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On 2 nonsplit extension groups associated with HS and $HS{:}2$

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Abstract: The group $HS{:}2$ is the full automorphism group of the Higman–Sims group HS. The groups $2^{4\cdot}S_6$ and $2^{5\cdot}S_6$ are maximal subgroups of HS and $HS{:}2$, respectively. The group $2^{4\cdot}S_6$ is of order 11520 and $2^{5\cdot}S_6$ is of order 23040 and each of them is of index 3850 in HS and $HS{:}2$, respectively. The aim of this paper is to first construct $\overline{G} = 2^{5\cdot}S_6$ as a group of the form $2^{4\cdot}S_6.2$ (that is, $\overline{G} = \overline{G}_1.2$) and then compute the character tables of these 2 nonsplit extension groups by using the method of Fischer–Clifford theory. We will show that the projective character tables of the inertia factor groups are not required. The Fischer–Clifford matrices of $\overline{G}_1$ and $\overline{G}$ are computed. These matrices together with the partial character tables of the inertia factors are used to compute the full character tables of these 2 groups. The fusion of $\overline{G}_1$ into $\overline{G}$ is also given.

Key words: Group extensions, Higman–Sims group, automorphism group, character table, Clifford theory, inertia groups, Fischer–Clifford matrices

1. Introduction

The Higman–Sims group, HS, is a sporadic simple group of order $2^9.3^2.5^3.7.11 = 44352000$. This is a group that was discovered in 1967 by Higman and Sims [16]. It is a simple group of index 2 in the group of automorphisms of the Higman–Sims graph. Higman and Sims were attending a presentation by Marshall Hall on the Hall–Janko group, J_2, which is a permutation group on 100 points with the stabilizer of a point a subgroup with the other 2 orbits of length 36 and 63. They then thought of a group of permutations on 100 points containing the Mathieu group M_{22}, which has a permutation representation on 22 and 77 points. From these 2 ideas they found HS, with a 1-point stabilizer isomorphic to M_{22}. Higman, in 1969 [15], independently discovered this group as a doubly transitive group acting on a certain "geometry" of 176 points. In his classical paper Conway [7] showed that HS is a subgroup of each of the Conway groups Co_1, Co_2, and Co_3. This group is also 1 of the 7 sporadic groups found in Co_1 but not in the Mathieu groups, and this set of groups is also known as the *second generation* of sporadic groups. The group $HS{:}2$ is of order $88704000 = 2^{10}.3^2.5^3.7.11$ and it is the full automorphism group of HS. The aim of this paper is to compute the Fischer–Clifford matrices of $\overline{G}_1$ and $\overline{G}$. We use these matrices and the partial character tables of each inertia factor group to compute the full character table of each group. In fact, we will show that the projective character tables of the inertia factor groups are not required. This work is taken from the dissertation of the second author [34] and the notations used are consistent with that of the $\mathbb{ATLAS}$ [8] and the $\mathbb{ATLAS}$ of group representations V3 [36].

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The method used is based on Fischer–Clifford theory. Let $\overline{G} = N{\cdot}G$, where $N \lhd \overline{G}$ and $\overline{G}/N \cong G$ is a group extension. The character table of $\overline{G}$ can be constructed once we have:

- the character tables (ordinary and projective) of the inertia factor groups,
- the fusions of classes of the inertia factors into classes of G,
- the Fischer–Clifford matrices of $\overline{G} = N{\cdot}G$.

We will see later that for the groups under discussion in this paper, the projective characters of the inertia factor groups are not involved (only ordinary characters of the inertia factor groups are needed); hence, we are only dealing with the special case of Fischer–Clifford theory, which we outline in the following text.

Let $\bar{g} \in \bar{G}$ be a lifting of $g \in G$ under the natural homomorphism $\bar{G} \longrightarrow G$ and $[g]$ be a conjugacy class of elements of G with representative g. Let $\{\theta_1, \theta_2, \ldots, \theta_t\}$ be a set of representatives of the orbits of $\bar{G}$ on $Irr(N)$ such that for $1 \leq i \leq t$, we have inertia groups $\bar{H}_i = I_{\bar{G}}(\theta_i)$ with the corresponding inertia factors H_i. For each $[g]$ we obtain the matrix $M(g)$ given by

$$M(g) = \begin{bmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{bmatrix} ,$$

where $M_i(g)$ is the submatrix corresponding to the inertia group $\bar{H}i$ and its inertia factor H_i. If $H_i \cap [g] = \emptyset$, then $M_i(g)$ will not exist and $M(g)$ does not contain $M_i(g)$. The size of the matrix $M(g)$ is $c(g) \times c(g)$, where $c(g)$ is the number of conjugacy classes of elements of $\bar{G}$ that correspond to the coset $\bar{g}N$. Then $M(g)$ is the *Fischer–Clifford matrix* of $\bar{G}$ corresponding to the coset $\bar{g}N$. The partial character table of $\bar{G}$ on the classes $\{x_1, x_2, \ldots, x_{c(g)}\}$ is given by

$$\begin{bmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix},$$

where the Fischer–Clifford matrix $M(g)$ is divided into blocks with each block corresponding to an inertia group $\bar{H}_i$ and $C_i(g)$ is the partial character table of H_i consisting of the columns corresponding to the classes that fuse into $[g]$ in G. We obtain the characters of $\bar{G}$ by multiplying the relevant columns of the characters of H_i by the rows of $M(g)$.

The theory of Fischer–Clifford matrices, which is based on Clifford theory (see [6]), was developed by B. Fischer ([11], [12], and [13]). This technique has also been discussed and applied to both split and nonsplit extension in several publications, for example in [1, 2, 4, 25, 29]. One can read more on Fischer–Clifford theory and projective characters in [10, 28, 27, 35] and [9, 18, 17, 20, 30, 31, 32], respectively. For the theory of characters one can also read [19].

1.1. The Conway groups

Leech created a lattice that gives the tightest lattice packing of spheres in 24 dimensions [21]. Conway analyzed the symmetry of this lattice in detail in [7] and discovered 3 previously unknown sporadic groups, namely the Co_1, Co_2, and Co_3. Let us give a definition of the Leech lattice, which is given as Theorem 5.1 in [37].

Definition 1.1 The *Leech lattice* Λ is a 24-dimensional even integral lattice containing no vectors of norm 2, 196560 vectors of norm 4, 16773120 vectors of norm 6, and 398034000 vectors of norm 8.

We first construct the biggest Conway group $Aut(\Lambda) = .O = 2.Co_1$ as a group of 24×24 matrices. All the vectors of norm 8 in the Leech lattice fall into congruence classes of 48 pairs of mutually perpendicular vectors called the *crosses* and we get 8292375 such crosses. When $.O$ acts on these crosses, the stabilizer of a cross is a group $2^{12}{:}M_{24}$, which is maximal in $.O$. So $.O$ is a group of order $8292375.2^{12}.|M_{24}|$. The group $.O$ is a perfect group with $Z(.O) = 2$. The quotient of this group by the center is a group denoted by $.1 = Co_1$ and is of order

$$|Co_1| = 4157776806543360000 = 2^{21}.3^9.5^4.7^2.11.13.23.$$

Note that the action of $.O$ on crosses is transitive and Co_1 is a simple group.

$.O$ also acts transitively on vectors of norm 4 having the products ± 4 or 0. These 3 orbits of $2^{12}{:}M_{24}$ on vectors of norm 4 are fused into a single orbit under $2.Co_1$. The stabilizer of a vector of norm 4 is denoted by Co_2, where

$$|Co_2| = 42305421312000 = 2^{18}.3^6.5^3.7.11.23.$$

Lastly, $.O$ is transitive on vectors of norm 6. The stabilizer of a vector of norm 6 is denoted by Co_3 and is of order

$$|Co_3| = 423054213122000 = 2^{10}.3^7.5^3.7.11.23.$$

From the $\mathbb{ATLAS}$ [8] we see that $Co_3 \leq Co_2 \leq Co_1$ with Co_2 and Co_3 both maximal subgroups of Co_1 and Co_3 a maximal subgroup of Co_2.

