{as } t \rightarrow \infty. \tag{12}$$

Indeed, setting  $x := t^{2/(d-2)}y$  into  $Q_t(x) = 0$  and multiplying by  $t^{-2d/(d-2)}$ , we obtain

$$y^d - py^2 + 2pt^{-d/(d-2)}y - pt^{-2d/(d-2)} = 0,$$

so Lemma 6 implies (11). On the other hand, writing the root of  $Q_t$  in the form  $x := (yt^{-d/2} + 1)/t$ , we find that

$$0 = t^d Q_t((yt^{-d/2} + 1)/t) = (yt^{-d/2} + 1)^d - py^2,$$

so, by Lemma 6,  $y$  is close to  $\pm 1/\sqrt{p}$  when  $t$  is large. This proves (12).

From  $\mathcal{R}(Q_t)^{d-1} = |\alpha_1|^{d-1}|\alpha_2|^{d-2} \dots |\alpha_{d-2}|^2|\alpha_{d-1}|$ , using (11), (12), in view of

$$\frac{2}{d-2}(d-1+d-2+\dots+2) - 1 = \frac{2}{d-2}\left(\frac{(d-1)d}{2} - 1\right) - 1 = d$$

we obtain

$$\mathcal{R}(Q_t)^{d-1} \sim p^{(d+1)/2}t^d \quad \text{as } t \rightarrow \infty$$

and also

$$\text{Sep}(Q_t) = \frac{\alpha_{d-1} - \alpha_d}{\alpha_{d-1}} \sim \frac{2}{\sqrt{pt^{d/2}}} \quad \text{as } t \rightarrow \infty. \tag{13}$$

Therefore,

$$\frac{\log(1/\text{Sep}(Q_t))}{\log \mathcal{R}(Q_t)} \rightarrow \frac{d/2}{d/(d-1)} = \frac{d-1}{2}$$

as  $t \rightarrow \infty$ .

In particular, this example yields the bound  $g_{\text{irr}}^*(d) \geq (d-1)/2$ . Furthermore, combining  $|\overline{Q}_t| \sim p^{1/(d-2)}t^{2/(d-2)}$  with (13) we see that  $|\overline{Q}_t|^{d(d-2)/4} \ll 1/\text{Sep}(Q_t)$ . This proves the first statement of Theorem 5.

The next construction is essentially due to Bugeaud and Dujella [2]. Let

$$C_k := \frac{1}{k+1} \binom{2k}{k}, \quad k = 0, 1, 2, \dots,$$

be the  $k^{\text{th}}$  Catalan number. The Catalan numbers for  $k = 0, 1, 2, \dots$  are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, \dots$$

It is well known that

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k} \tag{14}$$

and that the generating function of Catalan's numbers

$$c(x) := \sum_{k=0}^{\infty} C_k x^k$$

satisfies

$$c(x) - 1 = c(x)^2 x.$$

We next replace  $c(x)$  in the equality  $x^{-1} + c(x)(-x^{-1} + c(x)) = 0$ ,  $x \neq 0$ , by its truncated series and introduce a new parameter  $t$ . More precisely, for integers  $d \geq 2$  and  $t \geq 1$  consider the Laurent polynomial

$$G_t(x) := \frac{1}{x} + \left( \sum_{k=0}^{d-2} C_k x^k + \frac{x^{d-1}}{t} \right) \left( -\frac{1}{x} + \sum_{k=0}^{d-2} C_k x^k + \frac{x^{d-1}}{t} \right). \tag{15}$$

Note that the coefficient for  $x^{-1}$  in  $G_t(x)$  is zero, because  $C_0 = 1$ . The coefficient for  $x^n$ , where  $0 \leq n \leq d-3$ , in  $G_t(x)$  is equal to

$$-C_{n+1} + C_n C_0 + C_{n-1} C_1 + \dots + C_0 C_n,$$

which is zero again in view of (14). Consequently,

$$F_t(x) := \frac{t^2}{x^{d-2}} G_t(x) = x^d + 2t C_{d-2} x^{d-1} + \sum_{k=0}^{d-2} a_k(t) x^k \tag{16}$$

is a monic polynomial of degree  $d$  with integer coefficients. Here,

$$a_k(t) = 2C_{k-1}t + t^2 \sum_{j=k}^{d-2} C_j C_{d-2+k-j} \tag{17}$$



for  $k = 1, \dots, d - 2$  and

$$a_0(t) = -t + t^2 \sum_{j=0}^{d-2} C_j C_{d-2-j} = -t + C_{d-1} t^2. \tag{18}$$

