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Magnetic field-induced stability of a specific configuration and the asymptotic behavior of minimizers in nematic liquid crystals

Junichi ARAMAKI*
Division of Science, Faculty of Science and Engineering, Tokyo Denki University, Hatoyama-machi, Saitama 350-0394, Japan

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Abstract: We consider the stability of a specific nematic liquid crystal configuration under an applied magnetic field. We impose the strong anchoring condition, and we allow the boundary data to be nonconstant and also the applied field to be nonconstant. Thus, we shall extend the results of Lin and Pan in 2007. We show that for some specific configuration there exist 2 critical values H_n and H_{sh} of applied magnetic field. When the intensity of the magnetic field is smaller than H_n, the configuration of the energy is only a global minimizer, when the intensity is between H_n and H_{sh}, the configuration is not a global minimizer, but is weakly stable, and when the intensity is larger than H_{sh}, the configuration is not weakly stable. Moreover, we also examine the behavior of minimal values of energy and the asymptotic behavior of the global minimizer as the intensity tends to infinity.

Key words: Magnetic field-induced stability, variational problem, nematic liquid crystal

1. Introduction

The purpose of this paper is to use the Oseen–Frank model to examine the change in stability of a specific nematic liquid crystal configuration under an applied magnetic field. Thus, this paper considers the case where the specific boundary data and the applied field are nonconstant, and the asymptotic behavior of minimizers of free energy under the applied field.

On the other hand, the purpose of the paper by Lin and Pan [17] was to use the Landau–de Gennes model with magnetic effect instead, which is more general than the Oseen–Frank model, to examine the behavior of liquid crystals subject to the applied field. Hence they considered simple magnetic field and boundary data that are constants.

We emphasize that this paper treats the case where the magnetic field and the boundary data are nonconstant in the Oseen–Frank model. Similar arguments of this paper in the Landau–de Gennes model will appear in future work.

In a previous paper Aramaki [1], we considered the stability of a specific nematic liquid crystal configuration under a constant applied magnetic field. However, in the present paper we treat the case where the applied field may be nonconstant, and so the result is also an extension of a part of [17] and [1].

The effect of applied electric and magnetic fields on liquid crystals is an important problem in physics of liquid crystals. It is well known that as the magnetic field increases, passing a critical value, the configuration

*Correspondence: aramaki@mail.dendai.ac.jp
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will lose its stability. This phenomenon has been studied by many physicists and mathematicians, and previous works related to this paper include Aramaki [2, 3], [17], Atkin and Stewart [4, 5], and Cohen and Luskin [8] and the references therein.

Such a theory for molecular orientation in nematic liquid crystal was given by Ericksen and Leslie [11]. According to the theory, for nematic liquid crystals the bulk free energy without external field is given by

$$\mathcal{W}(\boldsymbol{n}) = \int_\Omega W(\nabla \boldsymbol{n}, \boldsymbol{n}) dx \tag{1.1}$$

where $\boldsymbol{n} = \boldsymbol{n}(x)$ is the unit vector field, which called the director field, at $x \in \Omega$, $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain occupied by the material, and $W(\nabla \boldsymbol{n}, \boldsymbol{n})$ is the Oseen–Frank energy density:

$$W(\nabla \boldsymbol{n}, \boldsymbol{n}) = K_1(\operatorname{div} \boldsymbol{n})^2 + K_2(\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n})^2 + K_3|\boldsymbol{n} \times \operatorname{curl} \boldsymbol{n}|^2 + \nu[\operatorname{Tr}(\nabla \boldsymbol{n})^2 - (\operatorname{div} \boldsymbol{n})^2] \tag{1.2}$$

where K_i $(i = 1, 2, 3)$ are positive constants representing the elastic coefficients, and ν is a real constant.

Throughout this paper, we impose the strong anchoring condition to the director field, that is to say, the Dirichlet boundary condition $\boldsymbol{n}(x) = \boldsymbol{e}_0(x)$ on the boundary $\partial\Omega$ where $\boldsymbol{e}_0 : \partial\Omega \to \mathbb{S}^2$ is a given smooth unit vector field. In the situation where liquid crystal material is subject to a static magnetic field $\boldsymbol{H}$, we must add a magnetic energy contribution to the energy $\mathcal{W}(\boldsymbol{n})$. It is well accepted that such a magnetic energy density is of the form $-\chi_a(\boldsymbol{H} \cdot \boldsymbol{n})^2$, where χ_a is a positive constant (cf. de Gennes and Prost [10, p. 287]). We assume that the magnetic field $\boldsymbol{H} = \boldsymbol{H}(x)$ is of the form $\boldsymbol{H}(x) = \sigma \boldsymbol{h}(x)$, where $\boldsymbol{h}(x) \in C^2(\overline{\Omega}, \mathbb{S}^2)$ is a unit vector field and $\sigma \geq 0$ is the intensity of $\boldsymbol{H}$.

According to Hardt et al. [13], for all $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$,

$$\begin{aligned}
\int_\Omega [\operatorname{Tr}(\nabla \boldsymbol{n})^2 - (\operatorname{div} \boldsymbol{n})^2] dx &= \int_\Omega \operatorname{div}[(\nabla \boldsymbol{n})\boldsymbol{n} - (\operatorname{div} \boldsymbol{n})\boldsymbol{n}] dx \\
&= \int_{\partial\Omega} [(\nabla \boldsymbol{n})\boldsymbol{n} - (\operatorname{div} \boldsymbol{n})\boldsymbol{n}] \cdot \boldsymbol{\nu} dS \\
&= \int_{\partial\Omega} [(\nabla_{\tan} \boldsymbol{n})\boldsymbol{n} - \operatorname{Tr}(\nabla_{\tan} \boldsymbol{n})\boldsymbol{n}] \cdot \boldsymbol{\nu} dS
\end{aligned}$$

where $\nabla_{\tan} \boldsymbol{n} = \nabla \boldsymbol{n} - (\nabla \boldsymbol{n})\boldsymbol{\nu} \otimes \boldsymbol{\nu}$ on $\partial\Omega$, $\boldsymbol{\nu}$ denotes the outward unit vector field at $\partial\Omega$ and dS is the surface element of $\partial\Omega$. Therefore, under the strong anchoring condition, the integral of the last term of (1.2):

$$\mathcal{S}(\boldsymbol{e}_0) := \int_\Omega [\operatorname{Tr}(\nabla \boldsymbol{u})^2 - (\operatorname{div} \boldsymbol{n})^2] dx \tag{1.3}$$

represents a surface energy that only depends on the boundary term $\boldsymbol{e}_0$ (cf. also Bauman et al. [6]), and so does not affect the problem of finding equilibrium configurations. Thus, we consider the total energy of the nematic state:

$$\mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}] = \mathcal{F}[\boldsymbol{n}] - \chi_a \sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n})^2 dx, \tag{1.4}$$

where

$$\mathcal{F}[\boldsymbol{n}] = \int_\Omega \{K_1(\operatorname{div} \boldsymbol{n})^2 + K_2(\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n})^2 + K_3|\boldsymbol{n} \times \operatorname{curl} \boldsymbol{n}|^2\} dx.$$

To describe the space of admissible director fields, let $W^{1,2}(\Omega,\mathbb{R}^3)$ be the usual Sobolev space of vector fields,

$$W^{1,2}(\Omega,\mathbb{S}^2)=\{\boldsymbol{u}\in W^{1,2}(\Omega,\mathbb{R}^3);|\boldsymbol{u}(x)|=1 \text{ a.e. in } \Omega\},$$

and

$$W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)=\{\boldsymbol{u}\in W^{1,2}(\Omega,\mathbb{S}^2);\boldsymbol{u}=\boldsymbol{e}_0 \text{ on } \partial\Omega\}.$$

We note that if $\boldsymbol{e}_0:\partial\Omega\to\mathbb{S}^2$ is a smooth vector field and $\partial\Omega$ is Lipschitzian, then $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ is a nonempty set (cf. [13]). Thus, we can define

$$C(\sigma)=C(\sigma,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e}_0)=\inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)}\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]. \tag{1.5}$$

We note that it follows from the standard variational method that $C(\sigma)$ is achieved in $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$. For the proof, see [17] or [2].

Now we consider the following variational problem:

$$\inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)}\int_\Omega|\nabla\boldsymbol{n}|^2dx. \tag{1.6}$$

It is well known that the minimizer of (1.6) exists and the critical point of (1.6) satisfies the following Euler–Lagrange equation:

$$\begin{cases}-\Delta\boldsymbol{n}=|\nabla\boldsymbol{n}|^2\boldsymbol{n} & \text{in } \Omega,\\ \boldsymbol{n}=\boldsymbol{e}_0 & \text{on } \partial\Omega.\end{cases} \tag{1.7}$$

Thus, $\boldsymbol{n}$ is a harmonic map from Ω to $\mathbb{S}^2$.

In this paper, we consider the stability of a specific configuration $\boldsymbol{e}\in C^2(\overline{\Omega},\mathbb{S}^2)$ satisfying

$$\operatorname{curl}\boldsymbol{e}=0,\ \boldsymbol{h}\cdot\boldsymbol{e}=0,\boldsymbol{e} \text{ is a unique minimizer of (1.6)}. \tag{H.1}$$

Next we assume that

$$\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2<c(\Omega) \tag{H.2}$$

where $c(\Omega)>0$ is the best constant such that the following Poincaré inequality holds:

$$c(\Omega)\int_\Omega|\boldsymbol{w}|^2dx\le\int_\Omega|\nabla\boldsymbol{w}|^2dx \quad \text{for all } \boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^3).$$

For example, define

$$\boldsymbol{e}(x_1,x_2,x_3)=\left(\frac{x_1-a_1}{\sqrt{(x_1-a_1)^2+x_2^2}},\frac{x_2}{\sqrt{(x_1-a_1)^2+x_2^2}},0\right) \tag{1.8}$$

$x_1-a_1>0$ for $x\in\overline{\Omega}$, $|a_1|$ large enough, $\boldsymbol{e}_0:=\boldsymbol{e}|_{\partial\Omega}$, and $\boldsymbol{h}=(0,0,1)$ or

$$\boldsymbol{h}(x_1,x_2,x_3)=\left(\frac{-x_2}{\sqrt{(x_1-a_1)^2+x_2^2}},\frac{x_1-a_1}{\sqrt{(x_1-a_1)^2+x_2^2}},0\right). \tag{1.9}$$

Then $\boldsymbol{e}$ satisfies (H.1), and for large $|a_1|$, (H.2) also holds. Precise arguments are given in section 2. We note that there are a lot of choices of a_1, and $\boldsymbol{h}$ satisfies the Maxwell equation $\operatorname{div}\boldsymbol{h}=0$.