1.2. The Higman–Sims group

We get the Higman–Sims group HS by showing that Co_3 acts transitively on the set S of 11178 vectors of norm 4 that have inner product -2 with vector v, when $v = (-2^{12}, 0^{12})$. The monomial group $2 \times M_{12}$ fixes v and has 6 orbits on S. When $u = (-5, -1^{23})$, the group M_{23} fixes u and has 5 orbits on S. The only way for both these sets of orbits to fuse into orbits for Co_3 is a single orbit of length 11178. Thus, the stabilizer in Co_3 of such a vector in S is a subgroup of index 11178. This is the Higman–Sims group HS of order

$$|HS| = 44352000 = 2^9.3^2.5^3.7.11 .$$

Moreover, if we let $w = (5, 1, 1^{22})$ and $x = (-1, -5, -1^{22})$, the stabilizer of the set $\{w, x\}$ is the monomial group $M_{22}{:}2$ and we get an involution of the group, which interchanges the 2 vectors. This results in HS extending to $HS{:}2$, which is a full automorphism group of HS. A complete list of maximal subgroups of the Conway groups is provided in Table 5.3 of [37]. For further reading one can also go to [7, 21, 24, 37].

We use [36] to find two 20×20 matrices a and b with a from class $2A$, b from class $5A$, and $HS =< a, b >$. Again using [36] we find two 20×20 matrices c, d from classes $2C$ and $5C$ of $HS{:}2$, respectively, with $HS{:}2 =< c, d >$. From the HS computed, $HS{:}2$ is an automorphism group of an isomorphic copy of it.

1.3. The groups $2^{4}{\cdot}S_6$ and $2^{5}{\cdot}S_6$

The group $HS{:}2$ has 3 conjugacy classes of subgroups of order 11520. The first is a group $2^{4}{\cdot}S_6$ that sits maximally inside of HS. The second is a group $2^4{:}S_6$ that is maximal in $\overline{M}_{22}$ and hence sits inside $HS{:}2$, but not inside HS. The third is a group of the form $2^5{:}A_6$ that is a maximal subgroup of $2^{5\cdot}S_6$. The group we are interested in, $2^{4\cdot}S_6$, is a maximal subgroup of $HS \leq HS{:}2$. For further reading on $2^4{:}S_6$ as a maximal subgroup of $\overline{M}_{22}$ one can read [23] and [35]. The group $2^{5\cdot}S_6$ is a group of order 23040 and it is a maximal subgroup of $HS{:}2$. The groups $2^{4\cdot}S_6$ and $2^{5\cdot}S_6$ are unique maximal subgroups of their form in HS and $HS{:}2$, respectively. Using generators a and b of HS and Programme G [34], we obtain elements a_1' and b_1' with $o(a_1') = 2$, $o(b_1') = 5$, and $\overline{G}_1' =< a_1', b_1' >= 2^{4\cdot}S_6$. Similarly using generators c and d of $HS{:}2$ and Programme H [34], we obtain two elements c' and d' with $o(c') = 2$, $o(d') = 5$ and $\overline{G'} =< c', d' >= 2^{5\cdot}S_6$. Our aim is to construct $\overline{G} = 2^{5\cdot}S_6$ as $\overline{G}_1.2$, where $\overline{G_1} = 2^{4\cdot}S_6$ and $\overline{G} = \overline{G}_1.2 \cong 2^{5\cdot}S_6$ are both inside $HS{:}2$. Since $\overline{G_1'}$ is in HS we seek for its isomorphic copy $\overline{G_1}$ in $HS{:}2$. The extension of $\overline{G}_1$ is $\overline{G}_1.2 = \overline{G}$ and $\overline{G} \cong \overline{G'}$.

Having obtained $\overline{G'}$, using GAP [14], we get 3 of its subgroups of order 11520. By methods of coset analysis [34], we determine that each of these 3 subgroups is of the form $2^4.S_6$. From these 3 subgroups, only 1, $\overline{G}_1$, is isomorphic to $\overline{G_1'}$ in HS. The group $\overline{G}_1$ has 7 generators, of which 5 are of order 2, 1 of order 5, and 1 of order 6. To this list of 7 generators we add 1 of the generators of $HS{:}2$ of order 2, namely c. The group generated by these 8 elements is $\overline{G} = 2^{4\cdot}S_6.2 = 2^{5\cdot}S_6$.

The groups $2^{4\cdot}S_6$ and $2^{5\cdot}S_6$ will be discussed fully in Sections 2 and 3, respectively.

2. The group $\overline{G}_1 = 2^{4\cdot}S_6$

From [36] we get two 20×20 matrices a and b over $GF(2)$ with $o(a) = 2$, $o(b) = 5$, $o(ab) = 11$, and $HS =< a, b >$. Again from [34] we get Programme G, where there are 2 inputs with $a = input[1]$ and $b = input[2]$. The program results in 2 outputs. Let $a_1' = output[1]$ and $b_1' = output[2]$. Then we have $o(a_1') = 2$, $o(b_1') = 5$, $o(a_1'b_1') = 6$, and $\overline{G'}_1 =< a_1', b_1' >= 2^{4\cdot}S_6$. Up to isomorphism, there is only 1 group of the type $2^{4\cdot}S_6$ that is a maximal subgroup of HS and this has 21 conjugacy classes of elements, of which 2 are classes of involutions.

Going back to [36], we get two 20×20 matrices c and d with $o(c) = 2$, $o(d) = 5$, and $HS{:}2 =< c, d >$. From [34] we again get Programme H where $c = input[1]$ and $d = input[2]$. Again from the program we get 2 outputs. Let $c_1' = output[1]$ and $d_1' = output[2]$. We get that $o(c_1') = 2$, $o(d_1') = 10$, $o(c_1'd_1') = 6$, and $\overline{G'} =< c_1', d_1' >= 2^{5\cdot}S_6$. Programmes G and H can also be found in [36].

Using GAP [14], we get 8 normal subgroups of $\overline{G'}$. Three of these groups (we call them $S1, S2, S3$) are of order 11520 and for each group the conjugacy class $2A$ has 15 elements; when S_6 acts on 2^4, we get 2 orbits of length 1 and 15 and hence all these groups are of the form $2^4.S_6$. One of them ($S2 = 2^4{:}S_6$), however, has 5 classes of involutions and is thus not a maximal subgroup of HS. The other one ($S3$, a split extension of 2^5 by A_6) has 24 conjugacy classes and again is not a maximal subgroup of HS. The group $S2$, from [23] and [35], is actually a maximal subgroup of $\overline{M}_{22}$. This leaves us with the group $S1 = \overline{G}_1 \cong \overline{G_1'}$. See Remark 2.1 for more details on groups $S1$, $S2$, and $S3$. The group $\overline{G}_1$ has 7 generators $a_1, a_2, a_3, a_4, a_5, a_6$, and a_7 with a_1 of order 2, a_2 of order 5, a_3 of order 6, and the rest of order 2. We use GAP to compute normal subgroups of $\overline{G}_1$ and it has only 1 proper normal subgroup, the elementary abelian group $N_1 = 2^4$. Our aim is to act $\overline{G}_1$ on N_1 and to do this we use Programme C [34]; this requires us to consider N_1 as a full row space V_1 of dimension 4 over $GF(2)$. The action of $\overline{G}_1$ on V_1 is multiplication of V_1 from the right. For this multiplication to be possible, this then requires us to rewrite $\overline{G}_1$ from a 20×20 representation to 4×4. To do this, we act

$\overline{G}_1$ on N_1 by acting the 7 generators $a_i,\ i=1,\cdots,7$ of $\overline{G}_1$ on the 4 generators $\lambda_i,\ i=1,\cdots,4$ of N_1.

Writing this action as maps we get:

$$a_1 : \lambda_1 \to \lambda_2,\ \lambda_2 \to \lambda_1,\ \lambda_3 \to \lambda_1\lambda_3\lambda_4,\ \lambda_4 \to \lambda_1\lambda_2\lambda_4;$$

$$a_2 : \lambda_1 \to \lambda_2,\ \lambda_2 \to \lambda_4,\ \lambda_3 \to \lambda_1\lambda_2,\ \lambda_4 \to \lambda_2\lambda_3\lambda_4;$$

$$a_3 : \lambda_1 \to \lambda_2\lambda_3\lambda_4,\ \lambda_2 \to \lambda_4,\ \lambda_3 \to \lambda_1\lambda_2\lambda_3,\ \lambda_4 \to \lambda_2.$$

For the rest, that is a_4 to a_7, we get:

$$a_i : \lambda_1 \to \lambda_1,\ \lambda_2 \to \lambda_2,\ \lambda_3 \to \lambda_3,\ \lambda_4 \to \lambda_4.$$

Writing this in matrix form we get:

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}; \quad \alpha_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}; \alpha_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

For the rest, α_4 to α_7, we get:

$$\alpha_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $G_1 =< \alpha_1, \alpha_2, \alpha_3 >\cong S_6$; that is, the action of $\overline{G}_1$ on N_1 is isomorphic to S_6.