The monic polynomial  $F_t(x)$  of degree  $d$  is irreducible if, say,  $t$  is a prime number. By Lemma 6, (17) and (18), as  $t \rightarrow \infty$ , the polynomial  $F_t(x)$  has  $d - 2$  roots  $\alpha_3, \dots, \alpha_d$  tending to  $d - 2$  roots of the polynomial

$$C_{d-1} + \sum_{k=1}^{d-2} x^k \sum_{j=k}^{d-2} C_j C_{d-2+k-j} = (x - \lambda_3) \dots (x - \lambda_d).$$

(In principle,  $\lambda_3, \dots, \lambda_d$  are not necessarily distinct, although in all examples with small  $d$  they are distinct.)

Let  $\xi$  be the root of the polynomial

$$E_t(x) := t \sum_{k=0}^{d-2} C_k x^k + x^{d-1}$$

satisfying

$$\xi \sim -t C_{d-2} \quad \text{as } t \rightarrow \infty. \tag{19}$$

Applying the mean value theorem to the function  $E_t(x)$  in the interval  $[\xi, \xi + \theta C_{d-2}^{3/2-d} t^{5/2-d}]$ , where  $\theta \in \mathbb{R}$  is fixed, in view of  $E_t(\xi) = 0$  and (19) we obtain

$$E_t(\xi + \theta C_{d-2}^{3/2-d} t^{5/2-d}) \sim \theta C_{d-2}^{3/2-d} t^{5/2-d} ((d-1)\xi^{d-2} + (d-2)C_{d-2} t \xi^{d-3}) \sim (-1)^d \theta \sqrt{\frac{t}{C_{d-2}}}$$

as  $t \rightarrow \infty$ . Now, by (15) and (16),

$$F_t(x) x^{d-1} = t^2 x G_t(x) = t^2 x \left( \frac{1}{x} + \frac{E_t(x)}{t} \left( -\frac{1}{x} + \frac{E_t(x)}{t} \right) \right) = t^2 - t E_t(x) + x E_t(x)^2.$$

Let us insert the root  $x$  of  $F_t$  written in the form  $x = \xi + \theta C_{d-2}^{3/2-d} t^{5/2-d}$  into  $1 - E_t(x) t^{-1} + x t^{-2} E_t(x)^2 = 0$ . By the above, we see that the left hand side tends to  $1 - \theta^2$  as  $t \rightarrow \infty$ . Hence  $\theta$  tends to 1 and  $-1$ , so that the remaining two roots  $\alpha_1, \alpha_2$  of  $F_t(x)$  satisfy

$$\alpha_1 - \xi \sim -C_{d-2}^{3/2-d} t^{5/2-d} \quad \text{and} \quad \alpha_2 - \xi \sim C_{d-2}^{3/2-d} t^{5/2-d}. \tag{20}$$

We are now in a position to prove Theorem 3. Set  $t := p k^d$  with a prime number  $p$  and a positive integer  $k$  and consider the polynomial  $P_k(x) := F_{p k^d}(k x) k^{-d}$ , where  $F_t(x)$  is defined in (16). By (17), (18) and the Eisenstein criterion applied to  $p$ , we see that  $P_k$  is a monic irreducible polynomial of degree  $d$ . Its roots are  $\beta_j = \alpha_j/k$ ,  $j = 1, \dots, d$ , where  $\alpha_j$  are the roots of  $F_t$ . Since  $t = p k^d$ , from (19) and (20) we derive that  $\beta_1, \beta_2 \sim -p C_{d-2} k^{d-1}$  and

$$\beta_2 - \beta_1 \sim 2 C_{d-2}^{3/2-d} p^{5/2-d} k^{-d^2+5d/2-1}$$

as  $k \rightarrow \infty$ . Thus

$$\text{Sep}(P_k) \leq |1 - \beta_1/\beta_2| \sim 2 p^{3/2-d} C_{d-2}^{1/2-d} k^{-d^2+3d/2}. \tag{21}$$