Finally we assume that

$$K_1 \le \min\{K_2, K_3\}. \tag{H.3}$$

Under the hypotheses (H.1), (H.2), and (H.3), we see that (Theorem 3.1 below) there exist 2 critical points $0 < H_n \le H_{sh} < \infty$ such that

(i) when $0 \le \sigma < H_{sh}$, $\boldsymbol{n} = \boldsymbol{e}$ is weakly stable,

(ii) when $0 \le \sigma < H_n$, $\boldsymbol{n} = \boldsymbol{e}$ is the only global minimizer of $\mathcal{F}_{\sigma \boldsymbol{h}}$ in $W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$. If $H_n < H_{sh}$, then $\boldsymbol{e}$ is a global minimizer of $\mathcal{F}_{H_n \boldsymbol{h}}$ and there exists at least one global minimizer of $\mathcal{F}_{H_n \boldsymbol{h}}$ except $\boldsymbol{e}$, and if $H_n < \sigma < H_{sh}$, then $\boldsymbol{n} = \boldsymbol{e}$ is not a global minimizer of $\mathcal{F}_{\sigma \boldsymbol{h}}$ in $W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$, but it is weakly stable (a local minimizer) of $\mathcal{F}_{\sigma \boldsymbol{h}}$,

(iii) If $\sigma > H_{sh}$, $\boldsymbol{n} = \boldsymbol{e}$ is not weakly stable.

The authors in [17] considered the Landau–de Gennes model with magnetic effect, which is more general than the Oseen–Frank model, to examine the stability of liquid crystals subject to the applied magnetic field. The Landau–de Gennes functional is given by

$$\mathcal{E}[\psi, \boldsymbol{n}] = \mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - \chi_a \sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n})^2 dx$$

where

$$\mathcal{G}[\psi, \boldsymbol{n}] = \int_\Omega \left\{ |\nabla_{q\boldsymbol{n}} \psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 \right\} dx,$$

which can describe nematic liquid crystals when $\psi = 0$ and describe smectic liquid crystals when $\psi \not\equiv 0$. They considered the case where $K_2 = K_3$ in $\mathcal{F}$ and the boundary data $\boldsymbol{n}_0$ is a constant vector, the magnetic field $\boldsymbol{h}$ is a constant vector, and found that there exist critical values H_{sh} and H_n of the intensity σ of applied magnetic field such that when $0 \le \sigma < H_n$, the configuration of the energy is the only global minimizer, when $H_n < \sigma < H_{sh}$, the configuration is not a global minimizer, but is weakly stable, and when $\sigma > H_{sh}$, the configuration is instable. However, the present paper treats $\mathcal{F}_{\sigma \boldsymbol{h}}$ with a general boundary data $\boldsymbol{e}_0$, which allows a unique harmonic extension that is curl-free and orthogonal to $\boldsymbol{H} = \sigma \boldsymbol{h}$, which may be nonconstant, and get 2 critical values H_{sh} and H_n, which have the same natures as [17]. Of course, the 2 formulas of H_{sh} and H_n are not identical. The analysis techniques we employ in this paper go back to the work of [17], who analyzed the energy in the situation where $\boldsymbol{e}_0$ and $\boldsymbol{h}$ are constant vectors. In our analysis, we improve their method to the case where $\boldsymbol{e}$ and $\boldsymbol{h}$ are nonconstant vector fields.

The plan of this paper is as follows. In section 2, we give examples satisfying (H.1) and (H.2), and preliminaries. In section 3, we consider the stability and instability of the configuration $\boldsymbol{e}$ according to the intensity σ of the magnetic field $\boldsymbol{H} = \sigma \boldsymbol{h}$ under the condition (H.1), (H.2), and (H.3). In section 4, we discuss the properties of the minimal energy $C(\sigma)$. Finally, in section 5, we examine the asymptotic behavior of the minimizers of $\mathcal{F}_{\sigma \boldsymbol{h}}$ as large σ.

2. Examples and preliminaries

First we shall show that there are many situations where (H.1) and (H.2) hold. Let Ω be a smooth bounded domain in $\mathbb{R}^3$, and define a vector field

$$\boldsymbol{e}(x) = \left(\frac{x_1 - a_1}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, \frac{x_2}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, 0 \right) \tag{2.1}$$

where $x_1 - a_1 > 0$ for all $x \in \overline{\Omega}$. There are a lot of choices of a_1. We define

$$\boldsymbol{e}_0 = \boldsymbol{e}\big|_{\partial\Omega} \tag{2.2}$$

and let $\boldsymbol{h}$ satisfy $\boldsymbol{h} \cdot \boldsymbol{e} = 0$, for example, $\boldsymbol{h} = (0, 0, 1)$ or

$$\boldsymbol{h} = \left(\frac{-x_2}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, \frac{x_1 - a_1}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, 0 \right).$$

By simple computations, we can show that $\boldsymbol{e}$ is a harmonic map from Ω into the unit sphere $\mathbb{S}^2$ and is curl-free; that is to say, $\boldsymbol{e}$ satisfies the equation (1.7) with $\boldsymbol{n} = \boldsymbol{e}$ and $\operatorname{curl} \boldsymbol{e} = 0$ in Ω, and $\boldsymbol{h}$ satisfies the Maxwell equation $\operatorname{div} \boldsymbol{h} = 0$ in Ω. For large $|a_1|$, putting $p = (1, 0, 0)$, we can see that $\boldsymbol{e}(\overline{\Omega}) \subset B_r(p) \subset \mathbb{S}^2$, where $B_r(p) = \{q \in \mathbb{S}^2; \text{dist } (q, p) \leq r\}$ and $0 < r < \pi/2$. Here $\text{dist}(q, p)$ denotes the geodesic distance on $\mathbb{S}^2$. We note that $B_r(p)$ satisfies the cut locus condition. That is to say, for any 2 points in $B_r(p)$, there exists a unique geodesic in $B_r(p)$ joining the 2 points.

According to Jäger and Kaul [15] and Hildebrandt et al. [14], the harmonic map $\boldsymbol{e}$ such that $\boldsymbol{e}(\overline{\Omega}) \subset B_r(p)$ with the Dirichlet data $\boldsymbol{e}_0$ exists and is unique. We can show that $\boldsymbol{e}$ is a unique minimizing harmonic map of (1.6) from the following proposition.

Proposition 2.1 *Let $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ be a minimizer of (1.6). Then $\boldsymbol{n}(\overline{\Omega}) \subset B_r(p)$. So we have $\boldsymbol{n} = \boldsymbol{e}$ in Ω.*

In order to prove this proposition, we need the lemma given by Jost [16]

Lemma 2.2 *Let B_0 and B_1 be closed subsets of $\mathbb{S}^2$ with $B_0 \subset B_1$ and assume that there exists a C^1 retraction map $\Pi : B_1 \to B_0$ such that*

$$|\nabla\Pi(x)(\boldsymbol{v})| < |\boldsymbol{v}| \quad \text{if } \boldsymbol{v} \in T_x\mathbb{S}^2,\ x \in B_1 \setminus B_0.$$

For a given boundary data $\boldsymbol{g} : \partial\Omega \to B_0$, if $\boldsymbol{n} : \Omega \to B_1$ is a energy minimizing map of (1.6) with the boundary data $\boldsymbol{g}$, then $\boldsymbol{n}(\Omega) \subset B_0$.

Using Lemma 2.2, we give an outline of the proof of Proposition 2.1.

Let $\boldsymbol{n}$ be a minimizer of (1.6). Then, by the Euler–Lagrange equation, $\boldsymbol{n} = (n_1, n_2, n_3)$ is a harmonic map with $\boldsymbol{n} = \boldsymbol{e}_0 = (e_{0,1}, e_{0,2}, 0)$ on $\partial\Omega$; that is to say, $\boldsymbol{n}$ satisfies (1.7). Define $\boldsymbol{u} = (u_1, u_2, u_3) = (|n_1|, n_2, n_3)$. Since $n_1 \in W^{1,2}(\Omega)$, it is well known that $u_1 = |n_1| \in W^{1,2}(\Omega)$ and $|\nabla|n_1|| = |\nabla n_1|$ a.e. in Ω. Since $\boldsymbol{u} = \boldsymbol{e}_0$ on $\partial\Omega$ and $e_{0,1} > 0$, $\boldsymbol{u}$ is also a minimizer of (1.6). Therefore $\boldsymbol{u}$ also satisfies the Euler–Lagrange equation. According to Schoen and Uhlenbeck [20], it follows that $\boldsymbol{u}$ is smooth near the boundary. Since u_1 also satisfies

$$\begin{cases} -\Delta u_1 = |\nabla \boldsymbol{n}|^2 u_1 & \text{in } \Omega, \\ u_1 = e_{0,1} & \text{on } \partial\Omega, \end{cases}$$

u_1 is a bounded nonnegative weak supersolution of Δ. Therefore, it follows from the weak Harnack inequality that

$$\operatorname*{ess\,inf}_{B_{\theta R}} u_1 \geq C \left(\frac{1}{|B_R|} \int_{B_R} u_1^p dx \right)^{1/p} \text{for all } B_R \subset \Omega \text{ and } 0 < \theta < 1,$$

where B_R denotes the ball with radius R and $p, C > 0$ depend only on Ω and $(1-\theta)^{-1}$ (cf. Gilbarg and Trudinger [12, Theorem 8.18], Chen and Wu [7, Chapter 4, Lemma 1.3]). Thus, we can see that $u_1 > 0$ in Ω. Therefore, $n_1 > 0$ in Ω or $n_1 < 0$ in Ω. Taking the boundary condition $n_1 = e_{0,1} > 0$ on $\partial\Omega$ into consideration, $n_1 > 0$ on $\overline{\Omega}$ (cf. [17] and [3]). Since n_1 is smooth near the boundary $\partial\Omega$, this implies that there exists $c > 0$ such that $n_1 \geq c$ on $\overline{\Omega}$. Thus, there exists $r < r' < \pi/2$ such that $n(\overline{\Omega}) \subset B_{r'}(p)$. Then the condition of Lemma 2.2 holds for $B_0 = B_r(p), B_1 = B_{r'}(p)$. So applying Lemma 2.2, we have $\boldsymbol{n}(\overline{\Omega}) \subset B_r(p)$. Note that $\boldsymbol{n}$ is smooth in $\overline{\Omega}$ (cf. Schoen and Uhlenbeck [19]). According to the uniqueness theorem in [15], we see that $\boldsymbol{n} = \boldsymbol{e}$ in Ω. For a complete proof, see [1].