Remark 2.1 Note that $N_1 = 2^4$ is generated by 4 commuting involutions from the class $2A$ of HS. From $\mathbb{ATLAS}$ we can see that $S1 = \overline{G}_1 = N_{HS}(N_1)$, $S2 = N_{\overline{M}_{22}}(N_1)$, and $N_{HS:2}(N_1) = 2^{5\cdot}S_6 = \overline{G}$. As observed, $S1, S2, S3$ are nonisomorphic maximal subgroups of $\overline{G}$ and that $S2$ and $S3$ do not sit inside HS. Our computations show that

$$\begin{aligned} S1 &= 2^{4\cdot}S_6 = \overline{G} \cap HS \leq_{max} HS \leq_{max} HS{:}\,2; \\ S2 &= 2^4{:}S_6 = \overline{G} \cap \overline{M}_{22} \leq_{max} \overline{M}_{22} \leq_{max} HS{:}\,2; \\ S3 &= 2^{4\cdot}(A_6 \times 2) \cong 2^5{:}A_6 \leq_{max} \overline{G} \leq_{max} HS{:}\,2; \\ S1 \cap S2 &= S2 \cap S3 = S1 \cap S3 = N_{M_{22}}(N_1) = 2^4{:}A_6 \leq_{max} M_{22} \leq_{max} HS. \end{aligned}$$

We compute the character table of $S1 = \overline{G}_1$ in Section 2.2 (Table 4) by using Fischer–Clifford theory. The character tables of $S2$ and $S3$ are given in Table 1 and Table 2 of [34], respectively. It is also interesting to note that the character tables of $S1$ and $S2$ have the same number of conjugacy classes. A pictorial view of Remark 2.1 is given in the Figure, where $A = 2^4.A_6$.

Lemma 2.2 $\overline{G} = S1 \cup S2 \cup S3$.

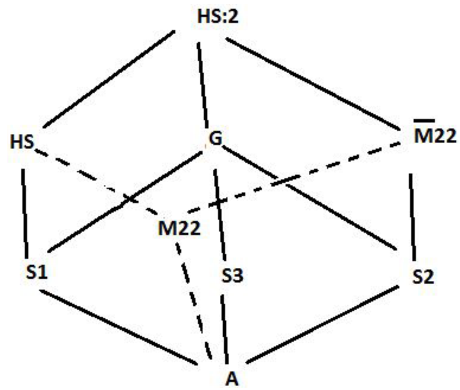

Figure. S1, S2, and S3.

Proof First we see that $\overline{G} \supseteq S1 \cup S2 \cup S3$, but we also have

$$S1 \cup S2 \cup S3 = (S1 - A) \cup (S2 - A) \cup (S3 - A) \cup A.$$

Hence:

$$\begin{aligned} |S1 \cup S2 \cup S3| &= |S1 - A| + |S2 - A| + |S3 - A| + |A| \\ &= 3 \times (16 \times 6! - 16 \times \frac{6!}{2}) + (16 \times \frac{6!}{2}) \\ &= 3 \times 16 \times 6! - 2 \times 16 \times \frac{6!}{2} \\ &= 2 \times 16 \times 6! \\ &= |2^5.S_6|. \end{aligned}$$

Thus, $2^5.S_6 = S1 \cup S2 \cup S3$. □

Theorem 2.3 *$HS{:}2$ has only 3 conjugacy classes of subgroups of type $2^4.A_6.2$. In particular, $S1$ and $S2$ are of type $2^4.A_6.2_1$ and $S3$ is of type $2^4.(A_6 \times 2)$.*

Proof From the $\mathbb{ATLAS}$ we can see that if $H \leq HS{:}2$ is of type $2^4.A_6.2$, then H must sit in one of the maximal subgroups of $HS{:}2$ of type $HS, \overline{M}_{22}$ or $2^{5\cdot}S_6$. Also, since $N_{HS:2}(S1) \supseteq N_{\overline{G}}(S1) = \overline{G}$ and $\overline{G}$ is maximal but not normal in $HS{:}2$, we have $N_{HS:2}(S1) = \overline{G}$. Hence, $[HS{:}2 : N_{HS:2}(S1)] = [HS{:}2 : \overline{G}] = 3850$. Similarly, since $N_{HS:2}(S2) = N_{HS:2}(S3) = \overline{G}$, we have $[HS{:}2 : N_{HS:2}(S2)] = [HS{:}2 : N_{HS:2}(S3)] = 3850$. Hence, we have 3 conjugacy classes for the subgroups of type $2^4.A_6.2$ in $HS{:}2$. Thus, the total number of subgroups of type $2^4.A_6.2$ in $HS{:}2$ is $3 \times 3850 = 11550$. □

2.1. Conjugacy classes and inertia factors of $\overline{G}_1$

Using GAP [14], we compute the conjugacy classes of $2^{4\cdot}S_6$. The action of $\overline{G}_1$ on N_1 is viewed as the action of G_1 on V_1. If G_1 acts on N_1, we get 2 orbits of length 1 and 15. From the $\mathbb{ATLAS}$ [8], by checking on the indices of maximal subgroups of S_6, we can see that there are 2 inertia factor groups, namely S_6 and $S_4 \times 2$. The full inertia groups are of the form $\overline{H_i} = 2^4.H_i$ of indices 1 and 15 in $2^{4\cdot}S_6$, respectively. We note that $H_1 \cong S_6$ and $H_2 \cong S_4 \times 2$. The character tables of H_1 and H_2 are easy to compute. The fusion of $S_4 \times 2$ into S_6 is given in Table 1. This technique has been used by various authors and several MSc and PhD students of the first author, such as Ali [1, 2], Mpono [28, 25], Rodrigues [33], and Whitely [35].

Table 1. The fusion of $S_4 \times 2$ into S_6.

$[x]_{S_4\times 2}$	$\longrightarrow$	$[g_1]_{S_6}$
$1A$		$1A$
$2A$		$2C$
$2B$		$2B$
$2C$		$2B$
$2D$		$2A$
$2E$		$2A$
$3A$		$3A$
$4A$		$4A$
$4B$		$4B$
$6A$		$6A$

We computed the conjugacy classes of $2^{4\cdot}S_6$ by using GAP [14] and then fused them into HS. Having the length of each coset, we use the fusion map to convert the conjugacy classes of $2^{4\cdot}S_6$ into the form that is required for the computation of Fischer–Clifford matrices (that is, into a form normally obtained by coset analysis). We give the conjugacy classes of $2^{4\cdot}S_6$ in Table 2.

2.2. Fischer–Clifford matrices and character table of $\overline{G}_1$

Most of the arguments used here and in the subsequent sections are very similar to the ones given in [26]. From the fusions and orbit lengths and centralizer orders, we compute the Fischer–Clifford matrix $M(1A)$ of $\overline{G}_1$; that is, $M(1A) = \begin{bmatrix} 1 & 1 \\ 15 & -1 \end{bmatrix}$.

Having computed $M(1A)$, we want to determine the type of partial character tables we are going to use for our computations. We will show that the ordinary character table of H_2 is required. We follow the methods used in [1, 2] and we use the character table of HS . Let $Irr(HS) = \{\Psi_i : 1 \leq i \leq 24\}$, where the notation is the same at that used in the $\mathbb{ATLAS}$ [8]. From the list we take the values of Ψ_2 and Ψ_3 on $1A$ and $2A$. We get:

$[x]_{HS}$	1A	2A
Ψ_2	22	6
Ψ_3	77	13

Let γ_1 and γ_2 be the rows of the Fischer–Clifford matrix $M(1A)$. Since

$$< (\psi_2)_N, 1_N >= \frac{1}{16}(22 + 15 \times 6) = \frac{112}{16} = 7,$$

we get the following decomposition: $22 = 7 + 15k$. Thus, $k = 1$ and hence $(\Psi_2)_N = 7\gamma_1 + \gamma_2$. Let $[x_1, \cdots, x_t]$ be the transpose of the partial entries for the projective characters of $H_2 \cong S_4 \times 2$ on $1A \in S_6$. Then

Table 2. Conjugacy classes of $2^{4\cdot}S_6$.

