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Polynomial root separation in terms of the Remak height

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Abstract: We investigate some monic integer irreducible polynomials which have two close roots. If $P(x)$ is a separable polynomial in $\mathbb{Z}[x]$ of degree $d \geqslant 2$ with the Remak height $\mathcal{R}(P)$ and the minimal distance between the quotient of two distinct roots and unity $\mathrm{Sep}(P)$, then the inequality $1/\mathrm{Sep}(P) \ll \mathcal{R}(P)^{d-1}$ is true with the implied constant depending on d only. Using a recent construction of Bugeaud and Dujella we show that for each $d \geqslant 3$ there exists an irreducible monic polynomial $P \in \mathbb{Z}[x]$ of degree d for which $\mathcal{R}(P)^{(2d-3)(d-1)/(3d-5)} \ll 1/\mathrm{Sep}(P)$. For $d=3$ the exponent $3/2$ is improved to $5/3$ and it is shown that the exponent 2 is optimal in the class of cubic (not necessarily monic) irreducible polynomials in $\mathbb{Z}[x]$.

Key words: Polynomial root separation, Mahler's measure, Remak height, discriminant

1. Introduction

Let

$$P(x) := a_d x^d + \dots + a_1 x + a_0 = a_d(x-\alpha_1)\dots(x-\alpha_d) \in \mathbb{C}[x], \quad a_d, a_0 \neq 0,$$

be a separable polynomial of degree $d \geqslant 2$. Throughout, let

$$\Delta(P) := a_d^{2d-2} \prod_{1 \leqslant i < j \leqslant d} (\alpha_i - \alpha_j)^2$$

be its *discriminant*,

$$H(P) := \max_{1 \leqslant j \leqslant d} |a_j|$$

its *height*,

$$M(P) := |a_d| \prod_{j=1}^{d} \max(1, |\alpha_j|)$$

its *Mahler measure* and

$$\mathcal{R}(P) := |a_d| \prod_{j=1}^{d} |\alpha_j|^{(d-j)/(d-1)},$$

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where $\alpha_1, \dots, \alpha_d$ are labeled so that $|\alpha_1| \geqslant |\alpha_2| \geqslant \dots \geqslant |\alpha_d|$, its *Remak height*. The last quantity in the context of polynomials first appeared in the paper of Remak [21] who proved the inequality

$$\sqrt{|\Delta(P)|} \leqslant d^{d/2} \mathcal{R}(P)^{d-1}. \tag{1}$$

This quantity also appears in [15], [20], [24] and is studied in detail in [9], [10], where it is named after Remak. In [8], it is shown that if $a_{ij} \in \mathbb{C}$ for $1 \leqslant i, j \leqslant d$ and the complex numbers z_j satisfy $|z_1| \geqslant |z_2| \geqslant \dots \geqslant |z_d|$, then

$$|\det(a_{ij} z_j^{i-1})_{1 \leqslant i,j \leqslant d}| \leqslant |z_1|^{d-1} |z_2|^{d-2} \dots |z_{d-1}| \prod_{j=1}^{d} \left(\sum_{i=1}^{d} |a_{ij}|^2 \right)^{1/2}. \tag{2}$$

This implies both (1) and Hadamard's inequality.

Note that in view of

$$\sqrt{M(P) \min(|a_d|, |a_0|)} \leqslant \mathcal{R}(P) \leqslant M(P) \tag{3}$$

(see [10]) the inequality (1) is at least as good as Mahler's inequality

$$\sqrt{|\Delta(P)|} \leqslant d^{d/2} M(P)^{d-1}.$$

In [16] Mahler also proved that

$$\operatorname{sep}(P) > \frac{\sqrt{3|\Delta(P)|}}{d^{d/2+1} M(P)^{d-1}}, \tag{4}$$

where

$$\operatorname{sep}(P) := \min_{i \neq j} |\alpha_i - \alpha_j|$$

is the minimal distance between two distinct roots of P. After the paper of Mahler various aspects of polynomial root separation have been investigated in [1]–[5], [7], [11]–[13], [18]–[20], [22].

In fact, in (4) one cannot replace $M(P)$ by $\mathcal{R}(P)$ (see the first example in Section 2 below), but instead finds the following.

Theorem 1 *For each $d \geqslant 2$ and each polynomial $P \in \mathbb{C}[x]$ of degree d, $P(0) \neq 0$, we have*

$$\operatorname{Sep}(P) > \frac{c_d \sqrt{|\Delta(P)|}}{\mathcal{R}(P)^{d-1}}, \tag{5}$$

where $\operatorname{Sep}(P) := \min_{i \neq j} |1 - \alpha_j/\alpha_i|$ *and*

$$c_d := \frac{\sqrt{3}}{d^{d/2+1} \sqrt{(1-1/d)(1-1/2d)}}. \tag{6}$$

The inequality (5) is due to Mignotte [19] (see also [7]). We shall give its short proof based on (2) in Section 4.

Note that for $d = 2$ we have

$$\operatorname{Sep}(P) = \frac{\sqrt{|\Delta(P)|}}{\mathcal{R}(P)},$$

which is better than (5). For $d = 3$ the constant $c_3 = 1/3\sqrt{5} = 0.14907\dots$ given in (6) can be improved to $1/4$. Furthermore, as in [22], the latter constant is best possible even if we restrict to the class of monic irreducible polynomials in $\mathbb{Z}[x]$.