We remark that if $\boldsymbol{e}$ is the vector field defined by (2.1), by a simple computation we can see that

$$|\nabla \boldsymbol{e}|^2 = \frac{1}{(x_1 - a_1)^2 + x_2^2}.$$

Therefore, for any bounded domain Ω, if we choose $|a_1|$ large enough, the condition (H.2) holds.

Next we have the following.

Proposition 2.3 *Assume that* (H.1) *and* (H.3) *hold. Then* $\boldsymbol{e}$ *is also a unique minimizer of* $\mathcal{F}$ *on* $W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$.

Proof We use the following identities. For all $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$,

$$|\nabla \boldsymbol{n}|^2 = (\operatorname{div} \boldsymbol{n})^2 + |\operatorname{curl} \boldsymbol{n}|^2 + \operatorname{Tr}(\nabla \boldsymbol{n})^2 - (\operatorname{div} \boldsymbol{n})^2, \tag{2.3}$$

$$|\operatorname{curl} \boldsymbol{n}|^2 = (\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n})^2 + |\boldsymbol{n} \times \operatorname{curl} \boldsymbol{n}|^2, \tag{2.4}$$

$$\operatorname{Tr}(\nabla \boldsymbol{n})^2 - (\operatorname{div} \boldsymbol{n})^2 = \operatorname{div}[(\nabla \boldsymbol{n})\boldsymbol{n} - (\operatorname{div} \boldsymbol{n})\boldsymbol{n}]. \tag{2.5}$$

Using (2.3), (2.4), (2.5) and (H.1), (H.3), we see that for any $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$,

$$\begin{aligned}
\mathcal{F}[\boldsymbol{n}] &\geq K_1 \int_\Omega \{(\operatorname{div} \boldsymbol{n})^2 + |\operatorname{curl} \boldsymbol{n}|^2\} dx \\
&= K_1 \int_\Omega |\nabla \boldsymbol{n}|^2 dx - K_1 \mathcal{S}(\boldsymbol{e}_0) \\
&\geq K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 dx - K_1 \mathcal{S}(\boldsymbol{e}_0) \\
&= K_1 \int_\Omega (\operatorname{div} \boldsymbol{e})^2 dx \\
&= \int_\Omega \{K_1 (\operatorname{div} \boldsymbol{e})^2 + K_2 (\boldsymbol{e} \cdot \operatorname{curl} \boldsymbol{e})^2 + K_3 |\boldsymbol{e} \times \operatorname{curl} \boldsymbol{e}|^2\} dx \\
&= \mathcal{F}[\boldsymbol{e}].
\end{aligned}$$

Thus, $\boldsymbol{e}$ is a minimizer of $\mathcal{F}$ on $W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$. Conversely, if $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ is a minimizer of $\mathcal{F}$, we see that $\mathcal{F}[\boldsymbol{n}] \leq \mathcal{F}[\boldsymbol{e}]$. This implies that

$$\int_\Omega |\nabla \boldsymbol{n}|^2 dx \leq \int_\Omega |\nabla \boldsymbol{e}|^2 dx.$$

By (H.1), we see that $\boldsymbol{n} = \boldsymbol{e}$. Thus, $\boldsymbol{e}$ is a unique minimizer of $\mathcal{F}$. □

Next we shall define the critical point and stability of $\mathcal{F}_{\sigma\boldsymbol{h}}$. For $\boldsymbol{n}_0 \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ and $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, we can write for small t

$$\boldsymbol{n}_t := \frac{\boldsymbol{n}_0 + t\boldsymbol{v}}{|\boldsymbol{n}_0 + t\boldsymbol{v}|} = \boldsymbol{n}_0 + t\boldsymbol{n}_1 + t^2\boldsymbol{n}_2 + +O(t^3) \tag{2.6}$$

where

$$\boldsymbol{n}_1 = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n}_0)\boldsymbol{n}_0, \tag{2.7}$$

$$\boldsymbol{n}_2 = -(\boldsymbol{v} \cdot \boldsymbol{n}_0)\boldsymbol{v} + \frac{1}{2}[3(\boldsymbol{v} \cdot \boldsymbol{n}_0)^2 - |\boldsymbol{v}|^2]\boldsymbol{n}_0.$$

Then we can see that (cf. [2])

$$\begin{aligned}\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_t] = \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_0] &+ 2t\Big\{\mathcal{A}(\boldsymbol{n}_0; \boldsymbol{v}) - \chi_a\sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n}_0)(\boldsymbol{h} \cdot \boldsymbol{n}_1)dx\Big\} \\ &+ t^2\Big\{\mathcal{B}(\boldsymbol{n}_0; \boldsymbol{v}) - \chi_a\sigma^2 \int_\Omega \{(\boldsymbol{h} \cdot \boldsymbol{n}_1)^2 + 2(\boldsymbol{h} \cdot \boldsymbol{n}_0)(\boldsymbol{h} \cdot \boldsymbol{n}_2)\}dx\Big\} + O(t^3),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}\mathcal{A}(\boldsymbol{n}_0; \boldsymbol{v}) = \int_\Omega &\{K_1(\operatorname{div} \boldsymbol{n}_0)(\operatorname{div} \boldsymbol{n}_1) \\ &+ K_2(\boldsymbol{n}_0 \cdot \operatorname{curl} \boldsymbol{n}_0)(\boldsymbol{n}_1 \cdot \operatorname{curl} \boldsymbol{n}_0 + \boldsymbol{n}_0 \cdot \operatorname{curl} \boldsymbol{n}_1) \\ &+ K_3(\boldsymbol{n}_0 \times \operatorname{curl} \boldsymbol{n}_0) \cdot (\boldsymbol{n}_1 \times \operatorname{curl} \boldsymbol{n}_0 + \boldsymbol{n}_0 \times \operatorname{curl} \boldsymbol{n}_1)\}dx, \\ \mathcal{B}(\boldsymbol{n}_0; \boldsymbol{v}) = \int_\Omega &\big[K_1\{(\operatorname{div} \boldsymbol{n}_1)^2 + 2(\operatorname{div} \boldsymbol{n}_0)(\operatorname{div} \boldsymbol{n}_2)\} \\ &+ K_2\{(\boldsymbol{n}_1 \cdot \operatorname{curl} \boldsymbol{n}_0 + \boldsymbol{n}_0 \cdot \operatorname{curl} \boldsymbol{n}_1)^2 \\ &+ 2(\boldsymbol{n}_0 \cdot \operatorname{curl} \boldsymbol{n}_0)(\boldsymbol{n}_2 \cdot \operatorname{curl} \boldsymbol{n}_0 + \boldsymbol{n}_1 \cdot \operatorname{curl} \boldsymbol{n}_1 + \boldsymbol{n}_0 \cdot \operatorname{curl} \boldsymbol{n}_2)\} \\ &+ K_3\{|\boldsymbol{n}_1 \times \operatorname{curl} \boldsymbol{n}_0 + \boldsymbol{n}_0 \times \operatorname{curl} \boldsymbol{n}_1|^2 \\ &+ 2(\boldsymbol{n}_0 \times \operatorname{curl} \boldsymbol{n}_0) \cdot (\boldsymbol{n}_2 \times \operatorname{curl} \boldsymbol{n}_0 \\ &+ \boldsymbol{n}_1 \times \operatorname{curl} \boldsymbol{n}_1 + \boldsymbol{n}_0 \times \operatorname{curl} \boldsymbol{n}_2)\}\big]dx.\end{aligned} \tag{2.9}$$

Definition 2.4 (i) *We call $\boldsymbol{n}_0 \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$, if for any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$,*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_t] = 0,$$

where $\boldsymbol{n}_t$ is defined by (2.6).

(ii) *Let $\boldsymbol{n}_0 \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ be a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$. Then we say that $\boldsymbol{n}_0$ is weakly stable, if for any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ there exists $T > 0$ such that for all $0 < t < T$, which may depend on $\boldsymbol{v}$,*

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_0] \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_t]$$

where $\boldsymbol{n}_t$ is defined by (2.6).

It easily follows from (2.8) that $\boldsymbol{n}_0 \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ is a critical point of $\mathcal{F}_{\sigma,\boldsymbol{h}}$ if and only if for all $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$,

$$\mathcal{A}(\boldsymbol{n}_0; \boldsymbol{v}) - \chi_a \sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n}_0)(\boldsymbol{h} \cdot \boldsymbol{n}_1) dx = 0. \tag{2.10}$$

Lemma 2.5 *The vector field* $\boldsymbol{e}$ *satisfying* (H.1) *is a critical point of* $\mathcal{F}_{\sigma\boldsymbol{h}}$ *for all* σ.