$[g]_{S_6}$	$[x]_{2^{4\cdot}S_6}$	$\lvert C_{2^{4\cdot}S_6}(x)\rvert$	$\longrightarrow HS$
1A	1A	11520	1A
	2A	768	2A
2A	2B	96	2B
	4A	384	4A
	4B	128	4B
2B	2C	64	2A
	4C	64	4B
	4D	32	4B
2C	2D	192	2A
	4E	64	4B
3A	3A	192	3A
	6A	24	6B
3B	3B	18	3A
4A	4F	16	4B
	8A	16	8A
4B	4G	16	4C
	8B	16	8A
5A	5A	5	5C
6A	6B	12	6A
	12A	12	12A
6B	6C	6	6B

$C_2(1A)M_2(1A)$ is a $t\times 2$ matrix; from the first entry of the first column we get $15x_1 = 15$. Hence, $x_1 = 1$ and this shows that the partial character table of H_2 that we need comes from the ordinary character table of H_2. Thus, we use the ordinary character table of $S_4 \times 2$.

To compute the Fischer–Clifford matrices, we use their general properties (which can also be found in [1], [28], and [35]) and the fusion of $S_4 \times 2$ into S_6, the centralizer orders of $2^{4\cdot}S_6$, the fusion of $\overline{G}$ into HS, together with restriction of HS to $\overline{G}$ that forces the signs of the Fischer–Clifford matrices. We give these in Table 3. Note the change of sign in $M(2A)$.

Table 3. The Fischer–Clifford matrices of $2^{4\cdot}S_6$.

$M(1A)=\begin{bmatrix}1 & 1\\ 15 & -1\end{bmatrix}$	$M(2A)=\begin{bmatrix}1 & 1 & 1\\ -1 & 1 & 1\\ 0 & -6 & 2\end{bmatrix}$
$M(2B)=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 2 & -2 & 0\end{bmatrix}$	$M(2C)=\begin{bmatrix}1 & 1\\ 3 & -1\end{bmatrix}$
$M(3A)=\begin{bmatrix}1 & 1\\ 3 & -1\end{bmatrix}$	$M(3B)=\begin{bmatrix}1\end{bmatrix}$
$M(4A)=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	$M(4B)=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$
$M(6A)=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	$M(5A)=M(6B)=\begin{bmatrix}1\end{bmatrix}$

For example, we calculate the partial character table corresponding to coset $2A \in S_6$. Let $C_1(2A), C_2(2A)$ be the partial character tables of the inertia factors for the classes that fuse to $2A \in S_6$. We have $M_1(2A) =$

$[1\ 1\ 1]$, $M_2(2A) = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -6 & 2 \end{bmatrix}$. Then the portions of the character table of $\overline{G} = 2^{4\cdot}S_6$ corresponding to the coset $2A$ are:

$$C_1(2A)M_1(2A) = \begin{bmatrix} 1 \\ -1 \\ -3 \\ 3 \\ -1 \\ 1 \\ -3 \\ 3 \\ -2 \\ 2 \\ 0 \end{bmatrix} [1\ 1\ 1] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \\ -2 & -2 & -2 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_2(2A)M_2(2A) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -2 & 0 \\ 2 & 0 \\ -3 & -1 \\ 3 & -1 \\ -3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -6 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -5 & 3 \\ 1 & 5 & -3 \\ -1 & 7 & -1 \\ 1 & -7 & 1 \\ 2 & -2 & -2 \\ -2 & 2 & 2 \\ 3 & 3 & -5 \\ -3 & 9 & 1 \\ 3 & -9 & -1 \\ -3 & -3 & 5 \end{bmatrix}.$$

We get the character table of $2^{4\cdot}S_6$ in Table 4, which can be compared to the one in GAP.

Table 4. The character table of $2^{4\cdot}S_6$.

	1A		2A			2B			2C		3A		3B	4A		4B		5a	6A		6B
	1a	2a	2b	4a	4b	2c	4c	4d	2d	4e	3a	6a	3b	4f	8a	4g	8b	5a	6b	12a	6c
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	-1	1	1	1	-1	-1	1	1	1	-1	-1	1	1	1	-1	-1	-1
χ_3	5	5	-3	-3	-3	1	1	1	1	1	2	2	-1	-1	-1	-1	-1	0	0	0	1
χ_4	5	5	3	3	3	1	1	1	-1	-1	2	2	-1	1	1	-1	-1	0	0	0	-1
χ_5	5	5	-1	-1	-1	1	1	1	3	3	-1	-1	2	1	1	-1	-1	0	-1	-1	0
χ_6	5	5	1	1	1	1	1	1	-3	-3	-1	-1	2	-1	-1	-1	-1	0	1	1	0
χ_7	9	9	-3	-3	-3	1	1	1	-3	-3	0	0	0	1	1	1	1	-1	0	0	0
χ_8	9	9	3	3	3	1	1	1	3	3	0	0	0	-1	-1	1	1	-1	0	0	0
χ_9	10	10	-2	-2	-2	-2	-2	-2	2	2	1	1	1	0	0	0	0	0	1	1	-1
χ_{10}	10	10	2	2	2	-2	-2	-2	-2	-2	1	1	1	0	0	0	0	0	-1	-1	1
χ_{11}	16	16	0	0	0	0	0	0	0	0	-2	-2	-2	0	0	0	0	1	0	0	0
χ_{12}	15	-1	-1	-5	3	3	-1	-1	3	-1	3	-1	0	1	-1	1	-1	0	1	-1	0
χ_{13}	15	-1	1	5	-3	3	-1	-1	-3	1	3	-1	0	-1	1	1	-1	0	-1	1	0
χ_{14}	15	-1	-1	7	-1	-1	3	-1	3	-1	3	-1	0	-1	1	-1	1	0	1	-1	0
χ_{15}	15	-1	1	-7	1	-1	3	-1	-3	1	3	-1	0	1	-1	-1	1	0	-1	1	0
χ_{16}	30	-2	2	-2	-2	2	2	-2	-6	2	-3	1	0	0	0	0	0	0	1	-1	0
χ_{17}	30	-2	-2	2	2	2	2	-2	6	-2	-3	1	0	0	0	0	0	0	-1	1	0
χ_{18}	45	-3	3	3	-5	1	-3	1	3	-1	0	0	0	1	-1	-1	1	0	0	0	0
χ_{19}	45	-3	-3	9	1	-3	1	1	-3	1	0	0	0	1	-1	1	-1	0	0	0	0
χ_{20}	45	-3	3	-9	-1	-3	1	1	3	-1	0	0	0	-1	1	1	-1	0	0	0	0
χ_{21}	45	-3	-3	-3	5	1	-3	1	-3	1	0	0	0	-1	1	-1	1	0	0	0	0

We compute the permutation characters of $HS{:}2$ when acting on $S1, S2$, and $S3$. For interest's sake we also include $\chi(HS|S1)$ and later we also give $\chi(2^5.S_6|Si)$, $i = 1, 2, 3$.

$\chi(HS|S1) = 1a + 22a + 77aa + 154a + 175a + 693a + 770a + 825a + 1056a = \chi(HS{:}2|2^5.S_6)$,

$\chi(HS{:}2|S1) = 1a + 1b + 22a + 22b + 77aa + 77bb + 154a + 154b + 175a + 175b + 693a + 693b + 770a + 770b + 825a + 825b + 1056a + 1056b$,

$\chi(HS{:}2|S2) = 1a + 22aa + 77aaa + 154a + 175a + 231a + 693a + 770aa + 825aa + 1056a + 1925a$,

$\chi(HS{:}2|S3) = 1a + 22a + 22b + 77aa + 77b + 154a + 175a + 231a + 693a + 770a + 770b + 825a + 825b + 1056a + 1925b$.