Theorem 2 *If $P(x) \in \mathbb{C}[x]$ is a separable cubic polynomial, $P(0) \neq 0$, then*

$$\mathrm{Sep}(P) > \frac{\sqrt{|\Delta(P)|}}{4\mathcal{R}(P)^2}. \tag{7}$$

Furthermore, for each $\varepsilon > 0$ there is a monic cubic irreducible polynomial $P(x) \in \mathbb{Z}[x]$ for which

$$\mathrm{Sep}(P) < (1+\varepsilon)\frac{\sqrt{|\Delta(P)|}}{4\mathcal{R}(P)^2}. \tag{8}$$

Note that, inequality (5) (unlike (4)) is symmetric with respect to the map $x \mapsto 1/x$ in the sense that we can replace $P(x)$ by its reciprocal polynomial $P^*(x) = \pm x^d P(1/x)$. Then $|\Delta(P)| = |\Delta(P^*)|$ and $\mathcal{R}(P) = \mathcal{R}(P^*)$, by Prop. 3.3 in [10]. Furthermore, $\mathrm{Sep}(P)$ is the minimal number among the following $d(d-1)/2$ real numbers

$$|1-\alpha_2/\alpha_1|, |1-\alpha_3/\alpha_1|, \dots, |1-\alpha_d/\alpha_{d-1}|,$$

because $|\alpha_1| \geqslant \dots \geqslant |\alpha_d|$ implies $|1-\alpha_i/\alpha_j| \geqslant |1-\alpha_j/\alpha_i|$ for $i < j$. So is also $\mathrm{Sep}(P^*)$, since the roots of P^* are $1/\alpha_d, \dots, 1/\alpha_1$. Hence $\mathrm{Sep}(P) = \mathrm{Sep}(P^*)$. Of course, $\mathrm{sep}(P)$ and $\mathrm{sep}(P^*)$ can be different.

Below, when the degree of P, i.e., d will be fixed, we shall write $u \ll v$ for positive quantities u, v if the inequality $u \leqslant cv$ holds with some constant $c = c(d)$ depending on d only. With this notation, one has

$$H(P) \leqslant 2^d M(P) \ll M(P) \leqslant \sqrt{\sum_{j=0}^{d} |a_j|^2} \leqslant \sqrt{(d+1)}H(P) \ll H(P), \tag{9}$$

so $H(P)$ and $M(P)$ are of the same size. Hence, for a separable polynomial $P(x) \in \mathbb{Z}[x]$ of degree d, from (4), (9) and (5) using $|\Delta(P)| \geqslant 1$ we find that

$$1/\mathrm{sep}(P) \ll H(P)^{d-1} \quad \text{and} \quad 1/\mathrm{Sep}(P) \ll \mathcal{R}(P)^{d-1}. \tag{10}$$

To investigate how sharp is the exponent $d-1$ in the first inequality of (10) the quantity

$$e_{\mathrm{irr}}(d) := \limsup_{H(P)\to\infty} \frac{\log(1/\mathrm{sep}(P))}{\log H(P)},$$

where the limsup is taken over all integer irreducible polynomials P of degree d, is introduced. Of course, by the first inequality of (10), it satisfies $e_{\mathrm{irr}}(d) \leqslant d-1$. A similar quantity, where the polynomial P is, in addition, monic, is denoted by $e^*_{\mathrm{irr}}(d)$. Clearly,

$$e^*_{\mathrm{irr}}(d) \leqslant e_{\mathrm{irr}}(d) \leqslant d-1.$$

It is straightforward that $e_{\rm irr}(2) = 1$ and $e^*_{\rm irr}(2) = 0$. It is also known that $e_{\rm irr}(3) = 2$ (see [12], [22]). The lower bounds for $e_{\rm irr}(d)$, $d \geqslant 4$, and for $e^*_{\rm irr}(d)$, $d \geqslant 3$, have been obtained in [1]–[4]. Currently, the best bound on $e_{\rm irr}(d)$ for each $d \geqslant 4$ is due to Bugeaud and Dujella [2]

$$e_{\rm irr}(d) \geqslant \frac{d}{2} + \frac{d-2}{4(d-1)}.$$

As for $e^*_{\rm irr}(d)$, their example gives the lower bound

$$e^*_{\rm irr}(d) \geqslant \frac{d}{2} + \frac{d-2}{4(d-1)} - 1$$

for $d \geqslant 7$, but for $d = 3, 5$ and $d \geqslant 4$ even, the best bounds are due to Bugeaud and Mignotte [4]

$$e^*_{\rm irr}(3) \geqslant 3/2, \quad e^*_{\rm irr}(5) \geqslant 7/4 \quad \text{and} \quad e^*_{\rm irr}(d) \geqslant (d-1)/2,$$

respectively.

By (9), the height $H(P)$ and the Mahler measure $M(P)$ are essentially of the same size, so we will not get anything new by considering a corresponding quantity with $M(P)$ in place of $H(P)$. However, by (3), the Remak height $\mathcal{R}(P)$ can be significantly smaller, i.e., $\sqrt{H(P)} \ll \mathcal{R}(P) \ll H(P)$. So one can study

$$g_{\rm irr}(d) := \limsup_{\mathcal{R}(P) \to \infty} \frac{\log(1/\mathrm{Sep}(P))}{\log \mathcal{R}(P)}$$

(resp. $g^*_{\rm irr}(d)$), where the limsup is taken over all (resp. all monic) integer irreducible polynomials P of degree d. Now, by the second inequality of (10), we obtain

$$g^*_{\rm irr}(d) \leqslant g_{\rm irr}(d) \leqslant d - 1$$

for each $d \geqslant 2$.

A simple example,

$$x^2 - (2t+1)x + t^2 + t - 1 = \left(x - t - \frac{1+\sqrt{5}}{2}\right)\left(x - t - \frac{1-\sqrt{5}}{2}\right)$$

with $t \in \mathbb{N}$ tending to infinity, shows that $g^*_{\rm irr}(2) \geqslant 1$, hence

$$g_{\rm irr}(2) = g^*_{\rm irr}(2) = 1.$$

For $d \geqslant 3$, by a construction based on the example of Bugeaud and Dujella [2], we can come closer to the upper bound $d-1$ with the quantity $g^*_{\rm irr}(d)$ compared to the quantities $e_{\rm irr}(d)$ and $e^*_{\rm irr}(d)$.

Theorem 3 *We have*

$$g^*_{\rm irr}(d) \geqslant \frac{(2d-3)(d-1)}{3d-5}$$

for each $d \geqslant 3$.

The next theorem sharpens the inequality of this theorem for $d = 3$ and evaluates the corresponding quantity for not necessarily monic polynomials.

Theorem 4 *We have* $g_{\mathrm{irr}}(3) = 2$ *and* $g^*_{\mathrm{irr}}(3) \geqslant 5/3$*.*

Clearly, for monic polynomials P of degree d we have

$$\mathcal{R}(P) \leqslant |\overline{P}|^{d/2},$$

where $|\overline{P}| := \max_{\alpha : P(\alpha)=0} |\alpha|$ is the *house* of P. Thus (10) implies

$$1/\mathrm{Sep}(P) \ll |\overline{P}|^{d(d-1)/2}$$

for monic integer separable polynomials P of degree d. In the opposite direction we prove the following.