Since  $\beta_j \sim \lambda_j k^{-1}$  as  $k \rightarrow \infty$  for  $j = 3, \dots, d$ , in view of

$$(d-1)(d-1+d-2) - (d-3+d-2+\dots+1) = (3d-5)d/2,$$

we find that

$$k^{(3d-5)d/2(d-1)} \ll \mathcal{R}(P_k) = |\beta_1||\beta_2|^{(d-2)/(d-1)} \dots |\beta_{d-1}|^{1/(d-1)} \ll k^{(3d-5)d/2(d-1)}. \tag{22}$$

Now, since  $\mathcal{R}(P_k) \rightarrow \infty$  as  $k \rightarrow \infty$ , combining (21) with (22) we find that

$$g_{\text{irr}}^*(d) \geq \frac{d^2 - 3d/2}{(3d-5)d/(2(d-1))} = \frac{(2d-3)(d-1)}{3d-5}.$$

This completes the proof of Theorem 3.

### 3. Proof of Theorem 4

Our proof of  $g_{\text{irr}}(3) = 2$  follows [22]. Let us begin, for example, with the polynomial

$$P(x) := x^3 - x - 1 = (x - \alpha)(x - \beta)(x - \gamma),$$

where  $\alpha = 1.32471\dots$  and  $\beta = -0.66235\dots + i0.56227\dots$ ,  $\gamma = -0.66235\dots - i0.56227\dots$  are two complex conjugate roots satisfying

$$|\beta| = |\gamma| < 1 \quad \text{and} \quad \Re(\beta) = \Re(\gamma) < 0.$$

Consider the sequence  $\alpha_1 := \alpha$  and

$$\alpha_{k+1} := 1/\{\alpha_k\} \quad \text{for} \quad k = 1, 2, 3, \dots$$

Then  $\alpha_k > 1$  and  $\alpha_k \in \mathbb{Q}(\alpha)$  for each  $k \in \mathbb{N}$ . Setting  $\beta_1 := \beta$ ,  $\gamma_1 := \gamma$  and  $q_k := [\alpha_k] \in \mathbb{N}$  (so that  $\alpha_{k+1} = 1/(\alpha_k - q_k)$ ), we also define two corresponding sequences

$$\beta_{k+1} = 1/(\beta_k - q_k) \quad \text{and} \quad \gamma_{k+1} = 1/(\gamma_k - q_k)$$

for  $k = 1, 2, 3, \dots$ . Note that, by the above construction, the continued fraction expansion for the cubic number  $\alpha_k$  is

$$\alpha_k = q_k + \frac{1}{q_{k+1} + \frac{1}{q_{k+2} + \dots}} \tag{23}$$

for each  $k \in \mathbb{N}$ .

It is easy to see that the ‘next’ polynomial  $P_k(x)$  obtained from  $P_{k-1}(x)$ , firstly, by replacing  $P_{k-1}(x)$  by  $P_{k-1}(x + q_{k-1})$  and then, secondly, by taking its reciprocal polynomial, namely,

$$P_k(x) = P_{k-1}^*(x + q_{k-1}) = a_k(x - \alpha_k)(x - \beta_k)(x - \gamma_k) \in \mathbb{Z}[x], \quad a_k \in \mathbb{N},$$

is irreducible, since so is  $P_{k-1}(x)$ . Furthermore, it is clear that

$$\sqrt{|\Delta(P_k)|} = \sqrt{|\Delta(P_{k-1})|} = \dots = \sqrt{|\Delta(P)|} = \sqrt{23}.$$

It is straightforward to check that for each  $k \in \mathbb{N}$  the roots  $\beta_k$  and  $\gamma_k = \overline{\beta_k}$  satisfy

$$|\beta_k| = |\gamma_k| < 1 \quad \text{and} \quad \Re(\beta_k) = \Re(\gamma_k) < 0.$$

Consequently,  $|\alpha_k - \beta_k| = |\alpha_k - \gamma_k| > \alpha_k$ , and so

$$\sqrt{23} = a_k^2 |\alpha_k - \beta_k| |\alpha_k - \gamma_k| |\beta_k - \gamma_k| > a_k^2 \alpha_k^2 |\beta_k| |1 - \gamma_k/\beta_k| \geq \mathcal{R}(P_k)^2 \text{Sep}(P_k). \tag{24}$$