3. Group $\overline{G} = 2^{5\cdot}S_6$

Having completed the computation of the full character table of $2^{4\cdot}S_6$, we now turn our attention to $2^{5\cdot}S_6$. We compute $2^{5\cdot}S_6 = 2^{4\cdot}S_6.2$ by adding the generator c of $HS{:}2$; that is, from $\overline{G}_1$ we get $\overline{G} =< a_1, a_2, a_3, a_4, a_5, a_6, a_7, c >$. Since $2^{5\cdot}S_6$ is the only group of its type that is a maximal subgroup of $HS{:}2$, we have $\overline{G} \cong \overline{G'}$, where $\overline{G'}$ was computed using Programme H. Our aim is to compute the full character table of $2^{5\cdot}S_6$. We first want to let $\overline{G}$ act on the elementary abelian group $N = 2^5$. We use GAP [14] to compute $N = 2^5$ as a normal subgroup of $\overline{G}$.

For the action of $\overline{G}$ we use Programme C [34]. We consider N as a full row vector space V of dimension 5 over $GF(2)$. For us to be able to act on a 5-dimensional vector space V it becomes necessary to rewrite $\overline{G}$ from a 20×20 to a 5×5 representation. To do this we first take the 8 generators of $\overline{G}$, namely a_1 to a_7 and c. We let these act on generators $\gamma_i,\ 1 = 1, \cdots, 5$ of our elementary abelian group $N = 2^5$.

Writing these as maps we get:

$$a_1 : \gamma_1 \to \gamma_1,\ \gamma_2 \to \gamma_3\gamma_4,\ \gamma_3 \to \gamma_1\gamma_3,\ \gamma_4 \to \gamma_1\gamma_2\gamma_3,\ \gamma_5 \to \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5;$$
$$a_2 : \gamma_1 \to \gamma_2\gamma_3\gamma_4,\ \gamma_2 \to \gamma_3,\ \gamma_3 \to \gamma_1\gamma_3,\ \gamma_4 \to \gamma_2,\ \gamma_5 \to \gamma_2\gamma_5;$$
$$a_3 : \gamma_1 \to \gamma_1\gamma_2,\ \gamma_2 \to \gamma_1\gamma_2\gamma_3\gamma_4,\ \gamma_3 \to \gamma_4,\ \gamma_4 \to \gamma_1\gamma_4,\ \gamma_5 \to \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5;$$
$$c : \gamma_1 \to \gamma_3,\ \gamma_2 \to \gamma_2,\ \gamma_3 \to \gamma_1,\ \gamma_4 \to \gamma_4,\ \gamma_5 \to \gamma_5.$$

For the rest, a_4 to a_7, we get:

$$a_i : \gamma_1 \to \gamma_1,\ \gamma_2 \to \gamma_2,\ \gamma_3 \to \gamma_3,\ \gamma_4 \to \gamma_4,\ \gamma_5 \to \gamma_5.$$

Writing this in matrix form we get:

$$\beta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

$$\beta_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \beta_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the rest, β_5 to β_8, we get that $\beta_i = I_5$.

Let $G =< \beta_1, \beta_2, \beta_3, \beta_4 >$; then $G \cong S_6$, which means that the action of $\overline{G}$ on N is isomorphic to S_6.

3.1. Conjugacy classes and inertia factors of $\overline{G}$

The action of $\overline{G}$ on N is reflected by the action of G on V. When G acts on V we get 4 orbits of conjugacy classes of elements of N, of lengths $1, 6, 10$, and 15. Let G^t be the set of all transpose of elements of G. The group G^t can also be generated by transpose matrices of each generator of G. When G^t acts on V, which is the equivalent of G acting on $Irr(N)$, by Brauer's theorem [5] we get 4 orbits, but these are of lengths $1, 1, 15$, and 15. These have corresponding point stabilizers H_1, H_2, H_3, and H_4. Let the full inertia groups be $\overline{H_i} = 2^5.H_i,\ i = 1, 2, 3, 4$. From the $\mathbb{ATLAS}$ [8], the corresponding inertia factor groups are $S_6,\ S_6,\ S_4 \times 2$, and $S_4 \times 2$. We have $H_1 \cong H_2 \cong S_6$ and $H_3 \cong H_4 \cong S_4 \times 2$. The character tables of S_6 and that of $HS{:}2$ are obtained from the $\mathbb{ATLAS}$ [8]. We also give the fusion of $S_4 \times 2$ into S_6 in Table 5.

Table 5. The fusion of $S_4 \times 2$ into S_6.

$[x]_{S_4\times 2}$	$\longrightarrow$	$[g_1]_{S_6}$
$1A$		$1A$
$2A$		$2C$
$2B$		$2B$
$2C$		$2B$
$2D$		$2A$
$2E$		$2A$
$3A$		$3A$
$4A$		$4A$
$4B$		$4B$
$6A$		$6A$

We computed the conjugacy classes of $2^{5}{\cdot}S_6$ by using GAP [14] and then fused them into $HS{:}2$. Having the length of each coset, we use the fusion map to convert the conjugacy classes of $2^{5}{\cdot}S_6$ into the form that is required for the computation of Fischer-Clifford matrices (that is, into a form normally obtained by coset analysis). We give the conjugacy classes of $2^{5}{\cdot}S_6$ in Table 6.

3.2. Fischer–Clifford matrices and character table of $\overline{G}$

From the fusions and orbit lengths and centralizer orders, we compute the Fischer–Clifford matrix $M(1A)$:

$$M(1A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 15 & 5 & -3 & -1 \\ 15 & -5 & 3 & -1 \end{bmatrix}.$$

Having computed $M(1A)$, we want to determine the type of partial character tables we are going to use for our computations. We follow the methods used in [34], which can also be found in [1]. We use the character table of $HS{:}2 =< a, b >$. Let $Irr(HS:2) = \{\Psi_i : 1 \leq i \leq 39\}$; the notation is the same at that used in the $\mathbb{ATLAS}$ [8]. We list the values of $\Psi_i, \leq i \leq 6$ on $1A, 2A, 2B$, and $2C$:

$\lvert C_{\overline{G}}(x)\rvert$	23040	3840	2304	1536
$[x]_{HS:2}$	1A	2A	2B	2C
Ψ_2	1	-1	-1	1
Ψ_3	22	0	8	6
Ψ_4	22	0	-8	6
Ψ_5	77	5	21	13
Ψ_6	77	-5	-21	13

Table 6. Conjugacy classes of $2^{5\cdot}S_6$.

$[g]_{S_6}$	$[x]_{2^5\cdot S_6}$	$\lvert C_{2^5\cdot S_6}(x)\rvert$	$\longrightarrow$ $HS{:}2$
1A	1A	23040	1A
	2A	3840	2D
	2B	2304	2C
	2C	1536	2A
2A	2D	768	2C
	4A	768	4A
	2E	256	2D
	4B	256	4B
	4C	192	4A
	2F	192	2B
2B	2G	128	2A
	2H	128	2D
	4D	128	4D
	4E	128	4B
	4F	64	4C
	4G	64	4A
2C	2I	384	2A
	2J	384	2C
	4H	64	4B
	4I	64	4D
3A	3A	144	3A
	6A	144	6C
	6B	48	6E
	6C	48	6B
3B	3B	36	3A
	6D	36	6A
4A	4J	32	4A
	8A	32	8C
	4K	32	4B
	8B	32	8A
4B	4L	32	4C
	8C	32	8A
	4M	32	4D
	8D	32	8D
5A	5A	10	5C
	10A	5	10D
6A	6E	24	6D
	6F	24	6A
	12A	24	12A
	12B	24	12B
6B	6G	12	6E
	6H	12	6A

Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the rows of the Fischer–Clifford matrix $M(1A)$. First we get

$$< (\Psi_2)_N, 1_N >= \frac{1}{32}(1 - 6 - 10 + 15) = 0,$$

$$< (\Psi_3)_N, 1_N >= \frac{1}{32}(22 \times 1 + 6 \times 0 + 10 \times 8 + 15 \times 6) = \frac{1}{32}(22 + 80 + 90) = 6,$$

$$< (\Psi_4)_N, 1_N >= \frac{1}{32}(22 \times 1 + 6 \times 0 + 10 \times (-8) + 15 \times 6) = \frac{1}{32}(22 - 80 + 90) = 1,$$

$$< (\Psi_5)_N, 1_N >= \frac{1}{32}(77\times 1 + 6\times 5 + 10\times 21 + 15\times 13) = \frac{1}{32}(77+30+210+195) = 16.$$