Theorem 5 *For each* $d \geqslant 4$ *there are infinitely many monic integer irreducible polynomials* $P \in \mathbb{Z}[x]$ *of degree* d *for which* $|\overline{P}|^{d(d-2)/4} \ll 1/\mathrm{Sep}(P)$*. Furthermore, there are infinitely many monic cubic integer irreducible polynomials* $P \in \mathbb{Z}[x]$ *for which* $|\overline{P}|^{5/2} \ll 1/\mathrm{Sep}(P)$*.*

For monic cubic polynomials we have $\mathcal{R}(P)^{5/3} \leqslant |\overline{P}|^{5/2}$, and so Theorem 5 implies the inequality $g^*_{\mathrm{irr}}(3) \geqslant 5/3$ of Theorem 4. In fact, by Proposition 7 below, the equality $g^*_{\mathrm{irr}}(3) = 5/3$ holds (and also the constant $5/2$ in Theorem 5 is optimal) if and only if Hall's conjecture [14] (asserting that there is an absolute constant $c > 0$ such that the Diophantine inequality $0 < |x^3 - y^2| < c\sqrt{x}$ has no solutions in positive integers) is true. A corresponding result for the equality $e^*_{\mathrm{irr}}(3) = 3/2$ is given in [4].

In Section 2 we give some examples (introduced in [16], [18], [2] or their variations) and prove the first statement of Theorem 5 and Theorem 3. In Section 3 prove Theorem 4 and the second statement of Theorem 5. Finally, in Section 4 we will prove Theorems 1 and 2.

2. Three examples

The following lemma is well known (see [17] or [23]).

Lemma 6 *Suppose* λ *is a root of the polynomial* $x^d + \sum_{i=0}^{d-1} c_i x^i$ *of multiplicity* m *and* $\varepsilon > 0$*. Then for* $|c_i - c_i'|$*,* $i = 0, \dots, d-1$*, sufficiently small the polynomial* $x^d + \sum_{i=0}^{d-1} c_i' x^i$ *has exactly* m *roots within* ε *of* λ*.*

As an illustration of his results in [16] Mahler considered the polynomial $x^d - 1$. Let us consider the polynomial

$$S_t(x) := x^d - t,$$

where t is a positive integer such that S_t is irreducible. (For instance, t can be a prime number.) Since $\alpha_j = e^{2\pi i(j-1)/d} t^{1/d}$ for each $j = 1, \dots, d$, we have

$$\mathcal{R}(S_t) = t^{1/2}, \quad M(S_t) = H(S_t) = t,$$

$$\sqrt{|\Delta(S_t)|} = d^{d/2} t^{(d-1)/2},$$

$$\mathrm{sep}(S_t) = 2\sin(\pi/d)t^{1/d}, \quad \mathrm{Sep}(S_t) = 2\sin(\pi/d).$$

Hence

$$\frac{\mathrm{Sep}(S_t)\mathcal{R}(S_t)^{d-1}d^{d/2+1}}{\sqrt{|\Delta(S_t)|}} = 2\sin(\pi/d)d < 2\pi.$$

In particular, the constant $\sqrt{3}$ in (6) cannot be replaced by the constant 2π. Moreover, from $\mathcal{R}(S_t^*) = \mathcal{R}(S_t) = t^{1/2}$, $\sqrt{|\Delta(S_t^*)|} = \sqrt{|\Delta(S_t)|} = d^{d/2}t^{(d-1)/2}$ and $\mathrm{sep}(S_t^*) = 2\sin(\pi/d)t^{-1/d}$ we deduce that

$$\frac{\mathrm{sep}(S_t^*)\mathcal{R}(S_t^*)^{d-1}}{\sqrt{|\Delta(S_t^*)|}} = \frac{2\sin(\pi/d)}{d^{d/2}t^{1/d}} < \varepsilon$$

for t large enough, so one cannot replace $M(P)$ by $\mathcal{R}(P)$ in (4).

The next example is due to Mignotte [18]. Fix a prime number p and consider the monic polynomial

$$Q_t(x) := x^d - p(tx-1)^2 \in \mathbb{Z}[x],$$

where t is a sufficiently large positive integer. This polynomial is irreducible, by Eisenstein's criterion. We claim that this polynomial has $d-2$ 'large' roots $\alpha_1, \dots, \alpha_{d-2}$ satisfying

$$\alpha_j \sim e^{2\pi i(\tau(j)-1)/(d-2)}p^{1/(d-2)}t^{2/(d-2)} \quad \text{as} \quad t \to \infty, \tag{11}$$

where τ is a permutation of the set $\{1, 2, \dots, d-2\}$, and two 'small' positive roots $\alpha_{d-1} > \alpha_d$ satisfying

$$\alpha_{d-1} - \frac{1}{t} \sim \frac{1}{\sqrt{p}t^{d/2+1}}, \quad \alpha_d - \frac{1}{t} \sim -\frac{1}{\sqrt{p}t^{d/2+1}} \quad \text{as} \quad t \to \infty. \tag{12}$$

Indeed, setting $x := t^{2/(d-2)}y$ into $Q_t(x) = 0$ and multiplying by $t^{-2d/(d-2)}$, we obtain

$$y^d - py^2 + 2pt^{-d/(d-2)}y - pt^{-2d/(d-2)} = 0,$$

so Lemma 6 implies (11). On the other hand, writing the root of Q_t in the form $x := (yt^{-d/2}+1)/t$, we find that

$$0 = t^d Q_t((yt^{-d/2}+1)/t) = (yt^{-d/2}+1)^d - py^2,$$

so, by Lemma 6, y is close to $\pm 1/\sqrt{p}$ when t is large. This proves (12).