If the sequences  $a_k \in \mathbb{N}$  and  $\alpha_k, k = 1, 2, 3, \dots$ , were both bounded from above then, as  $|\beta_k|, |\gamma_k| < 1$ , we would only have finitely many different polynomials  $P_k(x) \in \mathbb{Z}[x]$ . But then we must have  $\alpha_k = \alpha_j$  for some indices  $k > j \geq 1$ . By (23), this implies that the sequence  $q_k, k = 1, 2, 3, \dots$ , is ultimately periodic. So  $\alpha_1 = \alpha$  must be a quadratic number, a contradiction. This proves that at least one sequence  $a_k, k = 1, 2, 3, \dots$ , or  $\alpha_k, k = 1, 2, 3, \dots$ , is unbounded. Hence the sequence  $M(P_k) = a_k \alpha_k, k = 1, 2, 3, \dots$ , is unbounded. Thus, by (3),  $\mathcal{R}(P_k), k = 1, 2, 3, \dots$ , is unbounded and therefore (24) implies  $g_{\text{irr}}(3) \geq 2$ . Combining this with the upper bound  $g_{\text{irr}}(3) \leq 2$  we obtain  $g_{\text{irr}}(3) = 2$ .

Note that, by exactly the same argument, we can start with any Pisot number  $\alpha$  of degree  $d \geq 3$  with minimal polynomial  $P$  whose all other  $d - 1$  conjugates have negative real part. (For example, in [9] we have considered totally positive Pisot units  $\alpha$  of degree  $d$ . Then  $\alpha - 1$  is a Pisot number of degree  $d$  with its all remaining  $d - 1$  conjugates negative.) Putting

$$\alpha_{1,1} = \alpha, \quad \alpha_{1,k+1} = 1/\{\alpha_{1,k}\}, \quad k = 1, 2, 3, \dots,$$

we obtain the sequence of polynomials  $P_k, k = 1, 2, 3, \dots$ , with roots  $\alpha_{1,k}, \alpha_{2,k}, \dots, \alpha_{d,k}$  such that  $\alpha_{1,k}$  is a Pisot number,  $\alpha_{1,k} > 1 > |\alpha_{2,k}| \geq \dots \geq |\alpha_{d,k}|$ , and  $|\alpha_{1,k} - \alpha_{i,k}| > \alpha_{1,k}$  for  $i = 2, \dots, d - 1$ . It follows that

$$\mathcal{R}(P_k)^{d-1} \prod_{2 \leq i < j \leq d} |1 - \alpha_{j,k}/\alpha_{i,k}| < \sqrt{\Delta(P_k)} = \sqrt{\Delta(P)}.$$

Also, as above, all the numbers  $\alpha_{1,k}, k = 1, 2, 3, \dots$ , must be distinct, so the sequences  $M(P_k) = a_k \alpha_{1,k}, k = 1, 2, 3, \dots$ , and  $\mathcal{R}(P_k), k = 1, 2, 3, \dots$ , are unbounded. Of course, if  $\alpha$  is a Pisot number with negative conjugates, then the roots  $\alpha_{2,k}, \dots, \alpha_{d,k}$  are negative for each  $k \in \mathbb{N}$ .

We next turn to monic cubic polynomials with two close roots and use the ideas of [4]. Recall first that, by a result of Danilov [6], there exist two increasing sequences of positive integers  $x_k$  and  $y_k, k = 1, 2, 3, \dots$ , and an absolute constant  $c > 0$  such that

$$x_k^3 - y_k^2 \sim cx_k^{1/2} \quad \text{as} \quad k \rightarrow \infty. \tag{25}$$

(See formula (6) in [6], where there is misprint in the power of the polynomial  $t^2 + 6t - 11$ .) So Proposition 7 with  $w = 5/2$  implies the assertion of Theorem 5 for cubic polynomials and also the inequality  $g_{\text{irr}}^*(3) \geq 5/3$  of Theorem 4. Moreover, by Hall's conjecture [14],  $w$  is the largest real number with this property (although it is only known that  $w < 3$  which follows from an old result of Mordell), so equality  $g_{\text{irr}}^*(3) = 5/3$  is equivalent to Hall's conjecture.