Proof Since $\boldsymbol{e}$ satisfies that $\operatorname{curl} \boldsymbol{e} = 0$ in Ω, and $\boldsymbol{n}_1 = 0$ on $\partial\Omega$, it follows from the formula $\operatorname{curl}^2 \boldsymbol{n} = -\Delta \boldsymbol{n} + \nabla(\operatorname{div} \boldsymbol{n})$ for any $\boldsymbol{n} \in C^2(\Omega, \mathbb{R}^3)$, the divergence theorem and (H.1) that

$$\begin{aligned}
\mathcal{A}(\boldsymbol{e}; \boldsymbol{v}) &= \int_\Omega K_1 (\operatorname{div} \boldsymbol{e})(\operatorname{div} \boldsymbol{n}_1) dx \\
&= -\int_\Omega K_1 \nabla(\operatorname{div} \boldsymbol{e}) \cdot \boldsymbol{n}_1 dx \\
&= -\int_\Omega K_1 \Delta \boldsymbol{e} \cdot (\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}) dx \\
&= \int_\Omega K_1 |\nabla \boldsymbol{e}|^2 \boldsymbol{e} \cdot (\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}) dx = 0.
\end{aligned}$$

Moreover, it is clear that

$$\int_\Omega (\boldsymbol{h} \cdot \boldsymbol{e})(\boldsymbol{h} \cdot \boldsymbol{n}_1) dx = 0.$$

□

If $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ is a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$, $\boldsymbol{n}$ satisfies the following Euler–Lagrange equation (cf. [3]):

$$\begin{cases} -K_1 \nabla(\operatorname{div} \boldsymbol{n}) + (K_2 - K_3)\{(\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n}) \operatorname{curl} \boldsymbol{n} & \\ +\operatorname{curl} \big((\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n})\boldsymbol{n}\big)\} + K_3\{|\operatorname{curl} \boldsymbol{n}|^2 \boldsymbol{n} + \operatorname{curl}^2 \boldsymbol{n}\} & \\ -\chi_a \sigma^2 (\boldsymbol{h} \cdot \boldsymbol{n})\boldsymbol{h} - \lambda \boldsymbol{n} = 0 & \text{in } \Omega, \\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega \end{cases} \tag{2.11}$$

where $\lambda = \lambda(x)$ is the Lagrangean multiplier:

$$\begin{aligned} \lambda = \boldsymbol{n} \cdot [&-K_1 \nabla(\operatorname{div} \boldsymbol{n}) + (K_2 - K_3)\{(\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n}) \operatorname{curl} \boldsymbol{n} \\ &+ \operatorname{curl} \big((\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n})\boldsymbol{n}\big)\} + K_3\{|\operatorname{curl} \boldsymbol{n}|^2 \boldsymbol{n} + \operatorname{curl}^2 \boldsymbol{n}\} - \chi_a \sigma^2 (\boldsymbol{h} \cdot \boldsymbol{n})\boldsymbol{h}] \end{aligned}$$

In the particular case where $K_1 = K_2 = K_3 = K > 0$, (2.11) becomes

$$\begin{cases} -\Delta \boldsymbol{n} = |\nabla \boldsymbol{n}|^2 \boldsymbol{n} + \frac{\chi_a \sigma^2}{K}[(\boldsymbol{h} \cdot \boldsymbol{n})\boldsymbol{h} - (\boldsymbol{h} \cdot \boldsymbol{n})^2 \boldsymbol{n}] & \text{in } \Omega, \\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

For weak stability, we have the following.

Proposition 2.6 ([8]) *Let* $\boldsymbol{n}_0 \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ *be a critical point of* $\mathcal{F}_{\sigma \boldsymbol{h}}$*. Then*

(i) *If there exists* $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ *such that*

$$\mathcal{B}(\boldsymbol{n}_0; \boldsymbol{v}) - \chi_a \sigma^2 \int_\Omega \big\{(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{h}\cdot\boldsymbol{n}_0)(\boldsymbol{h}\cdot\boldsymbol{n}_2)\big\} dx < 0,$$

then $\boldsymbol{n}_0$ *is not weak stable.*

(ii) *If*

$$\mathcal{B}(\boldsymbol{n}_0; \boldsymbol{v}) - \chi_a \sigma^2 \int_\Omega \big\{(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{h}\cdot\boldsymbol{n}_0)(\boldsymbol{h}\cdot\boldsymbol{n}_2)\big\} dx > 0$$

for all $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ *which is not parallel to* $\boldsymbol{n}_0$ *on a set of positive measure, then* $\boldsymbol{n}_0$ *is weak stable.*

Since $\boldsymbol{e}$ is a critical point of $\mathcal{F}_{\sigma \boldsymbol{h}}$, we see that

$$\mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}_t] = \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}] + t^2 \left(\mathcal{B}(\boldsymbol{e}; \boldsymbol{v}) - \chi_a \sigma^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_1) dx \right) + O(t^3),$$

where $\boldsymbol{n}_t$ is defined by (2.6) with $\boldsymbol{n}_0 = \boldsymbol{e}$. Here we can compute $\mathcal{B}(\boldsymbol{e}; \boldsymbol{v})$ as follows:

$$\begin{aligned} \mathcal{B}(\boldsymbol{e}; \boldsymbol{v}) &= \int_\Omega [K_1\{(\operatorname{div} \boldsymbol{n}_1)^2 + 2(\operatorname{div} \boldsymbol{e})(\operatorname{div} \boldsymbol{n}_2)\} + K_2(\boldsymbol{e}\cdot\operatorname{curl} \boldsymbol{n}_1)^2 \\ &\qquad + K_3|\boldsymbol{e}\times\operatorname{curl} \boldsymbol{n}_1|^2] dx \\ &= \int_\Omega [K_1(\operatorname{div} \boldsymbol{n}_1)^2 + K_2(\boldsymbol{e}\cdot\operatorname{curl} \boldsymbol{n}_1)^2 + K_3|\boldsymbol{e}\times\operatorname{curl} \boldsymbol{n}_1|^2 \\ &\qquad - 2\nabla(\operatorname{div} \boldsymbol{e})\cdot\boldsymbol{n}_2] dx. \end{aligned}$$

Since $\nabla(\operatorname{div} \boldsymbol{e}) = \Delta \boldsymbol{e}$ as $\operatorname{curl} \boldsymbol{e} = 0$ and $|\boldsymbol{n}_1|^2 = |\boldsymbol{v}|^2 - (\boldsymbol{v}\cdot\boldsymbol{e})^2$, we have

$$\begin{aligned} -2\Delta \boldsymbol{e}\cdot\boldsymbol{n}_2 &= 2|\nabla \boldsymbol{e}|^2 \boldsymbol{e}\cdot\big(-(\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{v} + \frac{1}{2}[3(\boldsymbol{v}\cdot\boldsymbol{e})^2 - |\boldsymbol{v}|^2]\boldsymbol{e}\big) \\ &= -|\nabla \boldsymbol{e}|^2 |\boldsymbol{n}_1|^2. \end{aligned}$$

If we define

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{n}_1] = \int_\Omega \{K_1(\operatorname{div} \boldsymbol{n}_1)^2 + K_2(\boldsymbol{e}\cdot\operatorname{curl} \boldsymbol{n}_1)^2 + K_3|\boldsymbol{e}\times\operatorname{curl} \boldsymbol{n}_1|^2 dx, \tag{2.13}$$

we thus have

$$\mathcal{B}(\boldsymbol{e}; \boldsymbol{v}) = \mathcal{F}(\boldsymbol{e})[\boldsymbol{n}_1] - \int_\Omega K_1 |\nabla \boldsymbol{e}|^2 |\boldsymbol{n}_1|^2 dx. \tag{2.14}$$

We note that if we put $\boldsymbol{n}_1 = \boldsymbol{w}$, then $\boldsymbol{w} = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ and $\boldsymbol{w}\cdot\boldsymbol{e} = 0$ in Ω.

Proposition 2.7 *Assume that* (H.1), (H.2)*, and* (H.3) *hold. Then there exists a positive constant* c *such that*

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx \ge c\|\boldsymbol{w}\|^2_{W^{1,2}(\Omega, \mathbb{R}^3)}$$

for all $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$.

Proof Since $(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w})^2+|\boldsymbol{e}\times\operatorname{curl}\boldsymbol{w}|^2=|\operatorname{curl}\boldsymbol{w}|^2$ for any $\boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^3)$, we have

$$\begin{aligned}
&\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx\\
&\geq K_1\int_\Omega\{(\operatorname{div}\boldsymbol{w})^2+|\operatorname{curl}\boldsymbol{w}|^2\}dx-K_1\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2\int_\Omega|\boldsymbol{w}|^2dx\\
&=K_1\int_\Omega|\nabla\boldsymbol{w}|^2dx-K_1\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2\int_\Omega|\boldsymbol{w}|^2dx\\
&\geq K_1[1-c(\Omega)^{-1}\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2]\int_\Omega|\nabla\boldsymbol{w}|^2dx.
\end{aligned}$$

Using (H.2) and the Poincaré inequality, we get the result. □

Now we define the superheating critical value H_{sh}.

Definition 2.8 *Let* $K_1,K_2,K_3>0$ *and* (H.1), (H.2), and (H.3) *hold. Then we define*

$$H_{sh}^2=\frac{1}{\chi_a}\inf\Big\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx;\\ \boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^3),\boldsymbol{w}(x)\cdot\boldsymbol{e}(x)=0\ \textit{a.e. in}\ \Omega,\|\boldsymbol{h}\cdot\boldsymbol{w}\|_{L^2(\Omega)}=1\Big\}. \quad (2.15)$$

Although the idea of the definition and the nature of the critical field H_{sh} are similar to [17, Definition 3.1], the formula contains the extra term $-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx$ because the boundary data $\boldsymbol{e}_0$ is nonconstant. Of course, the formula in [17] also contains the term from the order parameter.