From $\mathcal{R}(Q_t)^{d-1} = |\alpha_1|^{d-1}|\alpha_2|^{d-2} \dots |\alpha_{d-2}|^2|\alpha_{d-1}|$, using (11), (12), in view of

$$\frac{2}{d-2}\big(d-1+d-2+\dots+2\big) - 1 = \frac{2}{d-2}\left(\frac{(d-1)d}{2} - 1\right) - 1 = d$$

we obtain

$$\mathcal{R}(Q_t)^{d-1} \sim p^{(d+1)/2}t^d \quad \text{as} \quad t \to \infty$$

and also

$$\mathrm{Sep}(Q_t) = \frac{\alpha_{d-1} - \alpha_d}{\alpha_{d-1}} \sim \frac{2}{\sqrt{p}t^{d/2}} \quad \text{as} \quad t \to \infty. \tag{13}$$

Therefore,

$$\frac{\log(1/\mathrm{Sep}(Q_t))}{\log \mathcal{R}(Q_t)} \to \frac{d/2}{d/(d-1)} = \frac{d-1}{2}$$

as $t \to \infty$.

In particular, this example yields the bound $g^*_{\mathrm{irr}}(d) \geqslant (d-1)/2$. Furthermore, combining $|\overline{Q_t}| \sim p^{1/(d-2)}t^{2/(d-2)}$ with (13) we see that $|\overline{Q_t}|^{d(d-2)/4} \ll 1/\mathrm{Sep}(Q_t)$. This proves the first statement of Theorem 5.

The next construction is essentially due to Bugeaud and Dujella [2]. Let

$$C_k := \frac{1}{k+1}\binom{2k}{k}, \quad k = 0, 1, 2, \ldots,$$

be the k^{th} Catalan number. The Catalan numbers for $k = 0, 1, 2, \ldots$ are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, \ldots.$$

It is well known that

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \tag{14}$$

and that the generating function of Catalan's numbers

$$c(x) := \sum_{k=0}^{\infty} C_k x^k$$

satisfies

$$c(x) - 1 = c(x)^2 x.$$

We next replace $c(x)$ in the equality $x^{-1} + c(x)(-x^{-1} + c(x)) = 0$, $x \neq 0$, by its truncated series and introduce a new parameter t. More precisely, for integers $d \geqslant 2$ and $t \geqslant 1$ consider the Laurent polynomial

$$G_t(x) := \frac{1}{x} + \Big(\sum_{k=0}^{d-2} C_k x^k + \frac{x^{d-1}}{t}\Big)\Big(-\frac{1}{x} + \sum_{k=0}^{d-2} C_k x^k + \frac{x^{d-1}}{t}\Big). \tag{15}$$

Note that the coefficient for x^{-1} in $G_t(x)$ is zero, because $C_0 = 1$. The coefficient for x^n, where $0 \leqslant n \leqslant d-3$, in $G_t(x)$ is equal to

$$-C_{n+1} + C_n C_0 + C_{n-1} C_1 + \cdots + C_0 C_n,$$

which is zero again in view of (14). Consequently,

$$F_t(x) := \frac{t^2}{x^{d-2}} G_t(x) = x^d + 2tC_{d-2}x^{d-1} + \sum_{k=0}^{d-2} a_k(t) x^k \tag{16}$$

is a monic polynomial of degree d with integer coefficients. Here,

$$a_k(t) = 2C_{k-1}t + t^2 \sum_{j=k}^{d-2} C_j C_{d-2+k-j} \tag{17}$$

for $k = 1, \ldots, d-2$ and

$$a_0(t) = -t + t^2 \sum_{j=0}^{d-2} C_j C_{d-2-j} = -t + C_{d-1} t^2. \tag{18}$$

The monic polynomial $F_t(x)$ of degree d is irreducible if, say, t is a prime number. By Lemma 6, (17) and (18), as $t \to \infty$, the polynomial $F_t(x)$ has $d-2$ roots $\alpha_3, \ldots, \alpha_d$ tending to $d-2$ roots of the polynomial

$$C_{d-1} + \sum_{k=1}^{d-2} x^k \sum_{j=k}^{d-2} C_j C_{d-2+k-j} = (x - \lambda_3) \ldots (x - \lambda_d).$$

(In principle, $\lambda_3, \ldots, \lambda_d$ are not necessarily distinct, although in all examples with small d they are distinct.)

Let ξ be the root of the polynomial

$$E_t(x) := t \sum_{k=0}^{d-2} C_k x^k + x^{d-1}$$

satisfying

$$\xi \sim -t C_{d-2} \quad \text{as} \quad t \to \infty. \tag{19}$$

Applying the mean value theorem to the function $E_t(x)$ in the interval $[\xi, \xi + \theta C_{d-2}^{3/2-d} t^{5/2-d}]$, where $\theta \in \mathbb{R}$ is fixed, in view of $E_t(\xi) = 0$ and (19) we obtain

$$E_t(\xi + \theta C_{d-2}^{3/2-d} t^{5/2-d}) \sim \theta C_{d-2}^{3/2-d} t^{5/2-d} ((d-1)\xi^{d-2} + (d-2) C_{d-2} t \xi^{d-3}) \sim (-1)^d \theta \sqrt{\frac{t}{C_{d-2}}}$$

as $t \to \infty$. Now, by (15) and (16),

$$F_t(x) x^{d-1} = t^2 x G_t(x) = t^2 x \Big(\frac{1}{x} + \frac{E_t(x)}{t} \Big(-\frac{1}{x} + \frac{E_t(x)}{t} \Big) \Big) = t^2 - t E_t(x) + x E_t(x)^2.$$

Let us insert the root x of F_t written in the form $x = \xi + \theta C_{d-2}^{3/2-d} t^{5/2-d}$ into $1 - E_t(x) t^{-1} + x t^{-2} E_t(x)^2 = 0$. By the above, we see that the left hand side tends to $1 - \theta^2$ as $t \to \infty$. Hence θ tends to 1 and -1, so that the remaining two roots α_1, α_2 of $F_t(x)$ satisfy

$$\alpha_1 - \xi \sim -C_{d-2}^{3/2-d} t^{5/2-d} \quad \text{and} \quad \alpha_2 - \xi \sim C_{d-2}^{3/2-d} t^{5/2-d}. \tag{20}$$