The remainder of this section is devoted to the proof of the following statement.

**Proposition 7** *Let  $w$  be a positive number satisfying  $5/2 \leq w < 3$ . Then the inequality  $|\overline{P}|^w \ll 1/\text{Sep}(P)$  has infinitely many solutions in monic cubic irreducible polynomials  $P \in \mathbb{Z}[x]$  if and only if the inequality  $0 < |x^3 - y^2| \ll x^{3-w}$  has infinitely many solutions in positive integers  $x, y$ .*

**Proof** Assume first that the inequality  $0 < |x_k^3 - y_k^2| \ll x_k^{3-w}$  holds for infinitely many pairs  $(x_k, y_k) \in \mathbb{N}^2$ . Consider the monic cubic polynomial

$$P_k(x) := x^3 - 3x_kx - 2y_k \in \mathbb{Z}[x]$$

with discriminant  $\Delta(P_k) = 108(x_k^3 - y_k^2)$ . Putting  $\delta_k := (x_k^3 - y_k^2)x_k^{w-3}/3$ , we have  $|\delta_k| \ll 1$ . Evaluating the polynomial  $P_k$  at  $x = -\sqrt{x_k} + z$  we find that

$$\begin{aligned} P_k(-\sqrt{x_k} + z) &= -x_k^{3/2} + 3x_kz - 3\sqrt{x_k}z^2 + z^3 + 3x_k^{3/2} - 3x_kz - 2y_k \\ &= 2(x_k^{3/2} - y_k) - 3\sqrt{x_k}z^2 + z^3 = \frac{2(x_k^3 - y_k^2)}{x_k^{3/2} + y_k} - 3\sqrt{x_k}z^2 + z^3. \end{aligned}$$

Therefore, since

$$\frac{2(x_k^3 - y_k^2)}{3(x_k^{3/2} + y_k)\sqrt{x_k}} \sim \frac{x_k^3 - y_k^2}{3x_k^2} = \frac{3\delta_k x_k^{3-w}}{3x_k^2} = \delta_k x_k^{1-w} \quad \text{as } k \rightarrow \infty,$$

for its two roots  $\alpha_k, \beta_k$  we have

$$\alpha_k + \sqrt{x_k} \sim -x_k^{1/2-w/2}\sqrt{\delta_k} \quad \text{and} \quad \beta_k + \sqrt{x_k} \sim x_k^{1/2-w/2}\sqrt{\delta_k}.$$

Thus the third root satisfies  $\gamma_k \sim 2\sqrt{x_k}$  as  $k \rightarrow \infty$ . Therefore, in both cases ( $\alpha_k, \beta_k$  are real or complex conjugate roots), we have  $\gamma_k > |\alpha_k| \geq |\beta_k|$  and

$$\text{sep}(P_k) = |\alpha_k - \beta_k| \sim 2\sqrt{|\delta_k|x_k^{1/2-w/2}}.$$

It follows that  $\text{Sep}(P_k) \sim 2\sqrt{|\delta_k|x_k^{-w/2}}$  and  $|\overline{P_k}| \sim 2x_k^{1/2}$ , giving the inequality  $|\overline{P_k}|^w \ll 1/\text{Sep}(P_k)$  for the monic cubic polynomials  $P_k$  defined above.

To complete the proof in one direction it remains to show that  $P_k$  are irreducible for  $k$  large enough. For a contradiction assume that  $P_k$  is reducible in  $\mathbb{Z}[x]$ . Then one of the roots  $\alpha_k, \beta_k$  or  $\gamma_k$  must be an integer. If at least two roots are integers then all three must be integers which is impossible in view of  $\beta_k - \alpha_k \rightarrow 0$ . So assume that one is an integer and two others are the roots of an irreducible polynomial  $Q(x) = x^2 + ux + v \in \mathbb{Z}[x]$ . By the same reason, as  $\beta_k - \alpha_k \rightarrow 0$ , these two cannot be  $\alpha_k, \beta_k$ , so one of the roots of  $Q$  is  $\gamma_k$ . Assume that the other root of  $Q$  is  $\beta_k$ . (The proof in case this is  $\alpha_k$  is the same.) Then  $\alpha_k, \beta_k$  are real negative numbers,  $u = -\gamma_k - \beta_k = \alpha_k$  and  $\Delta(Q) = u^2 - 4v = (\gamma_k - \beta_k)^2 \notin \mathbb{Z}^2$ . Thus

$$\beta_k - \alpha_k = \beta_k - u = \frac{-u - \sqrt{\Delta(Q)}}{2} - u = \frac{-3u - \sqrt{\Delta(Q)}}{2} \geq \frac{1}{2(-3u + \sqrt{\Delta(Q)})}.$$