Then we have, as in the first part of [17, Proposition 3.3],

Proposition 2.9 *We see that* $H_{sh}>0$ *and it is achieved.*

Proof From Proposition 2.7, we can see that $H_{sh}\geq 0$. Let $\{\boldsymbol{w}_j\}\subset W_0^{1,2}(\Omega,\mathbb{R}^3)$ with $\boldsymbol{w}_j\cdot\boldsymbol{e}=0$ in Ω and $\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|_{L^2(\Omega)}=1$ be a minimizing sequence of $\chi_aH_{sh}^2$. Then from Proposition 2.7, for large j, we have

$$c\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)}\leq\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2dx\leq\chi_aH_{sh}^2+1.$$

Therefore $\{\boldsymbol{w}_j\}$ is bounded in $W_0^{1,2}(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{w}_j\to\boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^3)$ weakly in $W^{1,2}(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$, and a.e. in Ω. Therefore, we see that $\boldsymbol{w}\cdot\boldsymbol{e}=0$ a.e. in Ω, $\boldsymbol{h}\cdot\boldsymbol{w}_j\to\boldsymbol{h}\cdot\boldsymbol{e}$ in $L^2(\Omega)$, and

$$\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2dx\to\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx.$$

Thus, we have

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx\leq\liminf_{j\to\infty}\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2dx\}=\chi_aH_{sh}^2.$$

Hence $\boldsymbol{w}$ is a minimizer of H_{sh}^2. From Proposition 2.7, $H_{sh}>0$. □

Next we define an another critical value.

Definition 2.10

$$H_n^2 = \frac{1}{\chi_a} \inf\left\{ \frac{\mathcal{F}[\boldsymbol{n}] - K_1 \|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}}; \boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0), \boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0 \ in \ \Omega \right\}. \tag{2.16}$$

Here we also note that since the boundary data is not constant, the formula needs the extra term $K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$ compared with [17, Definition 4.3].

Then we have (cf. [17, Theorem 4.5])

Proposition 2.11 *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and assume that* (H.1), (H.2), and (H.3) *hold. Then*

(i) $0 < H_n \leq H_{sh}$.

(ii) *If $H_n < H_{sh}$, then H_n is achieved.*

Proof Though the proof is similar to [1, Proposition 2.12], we repeat the proof for the reader's convenience.

Step 1. We prove that $H_n \leq H_{sh}$.

Let $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ satisfying $\boldsymbol{e}\cdot\boldsymbol{w} \equiv 0$ in Ω and $\boldsymbol{h}\cdot\boldsymbol{w} \not\equiv 0$ in Ω. Then we can write

$$\boldsymbol{n}_t = \frac{\boldsymbol{e} + t\boldsymbol{w}}{|\boldsymbol{e} + t\boldsymbol{w}|} = \boldsymbol{e} + t\boldsymbol{w} - \frac{1}{2}t^2|\boldsymbol{w}|^2\boldsymbol{e} + O(t^3).$$

Therefore

$$\mathcal{F}[\boldsymbol{n}_t] = \mathcal{F}[\boldsymbol{e}] + t^2\left\{ \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] + 2K_1 \int_\Omega (\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)dx \right\} + O(t^3),$$

and

$$\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_t)^2 dx = t^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{w})^2 dx + O(t^3).$$

Using (H.1), we have

$$2K_1 \int_\Omega (\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)dx = -K_1 \int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2 dx.$$

Hence

$$\begin{aligned}
\frac{\mathcal{F}[\boldsymbol{n}_t] - \mathcal{F}[\boldsymbol{e}]}{\|\boldsymbol{h}\cdot\boldsymbol{n}_t\|^2_{L^2(\Omega)}} &= \frac{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] + 2K_1\int_\Omega(\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t) \\
&= \frac{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t).
\end{aligned}$$

Since $\mathcal{F}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$, we see that

$$\begin{aligned}
\chi_a H_n^2 &\leq \frac{\mathcal{F}[\boldsymbol{n}_t] - \mathcal{F}[\boldsymbol{e}]}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}} \\
&= \frac{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t).
\end{aligned}$$

Letting $t \to 0$, we have

$$\chi_a H_n^2 \le \frac{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}}.$$

Since $W_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega,\mathbb{R}^3)$ is dense in $W_0^{1,2}(\Omega,\mathbb{R}^3)$, this implies that $H_n \le H_{sh}$.

Step 2. We prove (ii). Assume that $H_n < H_{sh}$. Let $\{\boldsymbol{n}_j\} \subset W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ with $\boldsymbol{h}\cdot\boldsymbol{n}_j \not\equiv 0$ be a minimizing sequence of H_n. Then, for large j,

$$\mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} = (\chi_a H_n^2 + o(1))\|\boldsymbol{h}\cdot\boldsymbol{n}_j\|^2_{L^2(\Omega)}.$$

Since $|\boldsymbol{h}\cdot\boldsymbol{n}_j| \le 1$, $\{\mathcal{F}[\boldsymbol{n}_j]\}$ is bounded. Therefore, $\{\operatorname{div}\boldsymbol{n}_j\}$ is bounded in $L^2(\Omega)$ and $\{\operatorname{curl}\boldsymbol{n}_j\}$ is bounded in $L^2(\Omega,\mathbb{R}^3)$. We note that $|\boldsymbol{n}_j| = 1$ a.e. in Ω and $\boldsymbol{n}_j = \boldsymbol{e}_0$ on $\partial\Omega$. Thus, it follows from Dautray and Lions [9] that $\{\boldsymbol{n}_j\}$ is bounded in $W^{1,2}(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{n}_j \to \widehat{\boldsymbol{n}}$ weakly in $W^{1,2}(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$, and a.e. in Ω. Hence $\widehat{\boldsymbol{n}} \in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ and

$$\begin{aligned}\mathcal{F}[\widehat{\boldsymbol{n}}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} &\le \liminf_{j\to\infty}\{\mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}\} \\ &= \chi_a H_n^2\|\boldsymbol{h}\cdot\widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.17}$$

If we show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$ in Ω, we can see that H_n is achieved.

Step 3. Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \equiv 0$ in Ω. Then, from (2.17), we see that

$$\mathcal{F}[\widehat{\boldsymbol{n}}] \le K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} = \mathcal{F}[\boldsymbol{e}].$$

Since $\boldsymbol{e}$ is a unique minimizer of $\mathcal{F}$ from (H.1) and Proposition 2.3, we have $\widehat{\boldsymbol{n}} = \boldsymbol{e}$. We write $\boldsymbol{n}_j = \boldsymbol{e} + \varepsilon_j\boldsymbol{w}_j$, where $\varepsilon_j = \|\boldsymbol{n}_j - \boldsymbol{e}\|_{W^{1,2}(\Omega,\mathbb{R}^3)}$, $\boldsymbol{w}_j \in W_0^{1,2}(\Omega,\mathbb{R}^3)$, and $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$. Here since $\boldsymbol{h}\cdot\boldsymbol{n}_j \not\equiv 0$, we see that $\boldsymbol{n}_j \not\equiv \boldsymbol{e}$. Thus, $\varepsilon_j > 0$. From Proposition 2.7 and Step 2,

$$o(1) = \mathcal{F}[\boldsymbol{n}_j] - \mathcal{F}[\boldsymbol{e}] = \varepsilon_j^2\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2 dx\} \ge c\varepsilon_j^2\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = c\varepsilon_j^2. \tag{2.18}$$

Since $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$, after passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ weakly in $W_0^{1,2}(\Omega,\mathbb{R}^3)$ and strongly in $L^4(\Omega,\mathbb{R}^3)$. Since $\boldsymbol{e}\cdot\boldsymbol{w}_j = -\frac{\varepsilon_j}{2}|\boldsymbol{w}_j|^2 \to 0$ strongly in $L^2(\Omega)$, we have $\boldsymbol{e}\cdot\widehat{\boldsymbol{w}} = 0$ a.e. in Ω. Therefore, it follows that

$$\begin{aligned}(\chi_a H_n^2 + o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|^2_{L^2(\Omega)} &= \frac{1}{\varepsilon_j^2}(\chi_a H_n^2 + o(1))\|\boldsymbol{e}\cdot\boldsymbol{n}_j\|^2_{L^2(\Omega)} \\ &= \frac{1}{\varepsilon_j^2}(\mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}) \\ &= \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2 dx. \end{aligned} \tag{2.19}$$

Hence, we have

$$\mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\widehat{\boldsymbol{w}}|^2 dx \le \liminf_{j\to\infty}\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2 dx\} = \chi_a H_n^2\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}. \tag{2.20}$$

Step 4. We show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \not\equiv 0$ in Ω. Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \equiv 0$ in Ω. Since $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$ and $\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|_{L^2(\Omega)} \to 0$, it follows from (2.19) and Proposition 2.7 that

$$\begin{aligned}(\chi_a H_{sh}^2 + o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|_{L^2(\Omega)} &= \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla \boldsymbol{e}|^2|\boldsymbol{w}_j|^2dx \\ &\geq c\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)}.\end{aligned}$$

Thus, we have $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \to 0$. This contradicts the fact that $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$. Since $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \not\equiv 0$, we have

$$\chi_a H_{sh} \leq \frac{\mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}] - K_1\int_\Omega |\nabla \boldsymbol{e}|^2|\widehat{\boldsymbol{w}}|^2dx}{\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|_{L^2(\Omega)}^2} \leq \chi_a H_n^2.$$

This contradicts the hypothesis of Step 2. Thus, we have $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$ in Ω and so H_n is achieved.

Step 5. We show that $H_n > 0$. When $H_n = H_{sh}$, it follows from Proposition 2.9 that $H_n = H_{sh} > 0$. Therefore, assume that $H_n < H_{sh}$. Then it follows from (ii) that H_n is achieved. Let $\boldsymbol{n}$ be a minimizer of H_n and assume that $H_n = 0$. Since $\mathcal{F}[\boldsymbol{n}] \geq \mathcal{F}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|_{L^2(\Omega)}$, we have

$$0 \leq \mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|_{L^2(\Omega)}^2 = \chi_a H_n^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|_{L^2(\Omega)} = 0.$$

Thus, from (H.1) we have $\boldsymbol{n} = \boldsymbol{e}$. This contradicts the fact that $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$ in Ω. □

3. Weak stability of the vector field in (H.1)

In this section we consider the stability of the vector field $\boldsymbol{e}$ in (H.1). The following theorem is similar to [17, Lemma 4.2 and Proposition 5.3]. However, since the boundary data $\boldsymbol{e}_0$ is nonconstant, the proof has to be modified carefully.