We are now in a position to prove Theorem 3. Set $t := pk^d$ with a prime number p and a positive integer k and consider the polynomial $P_k(x) := F_{pk^d}(kx) k^{-d}$, where $F_t(x)$ is defined in (16). By (17), (18) and the Eisenstein criterion applied to p, we see that P_k is a monic irreducible polynomial of degree d. Its roots are $\beta_j = \alpha_j / k$, $j = 1, \ldots, d$, where α_j are the roots of F_t. Since $t = pk^d$, from (19) and (20) we derive that $\beta_1, \beta_2 \sim -p C_{d-2} k^{d-1}$ and

$$\beta_2 - \beta_1 \sim 2 C_{d-2}^{3/2-d} p^{5/2-d} k^{-d^2 + 5d/2 - 1}$$

as $k \to \infty$. Thus

$$\mathrm{Sep}(P_k) \leqslant |1 - \beta_1/\beta_2| \sim 2 p^{3/2-d} C_{d-2}^{1/2-d} k^{-d^2 + 3d/2}. \tag{21}$$

Since $\beta_j \sim \lambda_j k^{-1}$ as $k \to \infty$ for $j = 3, \dots, d$, in view of

$$(d-1)(d-1+d-2) - (d-3+d-2+\cdots+1) = (3d-5)d/2,$$

we find that

$$k^{(3d-5)d/2(d-1)} \ll \mathcal{R}(P_k) = |\beta_1||\beta_2|^{(d-2)/(d-1)} \dots |\beta_{d-1}|^{1/(d-1)} \ll k^{(3d-5)d/2(d-1)}. \tag{22}$$

Now, since $\mathcal{R}(P_k) \to \infty$ as $k \to \infty$, combining (21) with (22) we find that

$$g^*_{\mathrm{irr}}(d) \geqslant \frac{d^2 - 3d/2}{(3d-5)d/(2(d-1))} = \frac{(2d-3)(d-1)}{3d-5}.$$

This completes the proof of Theorem 3.

3. Proof of Theorem 4

Our proof of $g_{\mathrm{irr}}(3) = 2$ follows [22]. Let us begin, for example, with the polynomial

$$P(x) := x^3 - x - 1 = (x-\alpha)(x-\beta)(x-\gamma),$$

where $\alpha = 1.32471\dots$ and $\beta = -0.66235\dots + i0.56227\dots$, $\gamma = -0.66235\dots - i0.56227\dots$ are two complex conjugate roots satisfying

$$|\beta| = |\gamma| < 1 \quad \text{and} \quad \Re(\beta) = \Re(\gamma) < 0.$$

Consider the sequence $\alpha_1 := \alpha$ and

$$\alpha_{k+1} := 1/\{\alpha_k\} \quad \text{for} \quad k = 1, 2, 3, \dots.$$

Then $\alpha_k > 1$ and $\alpha_k \in \mathbb{Q}(\alpha)$ for each $k \in \mathbb{N}$. Setting $\beta_1 := \beta$, $\gamma_1 := \gamma$ and $q_k := [\alpha_k] \in \mathbb{N}$ (so that $\alpha_{k+1} = 1/(\alpha_k - q_k)$), we also define two corresponding sequences

$$\beta_{k+1} = 1/(\beta_k - q_k) \quad \text{and} \quad \gamma_{k+1} = 1/(\gamma_k - q_k)$$

for $k = 1, 2, 3, \dots$. Note that, by the above construction, the continued fraction expansion for the cubic number α_k is

$$\alpha_k = q_k + \cfrac{1}{q_{k+1} + \frac{1}{q_{k+2}+\cdots}} \tag{23}$$

for each $k \in \mathbb{N}$.

It is easy to see that the 'next' polynomial $P_k(x)$ obtained from $P_{k-1}(x)$, firstly, by replacing $P_{k-1}(x)$ by $P_{k-1}(x + q_{k-1})$ and then, secondly, by taking its reciprocal polynomial, namely,

$$P_k(x) = P^*_{k-1}(x + q_{k-1}) = a_k(x - \alpha_k)(x - \beta_k)(x - \gamma_k) \in \mathbb{Z}[x], \quad a_k \in \mathbb{N},$$

is irreducible, since so is $P_{k-1}(x)$. Furthermore, it is clear that

$$\sqrt{|\Delta(P_k)|} = \sqrt{|\Delta(P_{k-1})|} = \cdots = \sqrt{|\Delta(P)|} = \sqrt{23}.$$

It is straightforward to check that for each $k \in \mathbb{N}$ the roots β_k and $\gamma_k = \overline{\beta_k}$ satisfy

$$|\beta_k| = |\gamma_k| < 1 \quad \text{and} \quad \Re(\beta_k) = \Re(\gamma_k) < 0.$$

Consequently, $|\alpha_k - \beta_k| = |\alpha_k - \gamma_k| > \alpha_k$, and so

$$\sqrt{23} = a_k^2|\alpha_k - \beta_k||\alpha_k - \gamma_k||\beta_k - \gamma_k| > a_k^2\alpha_k^2|\beta_k||1 - \gamma_k/\beta_k| \geqslant \mathcal{R}(P_k)^2\mathrm{Sep}(P_k). \tag{24}$$

If the sequences $a_k \in \mathbb{N}$ and α_k, $k = 1, 2, 3, \dots$, were both bounded from above then, as $|\beta_k|, |\gamma_k| < 1$, we would only have finitely many different polynomials $P_k(x) \in \mathbb{Z}[x]$. But then we must have $\alpha_k = \alpha_j$ for some indices $k > j \geqslant 1$. By (23), this implies that the sequence q_k, $k = 1, 2, 3, \dots$, is ultimately periodic. So $\alpha_1 = \alpha$ must be a quadratic number, a contradiction. This proves that at least one sequence a_k, $k = 1, 2, 3, \dots$, or α_k, $k = 1, 2, 3, \dots$, is unbounded. Hence the sequence $M(P_k) = a_k\alpha_k$, $k = 1, 2, 3, \dots$, is unbounded. Thus, by (3), $\mathcal{R}(P_k)$, $k = 1, 2, 3, \dots$, is unbounded and therefore (24) implies $g_{\mathrm{irr}}(3) \geqslant 2$. Combining this with the upper bound $g_{\mathrm{irr}}(3) \leqslant 2$ we obtain $g_{\mathrm{irr}}(3) = 2$.