As  $-3u = -3\alpha_k < 3\gamma_k$  and  $\sqrt{\Delta(Q)} = \gamma_k - \beta_k = \gamma_k + |\beta_k| < 2\gamma_k$ , this yields  $\text{sep}(P_k) = \beta_k - \alpha_k > 1/10\gamma_k$ , contrary to  $\text{sep}(P_k) \ll x_k^{1/2-w/2} \ll \gamma_k^{1-w} \ll \gamma_k^{-3/2}$ .

To prove the result in the opposite direction we assume that the inequality

$$\mathcal{R}(P)^{2w/3} \ll 1/\text{Sep}(P)$$

has infinitely many solutions in monic cubic irreducible polynomials  $P = P_k \in \mathbb{Z}[x]$ . Note that this assumption is weaker than required because  $\mathcal{R}(P)^{2w/3} \leq |\bar{P}|^w$ . Without restriction of generality (by replacing  $P_k(x)$  by  $P_k(6x)$ , if necessary, and omitting everywhere the index  $k$ ) we may assume that the coefficients of  $P(x) = x^3 + ax^2 + bx + c$  satisfy  $6|a, b, c$ . We claim that  $\mathcal{R}(P)^{2w/3} \ll 1/\text{Sep}(P)$  implies

$$\text{sep}(P) \ll |\bar{P}|^{1-w} \tag{26}$$

(possibly with another constant in  $\ll$ ).

Indeed, assume that  $\alpha, \beta, \gamma$  are the roots of  $P$  satisfying  $|\alpha| \leq |\beta| \leq |\gamma|$ . As  $\mathcal{R}(P)$  tends to infinity (there are only finitely many monic integer polynomials with  $\mathcal{R}(P)$  bounded),  $\text{Sep}(P)$  tends to zero; so let us consider only those  $P$  for which  $\text{Sep}(P) \leq 1/2$ . Evidently,  $\text{Sep}(P)$  is one of the numbers  $|1 - \alpha/\beta|$ ,  $|1 - \beta/\gamma|$  or  $|1 - \alpha/\gamma|$ .

In the first case,  $\text{Sep}(P) = |1 - \alpha/\beta|$ , using  $\text{sep}(P) \leq |\beta - \alpha| = |\beta|\text{Sep}(P)$ ,  $|\beta| \leq |\gamma|$  and  $w < 3$  we obtain

$$|\gamma|^{w-1}\text{sep}(P) \leq |\gamma|^{w-1}|\beta|\text{Sep}(P) \leq |\gamma|^{2w/3}|\beta|^{w/3}\text{Sep}(P) = \mathcal{R}(P)^{2w/3}\text{Sep}(P) \ll 1.$$

In the second case,  $\text{Sep}(P) = |1 - \beta/\gamma|$ , from  $\text{Sep}(P) \leq 1/2$  it follows that  $|\beta/\gamma| \geq 1/2$ , hence  $|\beta| \geq |\gamma|/2$ . Similarly, in the third case,  $\text{Sep}(P) = |1 - \alpha/\gamma|$ , we obtain  $|\alpha| \geq |\gamma|/2$ , so  $|\beta| \geq |\alpha| \geq |\gamma|/2$ . Therefore, in these two cases we have  $|\gamma|^{3/2} \ll |\gamma||\beta|^{1/2} = \mathcal{R}(P)$ , i.e.  $|\gamma| \ll \mathcal{R}(P)^{2/3}$ . From  $\text{sep}(P) \leq |\gamma|\text{Sep}(P)$  we conclude that

$$|\gamma|^{w-1}\text{sep}(P) \leq |\gamma|^w\text{Sep}(P) \ll \mathcal{R}(P)^{2w/3}\text{Sep}(P) \ll 1,$$

which gives (26) again.