Theorem 3.1 *Assume that* (H.1)*,* (H.2)*, and* (H.3) *hold. Then we have the following.*

(i) *When* $0 \leq \sigma < H_{sh}$, $\boldsymbol{n} = \boldsymbol{e}$ *is weakly stable.*

(ii) *When* $0 \leq \sigma < H_n$, $\boldsymbol{n} = \boldsymbol{e}$ *is the only global minimizer of* $\mathcal{F}_{\sigma\boldsymbol{h}}$ *in* $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$. *If* $H_n < H_{sh}$, *then* $\boldsymbol{e}$ *is a global minimizer of* $\mathcal{F}_{H_n\boldsymbol{h}}$ *and there exists at least one global minimizer of* $\mathcal{F}_{H_n\boldsymbol{h}}$ *except* $\boldsymbol{e}$, *and if* $H_n < \sigma < H_{sh}$, *then* $\boldsymbol{n} = \boldsymbol{e}$ *is not a global minimizer of* $\mathcal{F}_{\sigma\boldsymbol{h}}$ *in* $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$, *but it is weakly stable (a local minimizer) of* $\mathcal{F}_{\sigma\boldsymbol{h}}$.

(iii) *If* $\sigma > H_{sh}$, $\boldsymbol{n} = \boldsymbol{e}$ *is not weakly stable.*

Proof Although the proof is similar to [1], we have to revise it slightly. For the reader's convenience, we give a complete proof. Since $\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|_{L^2(\Omega)}^2$, if $\boldsymbol{n} \in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ is a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$, we have

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \leq \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|_{L^2(\Omega)}^2.$$

If $\boldsymbol{h}\cdot\boldsymbol{n} \equiv 0$ in Ω, we have

$$\mathcal{F}[\boldsymbol{n}] = \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \leq \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|_{L^2(\Omega)}^2 = \mathcal{F}[\boldsymbol{e}].$$

From Proposition 2.3, we have $\boldsymbol{n} = \boldsymbol{e}$. Therefore, global minimizer $\boldsymbol{n}$ with $\boldsymbol{n} \not\equiv \boldsymbol{e}$ satisfies $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$ in Ω.

When $0 \leq \sigma < H_n$, if $\mathcal{F}_{\sigma \boldsymbol{h}}$ has a global minimizer $\boldsymbol{n}$ that is not equal to $\boldsymbol{e}$, then $\boldsymbol{h} \cdot \boldsymbol{n} \not\equiv 0$ in Ω. Therefore, we see that

$$K_1 \|\mathrm{div}\, \boldsymbol{e}\|^2_{L^2(\Omega)} = \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}] \geq \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}] = \mathcal{F}[\boldsymbol{n}] - \chi_a \sigma^2 \|\boldsymbol{h} \cdot \boldsymbol{n}\|^2_{L^2(\Omega)}.$$

Hence,

$$\sigma^2 \geq \frac{1}{\chi_a} \frac{\mathcal{F}[\boldsymbol{n}] - K_1 \|\mathrm{div}\, \boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h} \cdot \boldsymbol{n}\|^2_{L^2(\Omega)}} \geq H_n^2.$$

This contradicts $\sigma < H_n$. Thus, the only global minimizer of $\mathcal{F}_{\sigma \boldsymbol{h}}$ is $\boldsymbol{n} = \boldsymbol{e}$.

When $\sigma > H_n$, choose $\widehat{\boldsymbol{n}} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ such that $\boldsymbol{h} \cdot \widehat{\boldsymbol{n}} \not\equiv 0$ in Ω and

$$\frac{1}{\chi_a} \frac{\mathcal{F}[\widehat{\boldsymbol{n}}] - K_1 \|\mathrm{div}\, \boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h} \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}} < H_n^2 + \delta < \sigma^2$$

for small $\delta > 0$. Then

$$\begin{aligned} \mathcal{F}_{\sigma \boldsymbol{h}}[\widehat{\boldsymbol{n}}] &= \mathcal{F}[\widehat{\boldsymbol{n}}] - \chi_a \sigma^2 \|\boldsymbol{h} \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)} \\ &< \mathcal{F}[\widehat{\boldsymbol{n}}] - \chi_a (H_n^2 + \delta) \|\boldsymbol{h} \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)} \\ &< K_1 \|\mathrm{div}\, \boldsymbol{e}\|^2_{L^2(\Omega)} = \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}]. \end{aligned}$$

Thus, $\boldsymbol{e}$ is not a global minimizer of $\mathcal{F}_{\sigma \boldsymbol{h}}$.

We shall examine weak stability. For any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, if we put $\boldsymbol{n}_0 = \boldsymbol{e}$ in (2.6),

$$\mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}_t] - \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}] = t^2 \left\{ \mathcal{F}(\boldsymbol{e})[\boldsymbol{n}_1] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{n}_1|^2 dx - \chi_a \sigma^2 \|\boldsymbol{h} \cdot \boldsymbol{n}_1\|^2_{L^2(\Omega)} \right\} + O(t^3),$$

where $\boldsymbol{n}_t = \boldsymbol{e} + t\boldsymbol{n}_1 + O(t^2)$, $\boldsymbol{n}_1 = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}$. Since $\boldsymbol{h} \cdot \boldsymbol{e} = 0$ in Ω, if

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}|^2 dx - \chi_a \sigma^2 \|\boldsymbol{h} \cdot \boldsymbol{v}\|^2_{L^2(\Omega)} > 0,$$

then $\boldsymbol{e}$ is weak stable.

If $\sigma > H_{sh}$, there exists $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ with $\boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0$ in Ω and $\|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)} = 1$ such that

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx < \chi_a \sigma^2.$$

If we put $\boldsymbol{n}_0 = \boldsymbol{e}$ and $\boldsymbol{v} = \boldsymbol{w}$ in (2.6), we have

$$\mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}_t] - \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}] = t^2 \left\{ \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx - \chi_a \sigma^2 \right\} + O(t^3).$$

Since $\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx < \chi_a \sigma^2$, it follows from Proposition 2.6 (i) that $\boldsymbol{e}$ is not weakly stable.

Assume that $H_n < H_{sh}$. We claim that $\mathcal{F}_{H_n\boldsymbol{h}}$ has a global minimizer $\boldsymbol{e}$ and $\boldsymbol{n}_{H_n}$ such that $\boldsymbol{n}_{H_n} \neq \boldsymbol{e}$. In fact, since H_n is achieved, let $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ be a minimizer of H_n. Here we use the result $C(0) = C(H_n)$ of Proposition 4.1 below. We have

$$\mathcal{F}[\boldsymbol{n}] - \chi_a H_n^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n})^2 dx = \mathcal{F}[\boldsymbol{e}] = \mathcal{F}_{H_n\boldsymbol{h}}[\boldsymbol{e}] = C(0) = C(H_n),$$

where $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$. Thus, $\boldsymbol{e}$ and $\boldsymbol{n}$ are minimizers of $\mathcal{F}_{H_n\boldsymbol{h}}$. On the other hand, we have

$$\mathcal{F}[\boldsymbol{n}] > \mathcal{F}[\boldsymbol{n}] - \chi_a H_n^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n})^2 dx = \mathcal{F}[\boldsymbol{e}].$$

Therefore, we have $\boldsymbol{n} \neq \boldsymbol{e}$.

Finally we show that if $H_n < H_{sh}$ and $H_n < \sigma < H_{sh}$, $\boldsymbol{e}$ is weakly stable. For any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ that is not parallel to $\boldsymbol{e}$, we have

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_t] - \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}] = t^2\Big\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}] \\ - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}|^2 d - \chi_a\sigma^2\|\boldsymbol{h}\cdot(\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}\|^2_{L^2(\Omega)}\Big\} + O(t^3).$$

Here we note that $(\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e})\cdot\boldsymbol{e} = 0$ and $\boldsymbol{h}\cdot(\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}) = \boldsymbol{h}\cdot\boldsymbol{v}$. Since $\sigma < H_{sh}$, we have

$$\chi_a\sigma^2\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{v})^2 dx < \mathcal{F}(\boldsymbol{e})[\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}|^2 dx.$$

Taking (H.2) and (H.3) into consideration, this inequality also holds for the case where $\boldsymbol{h}\cdot\boldsymbol{v} = 0$. Thus, $\boldsymbol{e}$ is weakly stable. This completes the proof. □

4. Estimates of $C(\sigma)$

In this section we shall estimate the minimal energy $C(\sigma)$ defined in (1.5) (cf. [17, Lemma 5.5 (i)]).