Note that, by exactly the same argument, we can start with any Pisot number α of degree $d \geqslant 3$ with minimal polynomial P whose all other $d-1$ conjugates have negative real part. (For example, in [9] we have considered totally positive Pisot units α of degree d. Then $\alpha - 1$ is a Pisot number of degree d with its all remaining $d-1$ conjugates negative.) Putting

$$\alpha_{1,1} = \alpha, \quad \alpha_{1,k+1} = 1/\{\alpha_{1,k}\}, \;\; k = 1, 2, 3, \dots,$$

we obtain the sequence of polynomials P_k, $k = 1, 2, 3, \dots$, with roots $\alpha_{1,k}, \alpha_{2,k}, \dots, \alpha_{d,k}$ such that $\alpha_{1,k}$ is a Pisot number, $\alpha_{1,k} > 1 > |\alpha_{2,k}| \geqslant \dots \geqslant |\alpha_{d,k}|$, and $|\alpha_{1,k} - \alpha_{i,k}| > \alpha_{1,k}$ for $i = 2, \dots, d-1$. It follows that

$$\mathcal{R}(P_k)^{d-1} \prod_{2 \leqslant i < j \leqslant d} |1 - \alpha_{j,k}/\alpha_{i,k}| < \sqrt{\Delta(P_k)} = \sqrt{\Delta(P)}.$$

Also, as above, all the numbers $\alpha_{1,k}$, $k = 1, 2, 3, \dots$, must be distinct, so the sequences $M(P_k) = a_k\alpha_{1,k}$, $k = 1, 2, 3, \dots$, and $\mathcal{R}(P_k)$, $k = 1, 2, 3, \dots$, are unbounded. Of course, if α is a Pisot number with negative conjugates, then the roots $\alpha_{2,k}, \dots, \alpha_{d,k}$ are negative for each $k \in \mathbb{N}$.

We next turn to monic cubic polynomials with two close roots and use the ideas of [4]. Recall first that, by a result of Danilov [6], there exist two increasing sequences of positive integers x_k and y_k, $k = 1, 2, 3, \dots$, and an absolute constant $c > 0$ such that

$$x_k^3 - y_k^2 \sim c x_k^{1/2} \quad \text{as} \quad k \to \infty. \tag{25}$$

(See formula (6) in [6], where there is misprint in the power of the polynomial $t^2 + 6t - 11$.) So Proposition 7 with $w = 5/2$ implies the assertion of Theorem 5 for cubic polynomials and also the inequality $g^*_{\mathrm{irr}}(3) \geqslant 5/3$ of Theorem 4. Moreover, by Hall's conjecture [14], w is the largest real number with this property (although it is only known that $w < 3$ which follows from an old result of Mordell), so equality $g^*_{\mathrm{irr}}(3) = 5/3$ is equivalent to Hall's conjecture.

The remainder of this section is devoted to the proof of the following statement.

Proposition 7 *Let w be a positive number satisfying $5/2 \leqslant w < 3$. Then the inequality $|\overline{P}|^w \ll 1/\mathrm{Sep}(P)$ has infinitely many solutions in monic cubic irreducible polynomials $P \in \mathbb{Z}[x]$ if and only if the inequality $0 < |x^3 - y^2| \ll x^{3-w}$ has infinitely many solutions in positive integers x, y.*

Proof Assume first that the inequality $0 < |x_k^3 - y_k^2| \ll x_k^{3-w}$ holds for infinitely many pairs $(x_k, y_k) \in \mathbb{N}^2$. Consider the monic cubic polynomial

$$P_k(x) := x^3 - 3x_k x - 2y_k \in \mathbb{Z}[x]$$

with discriminant $\Delta(P_k) = 108(x_k^3 - y_k^2)$. Putting $\delta_k := (x_k^3 - y_k^2)x_k^{w-3}/3$, we have $|\delta_k| \ll 1$. Evaluating the polynomial P_k at $x = -\sqrt{x_k} + z$ we find that

$$P_k(-\sqrt{x_k} + z) = -x_k^{3/2} + 3x_k z - 3\sqrt{x_k} z^2 + z^3 + 3x_k^{3/2} - 3x_k z - 2y_k$$

$$= 2(x_k^{3/2} - y_k) - 3\sqrt{x_k} z^2 + z^3 = \frac{2(x_k^3 - y_k^2)}{x_k^{3/2} + y_k} - 3\sqrt{x_k} z^2 + z^3.$$

Therefore, since

$$\frac{2(x_k^3 - y_k^2)}{3(x_k^{3/2} + y_k)\sqrt{x_k}} \sim \frac{x_k^3 - y_k^2}{3x_k^2} = \frac{3\delta_k x_k^{3-w}}{3x_k^2} = \delta_k x_k^{1-w} \quad \text{as} \quad k \to \infty,$$

for its two roots α_k, β_k we have

$$\alpha_k + \sqrt{x_k} \sim -x_k^{1/2-w/2}\sqrt{\delta_k} \quad \text{and} \quad \beta_k + \sqrt{x_k} \sim x_k^{1/2-w/2}\sqrt{\delta_k}.$$

Thus the third root satisfies $\gamma_k \sim 2\sqrt{x_k}$ as $k \to \infty$. Therefore, in both cases (α_k, β_k are real or complex conjugate roots), we have $\gamma_k > |\alpha_k| \geqslant |\beta_k|$ and

$$\mathrm{sep}(P_k) = |\alpha_k - \beta_k| \sim 2\sqrt{|\delta_k|} x_k^{1/2-w/2}.$$

It follows that $\mathrm{Sep}(P_k) \sim 2\sqrt{|\delta_k|} x_k^{-w/2}$ and $|\overline{P_k}| \sim 2x_k^{1/2}$, giving the inequality $|\overline{P_k}|^w \ll 1/\mathrm{Sep}(P_k)$ for the monic cubic polynomials P_k defined above.