Restricting the character Ψ_3 to N, since $< (\Psi_3)_N, 1_N >= 6$, we get the following equations, where a, b, c represent coefficients of $\gamma_2, \gamma_3, \gamma_4$, respectively.

$$\begin{aligned} 22 &= 6 + a + 15b + 15c, \\ 0 &= 6 - a + 5b - 5c, \\ 8 &= 6 - a - 3b + 3c, \\ 6 &= 6 + a - b - c. \end{aligned}$$

Solving we get: $a = 1, b = 0$, and $c = 1$. So we have the following decomposition:

$$(\Psi_3)_N = 6\gamma_1 + \gamma_2 + \gamma_4.$$

By considering the coefficients of γ_2 and γ_4 in the above decomposition, we deduce that we have irreducible characters χ_2 and $\chi_4 \in Irr(\overline{G})$ with $deg(\chi_2) = 1$ and $deg(\chi_4) = 15$. Since $deg(\chi_2) = 1$, we only need to use the ordinary character table of H_2. For $deg(\chi_4) = 15$, if $[x_1, x_2, \cdots, x_t]$ is the transpose of the partial entries for the projective characters of H_4 on $1A$, then $C_4(1A)M_4(1A)$ is a $t\times 4$ matrix with first set entry $15x_1 = 15$, and hence $x_1 = 1$. This shows that the partial character table of H_4 that we used contains a character of degree 1. Thus, the partial character table comes from an ordinary character table of H_4. Similarly, one can show that $< (\Psi_3)_N, \gamma_2 >= 6$. This gives us $(\Psi_3)_N = \gamma_1 + 6\gamma_2 + \gamma_3$. So again, H_1 and H_3 have partial character tables that each contain a character of degree 1. Therefore, the partial character tables of H_1 and H_3 are from ordinary character tables of S_6 and $S_4 \times 2$, respectively.

Using fusions, centralizer orders of $\overline{G}$, and properties of Fischer–Clifford matrices, we complete Table 7 of Fischer–Clifford matrices. The fusion of $\overline{G}$ into $HS{:}2$ together with the restriction of characters of $HS{:}2$ to $\overline{G}$ forces the signs of the Fischer–Clifford matrices and the order of the elements of the conjugacy classes of $\overline{G}$.

To compute the character table of $\overline{G}$, as an example consider the following. Let $C_1(2A), C_2(2A)$, $C_3(2A), C_4(2A)$ be the partial character tables of the inertia factors for the classes that fuse to $2A \in S_6$. Then the portions of the character table of $\overline{G} = 2^5{\cdot}S_6$ corresponding to the coset $2A$ are:

$$C_1(2A)M_1(2A) = \begin{bmatrix} 1 \\ -1 \\ -3 \\ 3 \\ -1 \\ 1 \\ -3 \\ 3 \\ -2 \\ 2 \\ 0 \end{bmatrix} [1\ 1\ 1\ 1\ 1\ 1] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & -3 & -3 & -3 & -3 & -3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -3 & -3 & -3 & -3 & -3 & -3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ -2 & -2 & -2 & -2 & -2 & -2 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Table 7. The Fischer–Clifford matrices of $2^{5\cdot}S_6$.

$M(1A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 15 & 5 & -3 & -1 \\ 15 & -5 & 3 & -1 \end{bmatrix}$	$M(2A) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ -6 & 6 & 2 & -2 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ -6 & -6 & 2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$
$M(2B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -2 & 2 & 2 & -2 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ -2 & -2 & 2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$	$M(2C) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 3 & -1 & -1 \\ -3 & 3 & 1 & -1 \end{bmatrix}$
$M(3A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & -3 & 1 & -1 \\ 3 & 3 & -1 & -1 \end{bmatrix}$	$M(3B) = M(5A) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
$M(4A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$	$M(4B) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$
$M(6A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$	$M(6B) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$C_2(2A)M_2(2A) = \begin{bmatrix} 1 \\ -1 \\ -3 \\ 3 \\ -1 \\ 1 \\ -3 \\ 3 \\ -2 \\ 2 \\ 0 \end{bmatrix} [-1\ 1\ -1\ 1\ -1\ 1] = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 3 & -3 & 3 & -3 & 3 & -3 \\ -3 & 3 & -3 & 3 & -3 & 3 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 3 & -3 & 3 & -3 & 3 & -3 \\ -3 & 3 & -3 & 3 & -3 & 3 \\ 2 & -2 & 2 & -2 & 2 & -2 \\ -2 & 2 & -2 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3(2A)M_3(2A) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -2 & 0 \\ 2 & 0 \\ -3 & -1 \\ 3 & -1 \\ -3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -6 & 6 & 2 & -2 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & 3 & -1 & -1 & -1 \\ 5 & -7 & -3 & 1 & 1 & 1 \\ 7 & -5 & -1 & 3 & -1 & -1 \\ -7 & 5 & 1 & -3 & 1 & 1 \\ -2 & -2 & -2 & -2 & 2 & 2 \\ -2 & -2 & -2 & -2 & 2 & 2 \\ 3 & -9 & -5 & -1 & 3 & 3 \\ 9 & -3 & 1 & 5 & -3 & -3 \\ -9 & 3 & -1 & -5 & 3 & -3 \\ -3 & 9 & 5 & 1 & -3 & -3 \end{bmatrix}.$$

$$C_4(2A)M_4(2A) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -2 & 0 \\ 2 & 0 \\ -3 & -1 \\ 3 & -1 \\ -3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -6 & -6 & 2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -7 & 3 & -1 & 1 & -1 \\ 5 & 7 & -3 & -1 & 1 & -1 \\ 7 & 5 & -1 & -3 & -1 & 1 \\ -7 & -5 & 1 & 3 & 1 & -1 \\ -2 & 2 & -2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 & -2 & 2 \\ 3 & 9 & -5 & 1 & 3 & -3 \\ 9 & 3 & 1 & -5 & -3 & 3 \\ -9 & -3 & -1 & 5 & 3 & -3 \\ -3 & -9 & 5 & -1 & -3 & 3 \end{bmatrix}.$$

We give the fusion of $\overline{G}_1$ into $\overline{G}$ in Table 8 and the character table of $\overline{G}$ in Table 9. Note that $\chi(2^{5\cdot}S_6|S1) = \chi_1 + \chi_2$, $\chi(2^{5\cdot}S_6|S2) = \chi_1 + \chi_{12}$ and $\chi(2^{5\cdot}S_6|S3) = \chi_1 + \chi_{13}$.

Table 8. The fusion of $2^{4\cdot}S_6$ into $2^{5\cdot}S_6$.

$[x]_{2^{4\cdot}S_6}$	$\longrightarrow$ $[g_1]_{2^{5\cdot}S_6}$	$[x]_{2^{4\cdot}S_6}$	$\longrightarrow$ $[g_1]_{2^{5\cdot}S_6}$
$1A$	$1A$	$4E$	$4H$
$2A$	$2C$	$4F$	$4K$
$2B$	$2F$	$4G$	$4L$
$2C$	$2G$	$5A$	$5A$
$2D$	$2I$	$6A$	$6B$
$3A$	$3A$	$6B$	$6F$
$3B$	$3B$	$6C$	$6G$
$4A$	$4A$	$8A$	$8C$
$4B$	$4B$	$8B$	$8B$
$4C$	$4E$	$12A$	$12A$
$4D$	$4F$		

Table 9. The character table of $2^{5\cdot}S_6$.