Proposition 4.1 *Suppose that* (H.1), (H.2), *and* (H.3) *hold. Then* $C(\sigma)$ *has the following properties.*

(i) $C(\sigma)$ *is a locally Lipschitz continuous and monotone decreasing function on* $[0,\infty)$. *Moreover,* $C(\sigma)$ *is strictly monotone decreasing on* $[H_n,\infty)$ *and*

$$C(\sigma) = C(0) = \mathcal{F}[\boldsymbol{e}] = K_1\int_\Omega (\operatorname{div}\boldsymbol{e})^2 dx \quad \textit{for } 0 \le \sigma \le H_n.$$

(ii) *There exist positive constants* C_1 *and* C_2 *depending only on* K_1, K_2, K_3, $\boldsymbol{h}, \boldsymbol{e}_0$, *and* Ω *such that*

$$|C(\sigma) + \chi_a\sigma^2|\Omega|| \le C_1\sigma + C_2. \tag{4.1}$$

Proof Define $\boldsymbol{k}(x) = \boldsymbol{h}(x) \times \boldsymbol{e}(x)$, so $(\boldsymbol{e}(x), \boldsymbol{k}(x), \boldsymbol{h}(x))$ is an orthonormal basis of $\mathbb{R}^3$. We write $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ by $\boldsymbol{n} = n_e\boldsymbol{e} + n_k\boldsymbol{k} + n_h\boldsymbol{h}$. Since $n_e^2 + n_k^2 + n_h^2 = 1$, we can write

$$\begin{aligned}\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] &= \int_\Omega \{K_1(\operatorname{div}\boldsymbol{n})^2 + K_2(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})^2 + K_3|\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2 dx \\ &\quad -\chi_a\sigma^2\int_\Omega n_h^2 dx \\ &= \mathcal{T}_\sigma[\boldsymbol{n}] - \chi_a\sigma^2|\Omega|,\end{aligned}$$

where

$$\mathcal{T}_\sigma[\boldsymbol{n}] = \int_\Omega \{K_1(\operatorname{div}\boldsymbol{n})^2 + K_2(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})^2 + K_3|\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2 dx + \chi_a\sigma^2\int_\Omega (n_e^2 + n_k^2)dx.$$

Choose a test field

$$\begin{aligned}\boldsymbol{n} &= (\cos\phi)\boldsymbol{e} + (\sin\phi)\boldsymbol{h} \\ &= (\cos\phi)e_1(x)\boldsymbol{e}_1 + (\cos\phi)e_2(x)\boldsymbol{e}_2 + (\cos\phi)e_3(x)\boldsymbol{e}_3 \\ &\quad +(\sin\phi)h_1(x)\boldsymbol{e}_1 + (\sin\phi)h_2(x)\boldsymbol{e}_2 + (\sin\phi)h_3(x)\boldsymbol{e}_3,\end{aligned}$$

where $\boldsymbol{e}(x) = \sum_{i=1}^3 e_i(x)\boldsymbol{e}_i$, $\boldsymbol{h}(x) = \sum_{i=1}^3 h_i(x)\boldsymbol{e}_i$ and $\boldsymbol{e}_1 = (1,0,0), \boldsymbol{e}_2 = (0,1,0), \boldsymbol{e}_3 = (0,0,1)$. Since $\boldsymbol{e}, \boldsymbol{h} \in C^2(\overline{\Omega}; \mathbb{R}^3)$, we see that $|\operatorname{div}\boldsymbol{n}|^2 \le C(|\nabla\phi|^2 + 1)$ for some constant C. Similarly we have

$$(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})^2 + |\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2 \le C(|\nabla\phi|^2 + 1).$$

Thus, if we write

$$\mathcal{T}_\sigma[\boldsymbol{n}] = \int_\Omega f_\sigma(\phi)dx,$$

we have $|f_\sigma(\phi)| \le C\overline{K}(|\nabla\phi|^2 + 1) + \chi_a\sigma^2(\cos\phi)^2$, where

$$\overline{K} = \max\{K_2, K_3\}.$$

As in [3] (cf. [17]), we have $\mathcal{T}_\sigma[\boldsymbol{n}] \le C_1\sigma + C_2$. Therefore, we have

$$C(\sigma) \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \le -\chi_a\sigma^2|\Omega| + C_1\sigma + C_2.$$

Since $\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = \mathcal{T}_\sigma[\boldsymbol{n}] - \chi_a\sigma^2|\Omega|$, from this inequality we have

$$\chi_a\sigma^2\int_\Omega (n_e^2 + n_k^2)dx \le C_1\sigma + C_2.$$

Hence, we have

$$\begin{aligned}\mathcal{T}_\sigma[\boldsymbol{n}] &= \mathcal{F}[\boldsymbol{n}] + \chi_a\sigma^2\int_\Omega (n_e^2 + n_k^2)dx \\ &\ge -\chi_a\sigma^2\int_\Omega (n_e^2 + n_k^2)dx \\ &\ge -C_1\sigma - C_2.\end{aligned}$$

Summing up, we see that (4.1) holds.

Next we shall prove that $C(\sigma)$ is a locally Lipschitz continuous function on $[0,\infty)$. Let $\boldsymbol{n}_\sigma$ and $\boldsymbol{n}_{\sigma_0}$ be minimizers of $\mathcal{F}_{\sigma\boldsymbol{h}}$ and $\mathcal{F}_{\sigma_0\boldsymbol{h}}$, respectively. Then we have

$$\begin{aligned} C(\sigma) &= \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma] \\ &\le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_{\sigma_0}] \\ &= \mathcal{F}_{\sigma_0\boldsymbol{h}}[\boldsymbol{n}_{\sigma_0}] - \chi_a(\sigma^2-\sigma_0^2)\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_{\sigma_0})^2dx \\ &= C(\sigma_0) - \chi_a(\sigma^2-\sigma_0^2)\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_{\sigma_0})^2dx. \end{aligned}$$

Thus, we have

$$C(\sigma)-C(\sigma_0) \le -\chi_a(\sigma^2-\sigma_0^2)\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_{\sigma_0})^2dx \le \chi_a|\sigma^2-\sigma_0^2||\Omega|. \tag{4.2}$$

Exchanging σ and σ_0, we see that

$$|C(\sigma)-C(\sigma_0)| \le \chi_a(\sigma+\sigma_0)|\sigma-\sigma_0||\Omega|.$$

Thus, $C(\sigma)$ is a locally Lipschitz function. From the first inequality of (4.2), we see that

$$C(\sigma)-C(\sigma_0) \le -\chi_a(\sigma^2-\sigma_0^2)\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_{\sigma_0})^2dx \le 0$$

if $\sigma\ge\sigma_0$. From this $C(\sigma)$ is monotone decreasing. When $0<\sigma<H_n$, since $\boldsymbol{e}$ is a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$, we have

$$C(\sigma)=\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}]=\mathcal{F}[\boldsymbol{e}]=C(0)=K_1\int_\Omega (\operatorname{div}\boldsymbol{e})^2dx.$$

By continuity, we have $C(H_n)=C(0)$.

If $\sigma>H_n$, the minimizers $\boldsymbol{n}_\sigma$ of $\mathcal{F}_{\sigma\boldsymbol{h}}$ satisfy $\boldsymbol{n}_\sigma\ne\boldsymbol{e}$. Moreover, we have $\boldsymbol{h}\cdot\boldsymbol{n}_\sigma\not\equiv 0$. In fact, if $\boldsymbol{h}\cdot\boldsymbol{n}_\sigma\equiv 0$, Since $\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma]\le\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}]$, we have $\mathcal{F}[\boldsymbol{n}_\sigma]\le\mathcal{F}[\boldsymbol{e}]$. From this we have

$$\int_\Omega\{K_1(\operatorname{div}\boldsymbol{n}_\sigma)^2+K_2(\boldsymbol{n}_\sigma\cdot\operatorname{curl}\boldsymbol{n}_\sigma)^2+K_3|\boldsymbol{n}_\sigma\times\operatorname{curl}\boldsymbol{n}_\sigma|^2\}dx\le K_1\int_\Omega(\operatorname{div}\boldsymbol{e})^2dx.$$

Moreover, as in the proof of Proposition 2.3, we have

$$\int_\Omega|\nabla\boldsymbol{n}_\sigma|^2dx\le\int_\Omega|\nabla\boldsymbol{e}|^2dx.$$

From our hypothesis (H.1), $\boldsymbol{n}_\sigma=\boldsymbol{e}$. This is a contradiction.

This implies that if $\sigma>\sigma_0>H_n$, we have

$$C(\sigma)-C(\sigma_0) \le -\chi_a(\sigma^2-\sigma_0^2)\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_{\sigma_0})^2dx < 0.$$

Thus, we see that $C(\sigma)$ is a strictly monotone decreasing function on $[H_n,\infty)$. This completes the proof. □

5. Asymptotics of the nematic state as the intensity of magnetic field tends to infinity

In this section we consider the asymptotic behavior of the global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$ as $\sigma \to \infty$. Although the arguments are similar to [17, Lemma 5.5 (ii) and (iii)], the analysis is rather complicated, because $\boldsymbol{h}$ and $\boldsymbol{e}$ are nonconstant.

Proposition 5.1 *Suppose that* (H.1), (H.2), *and* (H.3) *hold. Then we have the following.*

(i) $C(\sigma)/\sigma^2 \to -\chi_a|\Omega|$ *as* $\sigma \to \infty$.

(ii) *Let* $\boldsymbol{n}_\sigma$ *be a global minimizer of* $\mathcal{F}_{\sigma\boldsymbol{h}}$. *Then* $|\boldsymbol{h} \cdot \boldsymbol{n}_\sigma| \to 1$ *in* $L^2(\Omega)$ *as* $\sigma \to +\infty$.

(iii) *Assume that* $K_1 = K_2 = K_3$. *Then* $\boldsymbol{n}_\sigma \to \boldsymbol{h}$ *or* $-\boldsymbol{h}$ *in* $L^2(\Omega, \mathbb{R}^3)$ *as* $\sigma \to +\infty$.

Proof (i) is clear from Proposition 4.1.