To complete the proof in one direction it remains to show that P_k are irreducible for k large enough. For a contradiction assume that P_k is reducible in $\mathbb{Z}[x]$. Then one of the roots α_k, β_k or γ_k must be an integer. If at least two roots are integers then all three must be integers which is impossible in view of $\beta_k - \alpha_k \to 0$. So assume that one is an integer and two others are the roots of an irreducible polynomial $Q(x) = x^2 + ux + v \in \mathbb{Z}[x]$. By the same reason, as $\beta_k - \alpha_k \to 0$, these two cannot be α_k, β_k, so one of the roots of Q is γ_k. Assume that the other root of Q is β_k. (The proof in case this is α_k is the same.) Then α_k, β_k are real negative numbers, $u = -\gamma_k - \beta_k = \alpha_k$ and $\Delta(Q) = u^2 - 4v = (\gamma_k - \beta_k)^2 \notin \mathbb{Z}^2$. Thus

$$\beta_k - \alpha_k = \beta_k - u = \frac{-u - \sqrt{\Delta(Q)}}{2} - u = \frac{-3u - \sqrt{\Delta(Q)}}{2} \geqslant \frac{1}{2(-3u + \sqrt{\Delta(Q)})}.$$

As $-3u = -3\alpha_k < 3\gamma_k$ and $\sqrt{\Delta(Q)} = \gamma_k - \beta_k = \gamma_k + |\beta_k| < 2\gamma_k$, this yields $\mathrm{sep}(P_k) = \beta_k - \alpha_k > 1/10\gamma_k$, contrary to $\mathrm{sep}(P_k) \ll x_k^{1/2-w/2} \ll \gamma_k^{1-w} \ll \gamma_k^{-3/2}$.

To prove the result in the opposite direction we assume that the inequality

$$\mathcal{R}(P)^{2w/3} \ll 1/\mathrm{Sep}(P)$$

has infinitely many solutions in monic cubic irreducible polynomials $P = P_k \in \mathbb{Z}[x]$. Note that this assumption is weaker than required because $\mathcal{R}(P)^{2w/3} \leqslant |\overline{P}|^w$. Without restriction of generality (by replacing $P_k(x)$ by $P_k(6x)$, if necessary, and omitting everywhere the index k) we may assume that the coefficients of $P(x) = x^3 + ax^2 + bx + c$ satisfy $6|a, b, c$. We claim that $\mathcal{R}(P)^{2w/3} \ll 1/\mathrm{Sep}(P)$ implies

$$\mathrm{sep}(P) \ll |\overline{P}|^{1-w} \tag{26}$$

(possibly with another constant in $\ll$).

Indeed, assume that α, β, γ are the roots of P satisfying $|\alpha| \leqslant |\beta| \leqslant |\gamma|$. As $\mathcal{R}(P)$ tends to infinity (there are only finitely many monic integer polynomials with $\mathcal{R}(P)$ bounded), $\mathrm{Sep}(P)$ tends to zero; so let us consider only those P for which $\mathrm{Sep}(P) \leqslant 1/2$. Evidently, $\mathrm{Sep}(P)$ is one of the numbers $|1 - \alpha/\beta|$, $|1 - \beta/\gamma|$ or $|1 - \alpha/\gamma|$.

In the first case, $\mathrm{Sep}(P) = |1 - \alpha/\beta|$, using $\mathrm{sep}(P) \leqslant |\beta - \alpha| = |\beta|\mathrm{Sep}(P)$, $|\beta| \leqslant |\gamma|$ and $w < 3$ we obtain

$$|\gamma|^{w-1}\mathrm{sep}(P) \leqslant |\gamma|^{w-1}|\beta|\mathrm{Sep}(P) \leqslant |\gamma|^{2w/3}|\beta|^{w/3}\mathrm{Sep}(P) = \mathcal{R}(P)^{2w/3}\mathrm{Sep}(P) \ll 1.$$

In the second case, $\mathrm{Sep}(P) = |1 - \beta/\gamma|$, from $\mathrm{Sep}(P) \leqslant 1/2$ it follows that $|\beta/\gamma| \geqslant 1/2$, hence $|\beta| \geqslant |\gamma|/2$. Similarly, in the third case, $\mathrm{Sep}(P) = |1-\alpha/\gamma|$, we obtain $|\alpha| \geqslant |\gamma|/2$, so $|\beta| \geqslant |\alpha| \geqslant |\gamma|/2$. Therefore, in these two cases we have $|\gamma|^{3/2} \ll |\gamma||\beta|^{1/2} = \mathcal{R}(P)$, i.e. $|\gamma| \ll \mathcal{R}(P)^{2/3}$. From $\mathrm{sep}(P) \leqslant |\gamma|\mathrm{Sep}(P)$ we conclude that

$$|\gamma|^{w-1}\mathrm{sep}(P) \leqslant |\gamma|^{w}\mathrm{Sep}(P) \ll \mathcal{R}(P)^{2w/3}\mathrm{Sep}(P) \ll 1,$$

which gives (26) again.

Next, let us replace $P(x)$ by $P(x - a/3)$. This does not change either $\mathrm{sep}(P)$ or $\Delta(P)$. If α, β, γ were the roots of $P(x) = x^3 + ax^2 + bx + c$ satisfying $|\alpha| \leqslant |\beta| \leqslant |\gamma|$ (so that $\alpha + \beta + \gamma = -a$, and hence $3|\gamma| \geqslant |a|$) then the roots of $P(x - a/3)$ are $\alpha + a/3, \beta + a/3, \gamma + a/3$. The modulus of the largest of those three does not exceed $|\gamma| + |a|/3 \leqslant 2|\gamma| = 2|\overline{P}|$, so this change may increase the value of $|\overline{P}|$ at most twice. It follows that (26) holds for infinitely many monic cubic irreducible polynomials

$$P(x) = (x - a/3)^3 + a(x - a/3)^2 + b(x - a/3) + c = x^3 - (a^2/3 - b)x - (ab/3 - c - 2a^3/27).$$

Since $6|a, b, c$, we can write P in the form $P(x) = x^3 - 3px - 2q \in \mathbb{Z}[x]$ with integers $p := (a^2/3 - b)/3$, $q := (ab/3 - c - 2a^3/27)/2$ and with the roots α, β, γ satisfying $|\alpha| \leqslant |\beta| \leqslant |\gamma|$.