Next, let us replace  $P(x)$  by  $P(x - a/3)$ . This does not change either  $\text{sep}(P)$  or  $\Delta(P)$ . If  $\alpha, \beta, \gamma$  were the roots of  $P(x) = x^3 + ax^2 + bx + c$  satisfying  $|\alpha| \leq |\beta| \leq |\gamma|$  (so that  $\alpha + \beta + \gamma = -a$ , and hence  $3|\gamma| \geq |a|$ ) then the roots of  $P(x - a/3)$  are  $\alpha + a/3, \beta + a/3, \gamma + a/3$ . The modulus of the largest of those three does not exceed  $|\gamma| + |a|/3 \leq 2|\gamma| = 2|\bar{P}|$ , so this change may increase the value of  $|\bar{P}|$  at most twice. It follows that (26) holds for infinitely many monic cubic irreducible polynomials

$$P(x) = (x - a/3)^3 + a(x - a/3)^2 + b(x - a/3) + c = x^3 - (a^2/3 - b)x - (ab/3 - c - 2a^3/27).$$

Since  $6|a, b, c$ , we can write  $P$  in the form  $P(x) = x^3 - 3px - 2q \in \mathbb{Z}[x]$  with integers  $p := (a^2/3 - b)/3$ ,  $q := (ab/3 - c - 2a^3/27)/2$  and with the roots  $\alpha, \beta, \gamma$  satisfying  $|\alpha| \leq |\beta| \leq |\gamma|$ .

Now, since  $\gamma$  has the largest modulus among three roots satisfying  $\alpha + \beta + \gamma = 0$  and  $\text{sep}(P) \rightarrow 0$ , we must have  $\text{sep}(P) = |\alpha - \beta|$  and so  $\alpha, \beta$  tend to  $-\gamma/2$ . In particular, this implies  $2q = \alpha\beta\gamma \geq \gamma^3/5$ , so  $\gamma \ll q^{1/3}$ . Hence from  $\Delta(P) = 108(p^3 - q^2)$  using (26) and the irreducibility of  $P$  we find that

$$0 < \sqrt{108|p^3 - q^2|} = \sqrt{|\Delta(P)|} = |\alpha - \beta||\alpha - \gamma||\beta - \gamma| \ll \text{sep}(P)|\gamma|^2 \ll |\gamma|^{3-w} \ll q^{1-w/3}.$$

So the inequality  $0 < |p^3 - q^2| \ll q^{2-2w/3}$  has infinitely many solutions  $(p, q) \in \mathbb{N}^2$ . This implies the result in view of  $q^{2-2w/3} \ll (p^{3/2})^{2-2w/3} = p^{3-w}$ . □

**4. Proof of Theorems 1 and 2**

*Proof of Theorem 1.* To give a short proof of (5) we assume that  $\text{Sep}(P) = |1 - \alpha_l/\alpha_k|$  with  $k < l$ . Let us subtract the  $l^{\text{th}}$  column of the determinant  $\det(\alpha_j^{i-1})_{1 \leq i, j \leq d}$  from its  $k^{\text{th}}$  column. The element  $i \times k$  of the resulting determinant is equal to  $\alpha_k^{i-1} - \alpha_l^{i-1}$ . Taking out the factor  $1 - \alpha_l/\alpha_k$  out of each element of the  $k^{\text{th}}$  column we obtain

$$\det(\alpha_j^{i-1})_{1 \leq i, j \leq d} = (1 - \alpha_l/\alpha_k)\det(a_{ij}\alpha_j^{i-1})_{1 \leq i, j \leq d},$$

where  $a_{ij} := 1$  for  $j \neq k$  and  $a_{ik} := \alpha_k^{2-i}(\alpha_k^{i-1} - \alpha_l^{i-1})/(\alpha_k - \alpha_l)$ , because the element  $i \times k$  becomes

$$\frac{\alpha_k^{i-1} - \alpha_l^{i-1}}{1 - \alpha_l/\alpha_k} = \frac{(\alpha_k^{i-1} - \alpha_l^{i-1})\alpha_k^{i-1}}{(\alpha_k - \alpha_l)\alpha_k^{i-2}} = a_{ik}\alpha_k^{i-1}.$$