As in section 4, define $\boldsymbol{k}(x) = \boldsymbol{h}(x) \times \boldsymbol{e}(x)$. Then $(\boldsymbol{e}(x), \boldsymbol{k}(x), \boldsymbol{h}(x))$ is a orthonormal basis in $\mathbb{R}^3$. For $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$, we can write

$$\boldsymbol{n} = n_e\boldsymbol{e} + n_k\boldsymbol{k} + n_h\boldsymbol{h}, \quad n_e^2 + n_k^2 + n_h^2 = 1 \text{ a.e. in } \Omega.$$

We see that

$$\begin{aligned}
\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] &= \mathcal{F}[\boldsymbol{n}] - \chi_a\sigma^2\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n})^2 dx \\
&= \mathcal{F}[\boldsymbol{n}] - \chi_a\sigma^2\int_\Omega n_h^2 dx \\
&= \mathcal{F}[\boldsymbol{n}] - \chi_a\sigma^2\int_\Omega (1 - n_e^2 - n_k^2) dx \\
&= \mathcal{T}_\sigma[\boldsymbol{n}] - \chi_a\sigma^2|\Omega|.
\end{aligned}$$

Proof of (ii). We write $\boldsymbol{n}_\sigma = n_{\sigma,e}\boldsymbol{e} + n_{\sigma,k}\boldsymbol{k} + n_{\sigma,h}\boldsymbol{h}$. From (4.1), we see that

$$\mathcal{T}_\sigma[\boldsymbol{n}_\sigma] = \mathcal{F}[\boldsymbol{n}_\sigma] + \chi_a\sigma^2\int_\Omega (n_{\sigma,e}^2 + n_{\sigma,k}^2) dx \le C_1\sigma.$$

This implies that

$$\int_\Omega (n_{\sigma,e}^2 + n_{\sigma,k}^2) dx \le \frac{C_1}{\chi_a\sigma} \to 0$$

as $\sigma \to +\infty$. Thus, $\int_\Omega (1 - |n_{\sigma,h}|^2) dx \to 0$ as $\sigma \to +\infty$. Since $|n_{\sigma,h}| \le 1$, for any $1 < p < +\infty$,

$$\int_\Omega (1 - |n_{\sigma,h}|)^p dx \le 2^{p-1}\int_\Omega (1 - |n_{\sigma,h}|) dx \to 0.$$

Since $\boldsymbol{h} \cdot \boldsymbol{n}_\sigma = n_{\sigma,h}$, we see that $|\boldsymbol{h} \cdot \boldsymbol{n}_\sigma| \to 1$ in $L^2(\Omega)$ as $\sigma \to +\infty$.

Proof of (iii). Since $\boldsymbol{e}$ and $\boldsymbol{h}$ are not constant vectors, the analysis has to be carried out carefully. Thus, the proof is rather different to that in [17].

When $K_1 = K_2 = K_3 = K$, we shall show that $\boldsymbol{n}_\sigma$ has the following property: $n_{\sigma,\boldsymbol{h}} > 0$ in Ω or $n_{\sigma,\boldsymbol{h}} < 0$ in Ω or $n_{\sigma,\boldsymbol{h}} \equiv 0$ in Ω.

In fact, $\boldsymbol{n}_\sigma$ satisfies the Euler–Lagrange equation. That is to say, if we write $\boldsymbol{n}_\sigma = \boldsymbol{n}$ for brevity,

$$\begin{cases} -\Delta \boldsymbol{n} = |\nabla \boldsymbol{n}|^2 \boldsymbol{n} + b^2\sigma^2[n_h \boldsymbol{h} - n_h^2 \boldsymbol{n}] & \text{in } \Omega, \\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where $b^2 = \chi_a/K$. Define $\boldsymbol{u}_\sigma = (\operatorname{sign} n_{\sigma,h})\boldsymbol{n}_\sigma$ where $\operatorname{sign} t = 1$ if $t \geq 0$ and $\operatorname{sign} t = -1$ if $t < 0$ and write

$$\boldsymbol{u}_\sigma = u_{\sigma,e}\boldsymbol{e}(x) + u_{\sigma,k}\boldsymbol{k}(x) + u_{\sigma,h}\boldsymbol{h}(x).$$

Since $n_{\sigma,h} = 0$ on $\partial\Omega$, we see that $\boldsymbol{u}_\sigma = \boldsymbol{n}_\sigma = \boldsymbol{e}_0$ on $\partial\Omega$ and $\nabla \boldsymbol{u}_\sigma = \boldsymbol{n}_\sigma \nabla(\operatorname{sign} n_{\sigma,h}) + \operatorname{sign} n_{\sigma,h} \nabla \boldsymbol{n}_\sigma$. It is well known that

$$\nabla(\operatorname{sign} n_{\sigma,h}) = 2\nabla n_{\sigma,h}\delta_{\{n_{\sigma,h}=0\}} = 0.$$

Thus, $|\nabla \boldsymbol{u}_\sigma| = |\nabla \boldsymbol{n}_\sigma|$ and $\boldsymbol{u}_\sigma \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$. Therefore, $\boldsymbol{u}_\sigma$ is also a minimizer of $\mathcal{F}_{\sigma \boldsymbol{h}}$ and so $\boldsymbol{u}_\sigma$ satisfies the Euler–Lagrange equation

$$\begin{cases} -\Delta \boldsymbol{u}_\sigma = |\nabla \boldsymbol{u}_\sigma|^2 \boldsymbol{u}_\sigma + b^2\sigma^2\big((\boldsymbol{h}\cdot\boldsymbol{u}_\sigma)\boldsymbol{h} - (\boldsymbol{h}\cdot\boldsymbol{u}_\sigma)^2\boldsymbol{u}_\sigma\big) & \text{in } \Omega, \\ \boldsymbol{u}_\sigma = \boldsymbol{e}_0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

From the first equation of (5.2), we have

$$-(\Delta \boldsymbol{u}_\sigma)\cdot\boldsymbol{h} = |\nabla \boldsymbol{u}_\sigma|^2 u_{\sigma,h} + b^2\sigma^2(u_{\sigma,h} - u_{\sigma,h}^3). \tag{5.3}$$

Using the Leibniz formula, we can see that

$$\Delta u_{\sigma,h} = (\Delta \boldsymbol{u}_\sigma)\cdot\boldsymbol{h} = 2\mathrm{Tr}[\nabla \boldsymbol{u}_\sigma(\nabla \boldsymbol{h})^t] - \boldsymbol{u}_\sigma \cdot \Delta \boldsymbol{h} \tag{5.4}$$

where A^t denotes the transposed matrix for any matrix A. Since $\boldsymbol{e}\cdot\boldsymbol{h} = 0, \boldsymbol{k}\cdot\boldsymbol{h} = 0$, and $\boldsymbol{h}\cdot\boldsymbol{h} = 1$, we have

$$\begin{aligned} \Delta\boldsymbol{e}\cdot\boldsymbol{h} + 2\mathrm{Tr}[\nabla\boldsymbol{e}(\nabla\boldsymbol{h})^t] + \boldsymbol{e}\cdot\Delta\boldsymbol{h} &= 0, \\ \Delta\boldsymbol{k}\cdot\boldsymbol{h} + 2\mathrm{Tr}[\nabla\boldsymbol{k}(\nabla\boldsymbol{h})^t] + \boldsymbol{k}\cdot\Delta\boldsymbol{h} &= 0, \\ \Delta\boldsymbol{h}\cdot\boldsymbol{h} + 2\mathrm{Tr}[\nabla\boldsymbol{h}(\nabla\boldsymbol{h})^t] + \boldsymbol{h}\cdot\Delta\boldsymbol{h} &= 0. \end{aligned} \tag{5.5}$$

From (5.3), (5.4), and (5.5), we can get the equation

$$\Delta u_{\sigma,h} - 2\nabla\boldsymbol{h}(\boldsymbol{h})\cdot\nabla u_{\sigma,h} + (\Delta\boldsymbol{h}\cdot\boldsymbol{h})u_{\sigma,h} = -|\nabla\boldsymbol{u}_\sigma|^2 u_{\sigma,h} - b^2\sigma^2 u_{\sigma,h}(1 - u_{\sigma,h}^2) + g + \sum_{i=1}^3 \partial_i f^i,$$

where

$$\begin{aligned} g &= -u_{\sigma,e}\{(\Delta\boldsymbol{e}\cdot\boldsymbol{h}) - 2\mathrm{div}\,(\nabla\boldsymbol{h}(\boldsymbol{e}))\} - u_{\sigma,k}\{(\Delta\boldsymbol{k}\cdot\boldsymbol{h}) - 2\mathrm{div}\,(\nabla\boldsymbol{h}(\boldsymbol{k}))\}, \\ f^i &= 2\{((\nabla\boldsymbol{h}(\boldsymbol{e}))_i u_{\sigma,e} + ((\nabla\boldsymbol{h})(\boldsymbol{k}))_i u_{\sigma,k}\}. \end{aligned}$$

Clearly we see that $g \in L^{q/2}(\Omega)$ and $f^i \in L^q(\Omega)$ for any $q > 3$. By the weak Harnack inequality for nonnegative superharmonic function (cf. [12, Theorem 8.18]), for any $B_R(y) \subset \Omega$,

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} u_{\sigma,h}^p dx\right)^{1/p} \leq C\,\{\operatorname*{ess\,inf}_{B_{\theta R}(y)} u_{\sigma,\boldsymbol{h}} + k_\sigma(R)\}$$

for some $p \in [1,3)$ where

$$k_\sigma(R) = R^\delta \sum_{i=1}^{3} \|f^i\|_{L^q(\Omega)} + R^{2\delta}\|g\|_{L^{q/2}(\Omega)},$$

$\delta = 1 - 3/q$ and $0 < \theta < 1$. Here we note that the constant C is independent of σ. Since $u_{\sigma,e}, u_{\sigma,k} \to 0$ in $L^q(\Omega)$ as $\sigma \to \infty$ and $u_{\sigma,e}^2 + u_{\sigma,k}^2 \leq 1$, we see that $k_\sigma(R) \to 0$ as $\sigma \to \infty$. On the other hand, it follows from (ii) that

$$\left\{ \frac{1}{|B_R(y)|} \int_{B_R(y)} u_{\sigma,h}^p dx \right\}^{1/p} \to 1 \quad \text{as } \sigma \to \infty.$$

This implies that if $u_{\sigma,h} \not\equiv 0$, then $u_{\sigma,h} > 0$ in Ω. By (ii), $n_{\sigma,h} \not\equiv 0$ in Ω for large σ. Therefore, $n_{\sigma,h} > 0$ in Ω or $n_{\sigma,h} < 0$ in Ω. Assume that $n_{\sigma,h} > 0$ in Ω. By (ii), $n_{\sigma,h} \to 1$ in $L^2(\Omega)$ as $j \to \infty$. Hence $(n_e^2 + n_k^2)^{1/2} \to 0$ in $L^2(\Omega, \mathbb{R}^3)$. Thus, we have $\boldsymbol{n}_\sigma \to \boldsymbol{h}$ in $L^2(\Omega, \mathbb{R}^3)$; that is to say, $\boldsymbol{h} \cdot \boldsymbol{n}_\sigma \to 1$ in $L^2(\Omega)$. □

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