	$1A$				$2A$						$2B$					
	1a	2a	2b	2c	2d	4a	2e	4b	4c	2f	2g	2h	4d	4e	4f	4g
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
χ_3	5	5	5	5	-3	-3	-3	-3	-3	-3	1	1	1	1	1	1
χ_4	5	5	5	5	3	3	3	3	3	3	1	1	1	1	1	1
χ_5	5	5	5	5	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
χ_6	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1
χ_7	9	9	9	9	-3	-3	-3	-3	-3	-3	1	1	1	1	1	1
χ_8	9	9	9	9	3	3	3	3	3	3	1	1	1	1	1	1
χ_9	10	10	10	10	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
χ_{10}	10	10	10	10	2	2	2	2	2	2	-2	-2	-2	-2	-2	-2
χ_{11}	16	16	16	16	0	0	0	0	0	0	0	0	0	0	0	0
χ_{12}	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1
χ_{13}	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_{14}	5	-5	-5	5	3	-3	3	-3	3	-3	1	-1	1	-1	1	-1
χ_{15}	5	-5	-5	5	-3	3	-3	3	-3	3	1	-1	1	-1	1	-1
χ_{16}	5	-5	-5	5	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_{17}	5	-5	-5	5	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1
χ_{18}	9	-9	-9	9	3	-3	3	-3	3	-3	1	-1	1	-1	1	-1
χ_{19}	9	-9	-9	9	-3	3	-3	3	-3	3	1	-1	1	-1	1	-1
χ_{20}	10	-10	-10	10	2	-2	2	-2	2	-2	-2	2	-2	2	-2	2
χ_{21}	10	-10	-10	10	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2
χ_{22}	16	-16	-16	16	0	0	0	0	0	0	0	0	0	0	0	0
χ_{23}	15	5	-3	-1	-5	7	3	-1	-1	-1	-1	3	3	-1	-1	-1
χ_{24}	15	5	-3	-1	5	-7	-3	1	1	1	-1	3	3	-1	-1	-1
χ_{25}	15	5	-3	-1	7	-5	-1	3	-1	-1	3	-1	-1	3	-1	-1
χ_{26}	15	5	-3	-1	-7	5	1	-3	1	1	3	-1	-1	3	-1	-1
χ_{27}	30	10	-6	-2	-2	-2	-2	-2	2	2	2	2	2	2	-2	-2
χ_{28}	30	10	-6	-2	2	2	2	2	-2	-2	2	2	2	2	-2	-2
χ_{29}	45	15	-9	-3	3	-9	-5	-1	3	3	-3	1	1	-3	1	1
χ_{30}	45	15	-9	-3	9	-3	1	5	-3	-3	1	-3	-3	1	1	1
χ_{31}	45	15	-9	-3	-9	3	-1	-5	3	3	1	-3	-3	1	1	1
χ_{32}	45	15	-9	-3	-3	9	5	1	-3	-3	-3	1	1	-3	1	1
χ_{33}	15	-5	3	-1	-5	-7	3	1	-1	1	-1	-3	3	1	1	1
χ_{34}	15	-5	3	-1	5	7	-3	-1	1	-1	1	3	3	1	-1	1
χ_{35}	15	-5	3	-1	7	5	-1	-3	-1	1	3	1	-1	-3	-1	1
χ_{36}	15	-5	3	-1	-7	-5	1	3	1	-1	3	1	-1	-3	-1	1
χ_{37}	30	-10	6	-2	-2	2	-2	2	2	-2	2	-2	2	-2	-2	0
χ_{38}	30	-10	6	-2	2	-2	2	-2	-2	2	2	-2	2	-2	-2	2
χ_{39}	45	-15	9	-3	3	9	-5	1	3	-3	-3	-1	1	3	1	-1
χ_{40}	45	-15	9	-3	9	3	1	-5	-3	3	1	3	-3	-1	1	-1
χ_{41}	45	-15	9	-3	-9	-3	-1	5	3	-3	1	3	-3	-1	1	-1
χ_{42}	45	-15	9	-3	-3	-9	5	-1	-3	3	-3	-1	1	3	1	-1

Table 9. Continued.

	$2C$				$3A$				3B		$4A$			
	2i	2j	4h	4i	3a	6a	6b	6c	3b	6d	4j	8a	4k	8b
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1
χ_3	1	1	1	1	2	2	2	2	-1	-1	-1	-1	-1	-1
χ_4	-1	-1	-1	-1	2	2	2	2	-1	-1	1	1	1	1
χ_5	3	3	3	3	-1	-1	-1	-1	2	2	1	1	1	1
χ_6	-3	-3	-3	-3	-1	-1	-1	-1	2	2	-1	-1	-1	-1
χ_7	-3	-3	-3	-3	0	0	0	0	0	0	1	1	1	1
χ_8	3	3	3	3	0	0	0	0	0	0	-1	-1	-1	-1
χ_9	2	2	2	2	1	1	1	1	1	1	0	0	0	0
χ_{10}	-2	-2	-2	-2	1	1	1	1	1	1	0	0	0	0
χ_{11}	0	0	0	0	-2	-2	-2	-2	-2	-2	0	0	0	0
χ_{12}	1	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	1
χ_{13}	-1	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	-1
χ_{14}	1	-1	1	-1	2	-2	2	-2	-1	1	1	1	-1	-1
χ_{15}	-1	1	-1	1	2	2	2	-2	-1	1	-1	-1	1	1
χ_{16}	3	-3	3	-3	-1	1	-1	1	2	-2	-1	-1	1	1
χ_{17}	-3	3	-3	3	-1	1	-1	1	2	-2	1	1	-1	-1
χ_{18}	-3	3	-3	3	0	0	0	0	0	0	-1	-1	1	1
χ_{19}	3	-3	3	-3	0	0	0	0	0	0	1	-1	-1	1
χ_{20}	2	-2	2	-2	1	-1	1	-1	1	-1	0	0	0	0
χ_{21}	-2	2	-2	2	1	-1	1	-1	1	-1	0	0	0	0
χ_{22}	0	0	0	0	-2	2	-2	2	-2	2	0	0	0	0
χ_{23}	3	3	-1	-1	3	-3	-1	1	0	0	-1	1	-1	1
χ_{24}	-3	-3	1	1	3	-3	-1	1	0	0	1	-1	1	-1
χ_{25}	3	3	-1	-1	3	-3	-1	1	0	0	1	-1	1	-1
χ_{26}	-3	-3	1	1	3	-3	-1	1	0	0	-1	1	-1	1
χ_{27}	-6	-6	2	2	-3	3	1	-1	0	0	0	0	0	0
χ_{28}	6	6	-2	-2	-3	3	1	-1	0	0	0	0	0	0
χ_{29}	3	3	-1	-1	0	0	0	0	0	0	-1	1	-1	1
χ_{30}	-3	-3	1	1	0	0	0	0	0	0	-1	1	-1	1
χ_{31}	3	3	-1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{32}	-3	-3	1	1	0	0	0	0	0	0	1	-1	1	-1
χ_{33}	-3	3	1	-1	3	3	-1	-1	0	0	1	-1	-1	1
χ_{34}	3	-3	-1	1	3	3	-1	-1	0	0	-1	1	1	-1
χ_{35}	-3	3	1	-1	3	3	-1	-1	0	0	-1	1	1	-1
χ_{36}	3	-3	-1	1	3	3	-1	-1	0	0	1	-1	-1	1
χ_{37}	6	-6	-2	2	-3	-3	1	1	0	0	0	0	0	0
χ_{38}	-6	6	2	-2	-3	-3	1	1	0	0	0	0	0	0
χ_{39}	-3	3	1	-1	0	0	0	0	0	0	1	-1	-1	1
χ_{40}	3	-3	-1	1	0	0	0	0	0	0	1	-1	-1	1
χ_{41}	-3	3	1	-1	0	0	0	0	0	0	-1	1	1	-1
χ_{42}	3	-3	-1	1	0	0	0	0	0	0	-1	1	1	-1

Table 9. Continued.