In particular,  $a_{1k} = 0$  and

$$|a_{ik}| = |1 + \alpha_l/\alpha_k + \dots + (\alpha_l/\alpha_k)^{i-2}| \leq 1 + |\alpha_l/\alpha_k| + \dots + |(\alpha_l/\alpha_k)^{i-2}| \leq i - 1$$

for  $i = 2, \dots, d$ , since  $|\alpha_l| \leq |\alpha_k|$ . Thus, by (6),

$$\begin{aligned} \prod_{j=1}^d \left( \sum_{i=1}^d |a_{ij}|^2 \right)^{1/2} &\leq d^{(d-1)/2} \sqrt{1^2 + \dots + (d-1)^2} = d^{(d-1)/2} (d(d-1)(2d-1)/6)^{1/2} \\ &= d^{d/2+1} \sqrt{(1-1/d)(1-1/2d)}/\sqrt{3} = 1/c_d. \end{aligned}$$

Therefore, applying (2), we obtain

$$\begin{aligned} \sqrt{|\Delta(P)|} &= |a_d|^{d-1} |\det(\alpha_j^{i-1})_{1 \leq i, j \leq d}| = |a_d|^{d-1} \text{Sep}(P) |\det(a_{ij}\alpha_j^{i-1})_{1 \leq i, j \leq d}| \\ &< \text{Sep}(P)\mathcal{R}(P)^{d-1}/c_d, \end{aligned}$$

giving (5). □

*Proof of Theorem 2.* Assume that  $\text{Sep}(P) = |1 - \alpha_2/\alpha_1|$ . (The proof in two other cases is the same.)

Then

$$\frac{\sqrt{|\Delta(P)|}}{\text{Sep}(P)\mathcal{R}(P)^2} = \frac{|\alpha_1 - \alpha_2||\alpha_1 - \alpha_3||\alpha_2 - \alpha_3||\alpha_1|}{|\alpha_1 - \alpha_2||\alpha_1|^2|\alpha_2|} = |1 - \alpha_3/\alpha_1||1 - \alpha_3/\alpha_2|.$$

Since  $|1 - \alpha_3/\alpha_1| \leq 1 + |\alpha_3/\alpha_1| \leq 2$  and  $|1 - \alpha_3/\alpha_2| \leq 2$ , their product does not exceed 4. Furthermore, it is equal to 4 only if  $\alpha_3/\alpha_1 = \alpha_3/\alpha_2 = -1$ , which is impossible, because  $\alpha_1 \neq \alpha_2$ . Hence  $\sqrt{|\Delta(P)|}/\text{Sep}(P)\mathcal{R}(P)^2 < 4$ , giving (7).

To prove the lower bound (8), let us consider the polynomials

$$P_t(x) := (x + pt)(x - pt)^2 - p = (x - \alpha_t)(x - \beta_t)(x - \gamma_t),$$

where  $p$  is a fixed prime number and  $t$  runs through positive integers. By Eisenstein's criterion, the polynomial  $P_t$  is irreducible for each  $t \in \mathbb{N}$ . By Lemma 6, we have  $\alpha_t \sim -pt$  and  $\beta_t, \gamma_t \sim pt$  as  $t \rightarrow \infty$ . Furthermore, inserting  $x = pt + y/\sqrt{t}$  into  $P_t(x) = 0$  we find that

$$y^3 t^{-3/2} + 2p(y^2 - 1/2) = 0.$$

Hence Lemma 6 implies  $\beta_t - pt \sim -1/\sqrt{2t}$  and  $\gamma_t - pt \sim 1/\sqrt{2t}$  as  $t \rightarrow \infty$ . It follows that  $\beta_t - \gamma_t \sim \sqrt{2/t}$ ,

$$\text{Sep}(P_t) \sim \frac{\sqrt{2}}{pt^{3/2}}, \quad \mathcal{R}(P_t) \sim p^{3/2}t^{3/2} \quad \text{and} \quad \sqrt{|\Delta(P_t)|} \sim 4\sqrt{2}p^2t^{3/2}$$

as  $t \rightarrow \infty$ . Consequently,  $\text{Sep}(P_t)\mathcal{R}(P_t)^2/\sqrt{|\Delta(P_t)|} \rightarrow 1/4$  as  $t \rightarrow \infty$ . This completes the proof of (8).  $\square$

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