Now, since γ has the largest modulus among three roots satisfying $\alpha + \beta + \gamma = 0$ and $\mathrm{sep}(P) \to 0$, we must have $\mathrm{sep}(P) = |\alpha - \beta|$ and so α, β tend to $-\gamma/2$. In particular, this implies $2q = \alpha\beta\gamma \geqslant \gamma^3/5$, so $\gamma \ll q^{1/3}$. Hence from $\Delta(P) = 108(p^3 - q^2)$ using (26) and the irreducibility of P we find that

$$0 < \sqrt{108|p^3 - q^2|} = \sqrt{|\Delta(P)|} = |\alpha - \beta||\alpha - \gamma||\beta - \gamma| \ll \mathrm{sep}(P)|\gamma|^2 \ll |\gamma|^{3-w} \ll q^{1-w/3}.$$

So the inequality $0 < |p^3 - q^2| \ll q^{2-2w/3}$ has infinitely many solutions $(p, q) \in \mathbb{N}^2$. This implies the result in view of $q^{2-2w/3} \ll (p^{3/2})^{2-2w/3} = p^{3-w}$. $\square$

4. Proof of Theorems 1 and 2

Proof of Theorem 1. To give a short proof of (5) we assume that $\mathrm{Sep}(P) = |1-\alpha_l/\alpha_k|$ with $k<l$. Let us subtract the l^{th} column of the determinant $\det(\alpha_j^{i-1})_{1\leqslant i,j\leqslant d}$ from its k^{th} column. The element $i\times k$ of the resulting determinant is equal to $\alpha_k^{i-1}-\alpha_l^{i-1}$. Taking out the factor $1-\alpha_l/\alpha_k$ out of each element of the k^{th} column we obtain

$$\det(\alpha_j^{i-1})_{1\leqslant i,j\leqslant d} = (1-\alpha_l/\alpha_k)\det(a_{ij}\alpha_j^{i-1})_{1\leqslant i,j\leqslant d},$$

where $a_{ij}:=1$ for $j\neq k$ and $a_{ik}:=\alpha_k^{2-i}(\alpha_k^{i-1}-\alpha_l^{i-1})/(\alpha_k-\alpha_l)$, because the element $i\times k$ becomes

$$\frac{\alpha_k^{i-1}-\alpha_l^{i-1}}{1-\alpha_l/\alpha_k} = \frac{(\alpha_k^{i-1}-\alpha_l^{i-1})\alpha_k^{i-1}}{(\alpha_k-\alpha_l)\alpha_k^{i-2}} = a_{ik}\alpha_k^{i-1}.$$

In particular, $a_{1k}=0$ and

$$|a_{ik}| = |1+\alpha_l/\alpha_k+\dots+(\alpha_l/\alpha_k)^{i-2}| \leqslant 1+|\alpha_l/\alpha_k|+\dots+|(\alpha_l/\alpha_k)^{i-2}| \leqslant i-1$$

for $i=2,\dots,d$, since $|\alpha_l|\leqslant|\alpha_k|$. Thus, by (6),

$$\prod_{j=1}^{d}\left(\sum_{i=1}^{d}|a_{ij}|^2\right)^{1/2} \leqslant d^{(d-1)/2}\sqrt{1^2+\dots+(d-1)^2} = d^{(d-1)/2}(d(d-1)(2d-1)/6)^{1/2}$$

$$= d^{d/2+1}\sqrt{(1-1/d)(1-1/2d)}/\sqrt{3} = 1/c_d.$$

Therefore, applying (2), we obtain

$$\sqrt{|\Delta(P)|} = |a_d|^{d-1}\big|\det(\alpha_j^{i-1})_{1\leqslant i,j\leqslant d}\big| = |a_d|^{d-1}\mathrm{Sep}(P)\big|\det(a_{ij}\alpha_j^{i-1})_{1\leqslant i,j\leqslant d}\big|$$

$$< \mathrm{Sep}(P)\mathcal{R}(P)^{d-1}/c_d,$$

giving (5). □

Proof of Theorem 2. Assume that $\mathrm{Sep}(P)=|1-\alpha_2/\alpha_1|$. (The proof in two other cases is the same.) Then

$$\frac{\sqrt{|\Delta(P)|}}{\mathrm{Sep}(P)\mathcal{R}(P)^2} = \frac{|\alpha_1-\alpha_2||\alpha_1-\alpha_3||\alpha_2-\alpha_3||\alpha_1|}{|\alpha_1-\alpha_2||\alpha_1|^2|\alpha_2|} = |1-\alpha_3/\alpha_1||1-\alpha_3/\alpha_2|.$$

Since $|1-\alpha_3/\alpha_1|\leqslant 1+|\alpha_3/\alpha_1|\leqslant 2$ and $|1-\alpha_3/\alpha_2|\leqslant 2$, their product does not exceed 4. Furthermore, it is equal to 4 only if $\alpha_3/\alpha_1=\alpha_3/\alpha_2=-1$, which is impossible, because $\alpha_1\neq\alpha_2$. Hence $\sqrt{|\Delta(P)|}/\mathrm{Sep}(P)\mathcal{R}(P)^2<4$, giving (7).

To prove the lower bound (8), let us consider the polynomials

$$P_t(x) := (x+pt)(x-pt)^2-p = (x-\alpha_t)(x-\beta_t)(x-\gamma_t),$$

where p is a fixed prime number and t runs through positive integers. By Eisenstein's criterion, the polynomial P_t is irreducible for each $t\in\mathbb{N}$. By Lemma 6, we have $\alpha_t\sim -pt$ and $\beta_t,\gamma_t\sim pt$ as $t\to\infty$. Furthermore, inserting $x=pt+y/\sqrt{t}$ into $P_t(x)=0$ we find that

$$y^3t^{-3/2}+2p(y^2-1/2)=0.$$

Hence Lemma 6 implies $\beta_t - pt \sim -1/\sqrt{2t}$ and $\gamma_t - pt \sim 1/\sqrt{2t}$ as $t \to \infty$. If follows that $\beta_t - \gamma_t \sim \sqrt{2/t}$,

$$\mathrm{Sep}(P_t) \sim \frac{\sqrt{2}}{pt^{3/2}}, \quad \mathcal{R}(P_t) \sim p^{3/2}t^{3/2} \quad \text{and} \quad \sqrt{|\Delta(P_t)|} \sim 4\sqrt{2}p^2t^{3/2}$$

as $t \to \infty$. Consequently, $\mathrm{Sep}(P_t)\mathcal{R}(P_t)^2/\sqrt{|\Delta(P_t)|} \to 1/4$ as $t \to \infty$. This completes the proof of (8). $\square$

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