	$4B$				$5A$		6A				$6B$	
	4l	8c	4m	8d	5a	10a	6e	6f	12a	12b	6g	6h
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_3	-1	-1	-1	-1	0	0	0	0	0	0	1	1
χ_4	-1	-1	-1	-1	0	0	0	0	0	0	-1	-1
χ_5	-1	-1	-1	-1	0	0	-1	-1	-1	-1	0	0
χ_6	-1	-1	-1	-1	0	0	1	1	1	1	0	0
χ_7	1	1	1	1	-1	-1	0	0	0	0	0	0
χ_8	1	1	1	1	-1	-1	0	0	0	0	0	0
χ_9	0	0	0	0	0	0	1	1	1	1	-1	-1
χ_{10}	0	0	0	0	0	0	-1	-1	-1	-1	1	1
χ_{11}	0	0	0	0	1	1	0	0	0	0	0	0
χ_{12}	1	1	-1	-1	1	-1	1	-1	1	-1	-1	1
χ_{13}	1	1	-1	-1	1	-1	-1	1	-1	1	1	-1
χ_{14}	-1	-1	1	1	0	0	0	0	0	0	-1	1
χ_{15}	-1	-1	1	1	0	0	0	0	0	0	1	-1
χ_{16}	-1	-1	1	1	0	0	-1	1	-1	1	0	0
χ_{17}	-1	-1	1	1	0	0	1	-1	1	-1	0	0
χ_{18}	1	1	-1	-1	-1	1	0	0	0	0	0	0
χ_{19}	1	1	-1	-1	-1	1	0	0	0	0	0	0
χ_{20}	0	0	0	0	0	0	1	-1	1	-1	1	-1
χ_{21}	0	0	0	0	0	0	-1	1	-1	1	-1	1
χ_{22}	0	0	0	0	1	-1	0	0	0	0	0	0
χ_{23}	-1	1	-1	1	0	0	-1	1	1	-1	0	0
χ_{24}	-1	1	-1	1	0	0	1	-1	-1	1	0	0
χ_{25}	1	-1	1	-1	0	0	-1	1	1	-1	0	0
χ_{26}	1	-1	1	-1	0	0	1	-1	-1	1	0	0
χ_{27}	0	0	0	0	0	0	-1	1	1	-1	0	0
χ_{28}	0	0	0	0	0	0	1	-1	-1	1	0	0
χ_{29}	1	-1	1	-1	0	0	0	0	0	0	0	0
χ_{30}	-1	1	-1	1	0	0	0	0	0	0	0	0
χ_{31}	-1	1	-1	1	0	0	0	0	0	0	0	0
χ_{32}	1	-1	1	-1	0	0	0	0	0	0	0	0
χ_{33}	1	-1	-1	1	0	0	1	1	-1	-1	0	0
χ_{34}	1	-1	-1	1	0	0	-1	-1	1	1	0	0
χ_{35}	-1	1	1	-1	0	0	1	1	-1	-1	0	0
χ_{36}	-1	1	1	-1	0	0	-1	-1	1	1	0	0
χ_{37}	0	0	0	0	0	0	1	1	-1	-1	0	0
χ_{38}	0	0	0	0	0	0	-1	-1	1	1	0	0
χ_{39}	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{40}	1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{41}	1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{42}	-1	1	1	-1	0	0	0	0	0	0	0	0

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References

[1] Ali, F.: Fischer-Clifford Theory for Split and Non-Split Group Extensions. PhD, University of Natal, Pietermaritzburg 2001.

[2] Ali, F., Moori, J.: Fischer-Clifford matrices of the non-split group extension $2^6{:}U_4(2)$. Quaest. Math. 31, 27–36 (2008).

[3] Ali, F., Moori, J.: The Fischer-Clifford matrices and character table of a maximal subgroup of $\mathbf{Fi_{24}}'$. Algebra Colloq. 3, 389–414 (2010).

[4] Barraclough, R.W.: Some Calculations Related to the Monster Group. PhD, University of Birmingham, Birmingham 2005.

[5] Brauer, R.: Representations of finite groups. In: Lectures on Modern Mathematics (Ed.: T. L. Saaty) 133–175, J. Wiley and Sons (1963).

[6] Clifford, A.H.: Representations induced in an invariant subgroup. Ann. Math. 38, 533–550 (1937).

[7] Conway, J.H.: A perfect group of order 8315553613086720000 and the sporadic simple groups. In: Proceedings of the National Academy of Science of the United States of America 61, 398ğ400 (1968).

[8] Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: Atlas of Finite Groups. Oxford. Oxford University Press 1985.

[9] Darafsheh, M.R.: Recognition of the projective special linear group over GF(3). Acta Math. Sin. (Engl. Ser.) 26, 477–488 (2010).

[10] Darafsheh, M.R., Iranmanesh, A., Moosavi, S.: A rational property of the irreducible characters of a finite group. In: Groups St Andrews 2009 in Bath. Volume 1, 224–227. London Math. Soc. Lecture Note Ser. 387 (2011).

[11] Fischer, B.: Clifford Matrizen. Manuscript (1982).

[12] Fischer, B.: Unpublished manuscript (1985).

[13] Fischer, B.: Clifford matrices. In: Representation Theory of Finite Groups and Finite-Dimensional Lie Algebras (Eds.: G.O. Michler and C.M. Ringel), 1ğ16, Birkhäuser, Basel. (1991).

[14] GAP Group: GAP - Groups, Algorithms and Programming, Version 4.4. Aachen, St Andrews 2008. (`http://www-gap.dcs.st-and.ac.uk/~gap`).

[15] Higman, D.H.: On the simple group of D.H. Higman and C.C. Sims. Illinois J. Math. 13, 74–80 (1969).

[16] Higman, D.H., Sims, C.C.: A simple group of order 44352000. Mathematische Zeittschrift 105, 110ğ113 (1968).

[17] Humphreys, J.F.: Projective character tables for the finite simple groups of order less than one million. Comm. Algebra 7, 725–751 (1983).

[18] Iranmanesh, A., Faramarzi, A.: Projective representations of the group $G =< a, b, c | a^{p^2} = b^p = c^p = 1, b^{-1}ab = ac, c^{-1}ac = a^{p+1}, c^{-1}bc = b >$. Int. Math. J. 5, 75–83 (2004).

[19] Isaacs, I.M.: Character Theory of Finite Groups. San Diego. Academic Press 1976.

[20] Karpilovsky, G.: Projective representations of finite groups. New York. Marcel Dekker 1985.

[21] Leech, J.: Notes on sphere packings. Canad. J. Math. 19, 251–257 (1967).

[22] Moori, J.: On certain groups associated with the smallest Fischer group. J. London Math. Soc. 2, 61–67 (1981).

[23] Moori, J.: On the Groups G^+ and G of the forms $2^{10}{:}M_{22}$ and $2^{10}{:}\overline{M}_{22}$. PhD, University of Birmingham 1975.

[24] Moori, J.: Representation Theory. Lecture Notes, University of Natal, Pietermaritzburg.

[25] Moori, J., Mpono, Z.: The Fischer-Clifford matrices of the group $2^6{:}SP_6(2)$. Quaest. Math. 22, 257–298 (1999).

[26] Moori, J., Seretlo, T.: On a maximal subgroup of the Lyons group Ly. Submitted.

[27] Mpono, Z.E., Moori, J., Seretlo, T.T.: A group $2^7{:}S_8$ in $\overline{Fi}_{22}$. South East Asian Bull. Math. In press.

[28] Mpono, Z.: Fischer-Clifford Theory and Character Tables of Group Extensions. PhD, University of Natal, Pietermaritzburg 1998.

[29] Pahlings, H.: The character table of $2_+^{1+22}{}^{\cdot}Co_2$. J. Algebra 315, 301–325 (2007).

[30] Read, E.W.: Projective characters of the Weyl group of type F_4. J. London Math. Soc. 8, 83–93 (1974).

[31] Read, E.W.: The linear and projective characters of the finite reflection groups of type H_4. Quart. J. Math. Oxford 25, 73–79 (1974).

[32] Read, E.W.: On projective representations of finite reflection groups of type B_l and D_l. J. London Math. Soc. 10, 129–142 (1975).

[33] Rodrigues, B.G.: On the Theory and Examples of Group Extensions. MSc, University of Natal, Pietermaritzburg 1999.

[34] Seretlo, T.T.: Fischer Clifford Matrices and Character Tables of Certain Groups Associated with Simple Groups $O_{10}^+(2)$, HS and Ly. PhD, University of KwaZulu-Natal, Pietermaritzburg 2012.

[35] Whitley, N.S.: Fischer Matrices and Character Tables of Group Extensions. MSc, University of Natal, Pietermaritzburg 1994.

[36] Wilson, R.A., Walsh, R.P., Tripp, J., Suleiman, I., Rogers, S, Parker, R., Norton, S., Nickerson, S., Linton, S., Bray, J., Abbot, R.: Atlas of Finite Group Representation. 2006.

[37] Wilson, R. A.: The Finite Simple Groups. Berlin. Springer Verlag 2007.